Math 5B: Quiz 4

Name *Solutions* Perm # ____________

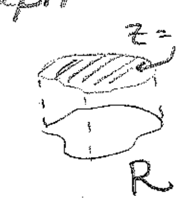
Circle your section number: 1 2 3 4 5 6

There is a second side to the quiz!

1. (a) If R is a region in the $x-y$ plane, what does $\iint_R dA$ represent geometrically?

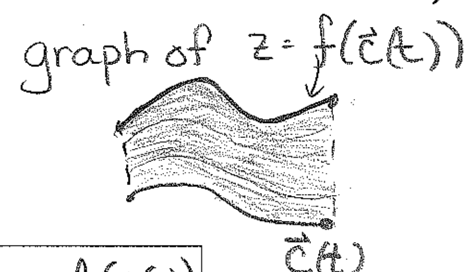
Volume above the region and below the graph of $z=1$; i.e., (area of R) $\times (1)$

$\boxed{\text{Area of the region } R}$



(b) If R is the rectangle $[a,b]\times[c,d]$, give a formula for computing $\iint_R f(x,y)dA$ (Hint: Use Fubini's Theorem):

$$\boxed{\iint_R f(x,y)dA = \int_a^b \left[\int_c^d f(x,y)dy\right]dx} \left(= \int_c^d \left[\int_a^b f(x,y)dx\right]dy \right)$$



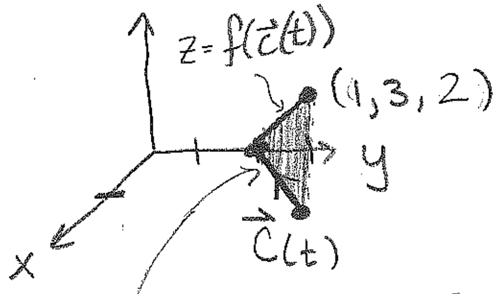
(c) What does $\int_c f ds$ represent geometrically?

$\boxed{\text{The area above } \vec{c}(t) \text{ and below } f(\vec{c}(t))}$

Problem 2 is on the back.

2. (a) If $\mathbf{c}$ is the straight line from $(1, 3)$ to $(0, 2)$ and $f(x, y) = 2x$, find $\int_{\mathbf{c}} f ds$.

graph:



length $= \sqrt{(2-3)^2+(0-1)^2}$
$= \sqrt{2}$

$f(0,2) = 0$
$f(1,3) = 2$

$\int_c f ds =$ area of triangle with height 2 and base $\sqrt{2}$

$$\boxed{\int_{\mathbf{c}} f ds = \sqrt{2}}$$

(b) Consider the curve given by the graph of the function $y = x^2$ from $(0, 0)$ to $(2, 4)$ and the vector field $\mathbf{F}(x, y) = 2x\,\mathbf{i} + \sqrt{y+1}\,\mathbf{j}$. Compute $\int_{\mathbf{c}} \mathbf{F} \cdot ds$.

Notice $\vec{F}$ is a gradient vector field
(you could check curl $\vec{F} = \vec{0}$)

$\vec{F} = \vec{\nabla} f$ where $f(x,y) = x^2 + \frac{2}{3}(y+1)^{3/2}$

$$\int_{\vec{c}} \vec{F} \cdot d\vec{s} = \int_{(0,0)}^{(2,4)} \vec{\nabla} f \cdot d\vec{s} = f(2,4) - f(0,0)$$

$$= \left[(2)^2 + \tfrac{2}{3}(4+1)^{3/2}\right] - \left[0^2 + \tfrac{2}{3}(0+1)^{3/2}\right]$$

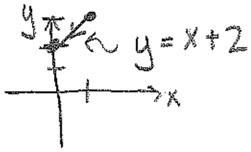
$$= 4 + \tfrac{2}{3}\, 5\sqrt{5} - \tfrac{2}{3} = \tfrac{10}{3} + \tfrac{10}{3}\sqrt{5}$$

$$\boxed{\int_{\mathbf{c}} \mathbf{F} \cdot ds = \tfrac{10}{3}\left(1 + \sqrt{5}\right)}$$

Or, parameterize each curve and compute:

2. (a) If $\mathbf{c}$ is the straight line from $(1,3)$ to $(0,2)$ and $f(x,y) = 2x$, find $\int_{\mathbf{c}} f ds$.

one possible parameterization (there are many)



is $\vec{c}(t) = (-t, -t+2) \quad t \in [-1, 0]$

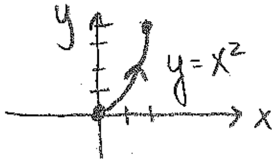
$\vec{c}\,'(t) = (-1,-1) \Rightarrow \|\vec{c}\,'(t)\| = \sqrt{2}$

$$\int_{\vec{c}} f ds = \int_{-1}^{0} f(\vec{c}(t)) \|\vec{c}\,'(t)\| dt = \int_{-1}^{0} 2(-t)\sqrt{2}\, dt$$

$$= -2\sqrt{2} \int_{-1}^{0} t\, dt = -2\sqrt{2}\left(t^2/2\right)\Big|_{-1}^{0} = 0 + \sqrt{2}$$

$$\boxed{\int_{\mathbf{c}} f ds = \sqrt{2}}$$

(b) Consider the curve given by the graph of the function $y = x^2$ from $(0,0)$ to $(2,4)$ and the vector field $\mathbf{F}(x,y) = 2x\,\mathbf{i} + \sqrt{y+1}\,\mathbf{j}$. Compute $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$.



$\vec{c}(t) = (t, t^2) \quad t \in [0,2]$

$\vec{c}\,'(t) = (1, 2t)$

$$\int_{\vec{c}} \vec{F} \cdot d\vec{s} = \int_0^2 \vec{F}(\vec{c}(t)) \cdot \vec{c}\,'(t)\, dt$$

$$= \int_0^2 \left(2t, \sqrt{t^2+1}\right) \cdot (1, 2t)\, dt$$

$$= \int_0^2 \left[2t + 2t\sqrt{t^2+1}\right] dt$$

$$= t^2 + \tfrac{2}{3}(t^2+1)^{3/2} \Big|_0^2 = 4 + \tfrac{2}{3}(5)^{3/2} - \tfrac{2}{3}$$

$$= 10/3 + 2/3 \cdot 5\sqrt{5}$$

$$\boxed{\int_{c} \mathbf{F} \cdot d\mathbf{s} = \frac{10}{3}\left(1 + \sqrt{5}\right)}$$