Math 5B: Supplemental notes on lecture and Chapters 3, 5, 6, 7, 8.1

You should be able to read these notes along with your book (and refer to the pictures in the book when needed). This is a brief summary of the topics we've covered in the second half of the course; you can use this (and the notes from the first half of the course) to review for the final! Of course, you should also review your lecture notes, examples from class, and homework problems.

Chapter 3

Curves

Curves are one-dimensional objects, although they sit in two- or three-dimensional space, and they can therefore be described by a single *parameter*. There are many different ways to describe the same curve; a *parameterization* is simply a function $\mathbf{c}(t)$ defined for $t \in [a, b]$ that, for every value of t (the parameter), returns the (two- or three-, or more!, dimensional) point on the curve. We often think of the parameter as time: we start at the *initial point* $\mathbf{c}(a)$, then as the time t increases, we trace out the points along the path $\mathbf{c}(t)$, ending at $t = b$ at the terminal point $\mathbf{c}(b)$. (See Figures 2.52 and 2.53 in Section 2.5.)

We had previously seen parametric equations of a line, and now we have examples of various parametric equations for circles, ellipses, spirals, etc. (See Section 3.1).

The *velocity vector*, given by $\mathbf{v}(t) = \mathbf{c}'(t)$, always gives the direction of the tangent line of the curve at the point $\mathbf{c}(t)$. We can use this fact to find parametric equations of the tangent line at any given point: See Example 3.13. We also saw a few physical applications: for instance, given the *acceleration vector* ($\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{c}''(t)$), we could integrate to find the position function $\mathbf{c}(t)$. (Example 3.14)

Finally, we found a formula for the length of a curve given by the parameterization $\mathbf{c}(t)$. The key fact is that each small piece of the curve is approximated by the tangent vector $\mathbf{c}'(t)\Delta t$. See Figures 3.25 and 5.17. If you understand this geometrically, the formulas for the length, as well as the formulas for the path integral, make a lot more sense!! (*Note: Analytically, this fact is really just the mean value theorem!*) Knowing this, we sum up each small bit of length along the curve by computing the integral:

$$\ell(\mathbf{c}) = \int_a^b \|\mathbf{c}'(t)\|\, dt.$$

Of course, this definition must give the same number no matter which parameterization we used to describe the curve since the length is an intrinsic physical property of the curve. (*Mathematically, given two parameterizations for the same curve we can make a change of variables to change from one to the other and see that the integrals stay the same, but this is beyond what we did in class.*)

If instead of integrating all the way to b, we stop at t, we find the length of the curve between $\mathbf{c}(a)$ and $\mathbf{c}(t)$. This gives the *arc-length function*, defined for $t \in [a, b]$:

$$s(t) = \int_a^t \|\mathbf{c}'(\tau)\| d\tau.$$

(Notice we switch to a different dummy variable inside the integral since we use t as a variable for the function.) From the definition, we always know the values $s(a) = 0$ and $s(b) = \ell(\mathbf{c})$. Viewing s as a parameter for the curve, we see that at each point $s \in [0, \ell]$, the value of s is exactly the same as the length traced out by the curve from the starting point $\mathbf{c}(0)$ to the point $\mathbf{c}(s)$: Physically, this means that if we think of s as describing the time it takes a particle following this curve to get to $\mathbf{c}(s)$, the particle is always traveling at speed one! In other words, for the arc-length parameterization, $\|\mathbf{c}'(s)\| = 1$. Examples 3.29 and 3.30 show how to reparameterize a curve by arc-length. (Having the arc-length parameterization for a curve is very nice since the speed is then always constant; unfortunately, computing the arc-length function by hand is typically very difficult because of the square root that appears in the integral!)

Chapter 5 and Chapter 7

Path Integrals

The path integral of a scalar function extends the usual 1-d integral to an integral of a function over a curve. For example, consider a curve in the x-y plane: $\mathbf{c}(t) = (x(t), y(t))$, $t \in [a, b]$. If we have a function $f(x, y)$, we can think of only the values of f at points on the curve: $f(\mathbf{c}(t)) = f(x(t), y(t))$. If we graphed these points ($z = f(\mathbf{c}(t))$), we'd be graphing a curve in space that lies above the curve $\mathbf{c}(t)$. As shown in Figure 5.12, the path integral gives the area under the curve. Recall that each bit of length along the curve is approximated by $\mathbf{c}'(t)\Delta t$. The height, of course, is just given by the function, so the correct definition of the path integral of f along $\mathbf{c}$ is

$$\int_{\mathbf{c}} f ds = \int_a^b f(\mathbf{c}(t))\|\mathbf{c}'(t)\| dt.$$

Intuitively, this is summing up each bit of area = height*length. Just as for length, since this integral represents a fixed area, it gives the same numerical answer no matter what parameterization we use to describe the curve. (*Notice that if we had a curve parameterized by arclength, we'd just have something that looks like the integral of f since the curve would have speed one at every point: $\int_{\mathbf{c}} f\, ds = \int_0^\ell f(\mathbf{c}(s))\, ds$; this explains why use the "ds" in the notation for the path integral.*)

We also defined the path integral of a vector function. The physical motivation was to add up the work done by a force field $\mathbf{F}$ along a curve $\mathbf{c}(t)$. We only care about the values of the force on the curve $\mathbf{F}(\mathbf{c}(t))$, and we only care about the part of the force that points tangent to the curve! (This is since the work done is the dot product of the force with the direction the particle is moving.) In the end, we have a number representing work given by the path integral

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt.$$

Notice that we always dot the force with the velocity vector, which is tangent to the line at each point! If the force is perpendicular to the curve at each point, no work is done, and the dot product is largest if the force is tangent to the curve at each point.

The path integral of a vector function also does not depend on how the curve is parameterized, but it does depend on the *orientation* of the curve. If the particle went the opposite direction, the work done by the force would have to change sign! (See Theorem 5.3.)

One bit of notation that you will often see uses components: If $\mathbf{F}(x, y) = (F_1(x, y), F_2(x, y))$ and the curve $\mathbf{c}(t) = (x(t), y(t))$ for $t \in [a, b]$,

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{c}} F_1\, dx + F_2\, dy = \int_a^b \left[F_1(x, y)\frac{dx}{dt} + F_2(x, y)\frac{dy}{dt} \right] dt$$

Gradient Vector Fields

Usually, the path integral of a vector field changes when the path changes, even if it starts and ends at the same points. However, we have a version of the *Fundamental Theorem of Calculus*: If $\mathbf{F}$ is a *gradient vector field* (that is, if $\mathbf{F} = \nabla f$ for some scalar function f), the integral depends only on the endpoints of the path:

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = f(\mathbf{c}(b)) - f(\mathbf{c}(a)).$$

This means that if $\mathbf{F}$ is a gradient vector field and the curve is *closed*, then $\oint_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = 0$.

One way to tell that $\mathbf{F}$ is a gradient vector field (other than guess a function f that works!) is to check its curl:

Theorem 5.4: If curl $\mathbf{F} = \mathbf{0}$ at every point, then $\mathbf{F} = \nabla f$ for some scalar function f. (And if curl $\mathbf{F} \neq \mathbf{0}$, then $\mathbf{F}$ is not a gradient vector field.)

(*This assumes* $\mathbf{F}$ *is defined everywhere on* $\mathbb{R}^2$ *or* $\mathbb{R}^3$*; it also holds if* $\mathbf{F}$ *is defined everywhere on a nice simply-connected set.*)

Surface Integrals

Surfaces are two-dimensional, although they sit in three-dimensional space. As we did for one-dimensional cuves, this means we can *parameterize* a surface using two parameters: A parameterization for a surface is a function of two variables, say u and v, that for each pair of parameters (u, v) (in a domain D), returns a point on the surface $\mathbf{r}(u, v)$. In terms of components, each point on the surface is described by its three components x, y and z:

$$\mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v)).$$

See Figures 7.4, 7.6, 7.8, and 7.10 for examples of parameterizations of surfaces such as the cylinder, cone, and sphere. Notice the parameterization for the sphere uses *spherical coordinates* (See Definition 2.4 and Figure 2.72 from Section 2.8 for a description of these coordinates!)

For curves, we needed to know the length of each small part of the curve; for surfaces, we need to know the area of each small part of the surface (See Figure 7.49). We approximate this area by the small piece of the tangent plane. Notice that two tangent vectors that span the tangent plane at each point are given by the partial derivatives of $\mathbf{r}(u, v)$ (this is similar to how the tangent vector to a curve is given by the derivative of $\mathbf{c}(t)$!):

$$\begin{aligned} \mathbf{T}_u &= \frac{\partial \mathbf{r}}{\partial u} = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right) \\ \mathbf{T}_v &= \frac{\partial \mathbf{r}}{\partial v} = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right) \end{aligned}$$

The normal vector to the surface at each point (u, v) is then found by taking the cross product:

$$\mathbf{N}(u, v) = \mathbf{T}_u(u, v) \times \mathbf{T}_v(u, v).$$

To approximate each small bit of area, we need to shrink both tangent vectors (by the change in u and the change in v respectively) so that they span only a small parallelogram on the tangent plane, the area of which is given by $\|\mathbf{N}(u, v)\| \Delta u \Delta v$ (use the formula for the area of a parallelogram from Chapter 1!). Using this we can define the *surface area* of a surface S:

$$\iint_S dS = \iint_D \|\mathbf{N}(u, v)\| \, dA \left(= \iint_D \|\mathbf{N}(u, v)\| \, du dv \right).$$

where D is the domain of the (u, v) parameters. (The double integral over a domain is defined in Chapter 6; see below.) Similar to how we defined path integrals, we also define the *surface integral of f over S*:

$$\iint_S f \, dS = \iint_D f(\mathbf{r}(u, v)) \|\mathbf{N}(u, v)\| \, dA$$

where $\mathbf{r}(u, v)$ is any parameterization of the surface and the normal vector is defined as above.

Note: We didn't cover this in class, so it won't be on the test, but we can also define the surface integral of a vector field. We simply take the dot product of the vector field with the normal vector at every point; this means we are only interested in the amount of the vector $\mathbf{F}$ *that points out of the surface. The resulting integral physically represents the* flux *of* $\mathbf{F}$ *across the surface. You will see such integrals again in 5C and in classes on fluid mechanics or electricity and magnetism.*

Chapter 6

Double and triple integrals

The double integral of a function $f(x, y)$ on a domain D (in the x-y plane) is defined as the volume under the graph of the surface $z = f(x, y)$ above D. (Section 6.1; Figures 6.2, 6.4.) This is the same as summing up (meaning integrating!) cross-sectional areas (Figure 6.18) – just as you did for volumes of revolution in calculus. Integrating up cross-sectional areas is equivalent to evaluating the iterated integral since each cross-sectional area is found by computing an integral in either the x- or y-direction. (See the discussion on page 378, above Theorem 6.3: Fubini's Theorem.)

See Sections 6.2, 6.3, and the examples from class, for how to evaluate double integrals over various domains that aren't rectangles. It usually boils down to deciding which direction to integrate in first, and making sure the bounds start and end at the right values for the domain! (Drawing a picture of the region you're integrating over will almost always be helpful.) Switching the order of integration is often useful and, as we've seen, can sometimes turn a harder integral into an easier one.

Triple integrals define how to integrate a function of three variables $f(x, y, z)$ over a domain in three-dimensional space. They are also computed by doing the iterated integrals. (See Section 6.5 for some examples.)

Change of Variables

A change of variables can be thought of as transforming old coordinates (u, v) to new coordinates (x, y). One useful change of variables we've already seen is polar coordinates: the formulas $x(r, \theta) = r\cos\theta$ and $y(r, \theta) = r\sin\theta$ define how to get (x, y) from (r, θ). (Of course, we could also go backwards; how do you get $r(x, y)$ and $\theta(x, y)$ if you know the (x, y) point?)

Changing variables in the integral is similar to u-substitution. Instead of describing the integral in the (x, y) variables, we need to describe everything in terms of the (u, v) variables: This means writing the function in terms of the (u, v) variables, finding the corresponding domain D^* for the (u, v) variables, and finally describing how the area ("$dx\,dy$") changes, which requires the absolute value of the *Jacobian*:

$$\iint_D f(x, y)\,dx\,dy = \iint_{D^*} f(x(u, v), y(u, v)) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\,dv.$$

We discussed this a lot in class, and went over several examples; you may also review the examples in Section 6.4 and from the homework. If you want more practice using this formula, look at Exercises 6.4, #1–#5 (polar coordinates) and #22–#28.

Section 8.1

Green's Theorem

Green's Theorem relates a double integral of a special combination of derivatives of the components of a vector field $\mathbf{F}$ on a domain D to the path integral of the vector field around the (positively oriented) boundary of the domain D. (The *boundary* of D is often denoted by ∂D). If we write $\mathbf{F}(x, y) = (P(x, y), Q(x, y))$,

$$\int_{\mathbf{c}} \mathbf{F}\cdot d\mathbf{s} = \int_{\mathbf{c}} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

where $\mathbf{c} = \partial D$. (Remember that the first equality is just two different ways to write the path integral of $\mathbf{F}$.) We think of this as a type of fundamental theorem of calculus since the theorem allows us to take away both one integral and one derivative from the right-hand-side as if they "cancelled" each other out.

One important thing to remember about the boundary curve in Green's Theorem is that it has positive orientation: that is, the outer curve around the boundary goes counter-clockwise and any inner curves must go clockwise. An easy way to remember this is that if you walk around the boundary in the positive direction, you always keep the domain on your left-hand side. Notice that our usual parameterization of the unit circle centered at $(0,0)$ – $\mathbf{c}(t) = (\cos\theta, \sin\theta)$ – has positive orientation. How can we parameterize the unit circle going in the opposite (clockwise) direction?

We can rewrite Green's theorem using the curl of $\mathbf{F}$ (see pg. 506):

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \iint_D \text{curl } \mathbf{F} \cdot \mathbf{k} \, dA$$

Physically, this says that the circulation of a vector field around a closed curve is related to its curl inside the curve.

We often use Green's theorem to turn a line integral into a simpler-to-compute double integral. There is also a nice application that allows us to find the area of a domain D by integrating along its boundary instead! Whenever we have a vector field (P, Q) that happens to have $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$, we know by Green's theorem that $\iint 1 \, dA = \int_{\mathbf{c}=\partial D} P\,dx + Q\,dy$. For instance, we have the following three formulas,

$$\begin{aligned} \text{Area of } D = \iint 1\,dA &= \frac{1}{2}\int_{\mathbf{c}} x\,dy - y\,dx \\ &= \int_{\mathbf{c}} x\,dy \\ &= -\int_{\mathbf{c}} y\,dx. \end{aligned}$$

Computing any one of the line integrals above must give the area of D. (See Example 8.5.)

In case anyone wanted to see all the details, this last section is a write-up of the proof we outlined in class.

A simple proof of Green's theorem, in the case when the domain is the unit square, makes it very clear that the result is related to the fundamental theorem of calculus. Let $D = [0,1] \times [0,1]$, and $\mathbf{F}(x,y) = (P(x,y), Q(x,y))$ be a vector field. We will show that Green's theorem holds by computing both sides. First, using properties of double integrals, and Fubini's theorem,

$$\begin{aligned} \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy &= \int_0^1 \left(\int_0^1 \frac{\partial Q}{\partial x}\,dx\right) dy - \int_0^1 \left(\int_0^1 \frac{\partial P}{\partial y}\,dy\right) dx \\ &= \int_0^1 \left(Q(x,y)\big|_{x=0}^1\right) dy - \int_0^1 \left(P(x,y)\big|_{y=0}^1\right) dx \\ &= \int_0^1 [Q(1,y) - Q(0,y)]\,dy - \int_0^1 [P(x,1) - P(x,0)]\,dx \\ &= \int_0^1 [Q(1,t) - Q(0,t) - P(t,1) + P(t,0)]\,dt \end{aligned}$$

The last step just changes the dummy variables inside the integrals; the middle step used the fundamental theorem of calculus to say that the integral of a derivative just gives the function back (evaluated at the endpoints).

On the other hand, the counter clockwise curve around the square is made up of four lines, each of which we can parameterize. Start with the line along the $x-$axis that goes from (0,0) to (1, 0): $\mathbf{c_1}(t) = (t, 0),\ t \in [0, 1]$. Then, $\mathbf{c_2}(t) = (1, t), t \in [0, 1]$ points upward along the line $x = 1$. The third line points to the left, so we have to be careful with the minus sign when parameterizing it: $\mathbf{c_3}(t) = (-t, 1), t \in [-1, 0]$ parameterizes the top line of the square, starting at (1, 1) and ending at (0, 1). Finally, the last line points down along the $y-$axis: $\mathbf{c_4}(t) = (0, -t), t \in [-1, 0]$.

$$\begin{aligned}
\int_{\mathbf{c}} P\,dx + Q\,dy &= \int_{\mathbf{c_1}} P\,dx + Q\,dy + \int_{\mathbf{c_2}} P\,dx + Q\,dy + \int_{\mathbf{c_3}} P\,dx + Q\,dy + \int_{\mathbf{c_4}} P\,dx + Q\,dy \\
&= \int_0^1 (P(t,0), Q(t,0)) \cdot (1,0)\,dt + \int_0^1 (P(1,t), Q(1,t)) \cdot (0,1)\,dt \\
&\quad + \int_{-1}^0 (P(-t,1), Q(-t,1)) \cdot (-1,0)\,dt + \int_{-1}^0 (P(0,-t), Q(0,-t)) \cdot (0,-1)\,dt \\
&= \int_0^1 [P(t,0) + Q(1,t)]\,dt - \int_{-1}^0 [P(-t,1) + Q(0,-t)]\,dt \\
&= \int_0^1 [P(t,0) + Q(1,t)]\,dt - \int_0^1 [P(u,1) + Q(0,u)]\,du.
\end{aligned}$$

The last step simply changed variables ($u = -t$) in the integral. Putting the integrals back together, we see that we get exactly the same expression we found above: $\int_0^1 [Q(1,t) - Q(0,t) - P(t,1) + P(t,0)]\,dt$. Therefore, we proved that

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy = \int_{\mathbf{c}} P\,dx + Q\,dy,$$

where D is the unit square. It takes more work to prove Green's theorem for general domains, but this proof shows the important idea that Green's theorem is related to the fundamental theorem of calculus!