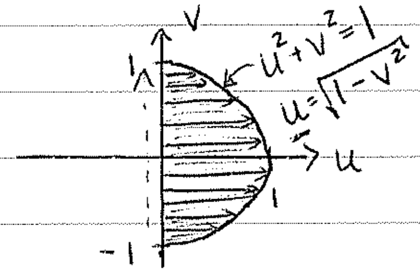
Lecture Notes 11/22

4.3 Changing the order of integration

<u>Ex</u>: Find $\iint_R u\,du\,dv$

where R is the region $u^2+v^2 \le 1$, $u \ge 0$.

1.



For each v from -1 to 1, the limits of integration for u go from 0 to $+\sqrt{1-v^2}$.

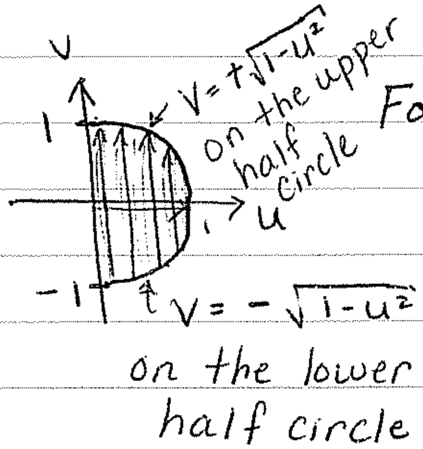
$$\iint_R u\,du\,dv = \int_{-1}^{1}\left(\int_0^{\sqrt{1-v^2}} u\,du\right)dv$$

$$= \int_{-1}^{1} \frac{u^2}{2}\Big|_0^{\sqrt{1-v^2}} dv = \int_{-1}^{1} \frac{1-v^2}{2}\,dv$$

$$= \frac{1}{2}\left(v - \frac{v^3}{3}\right)\Big|_{-1}^{1} = \underline{\underline{\frac{2}{3}}}$$

OR

2.



For each u from 0 to 1, v goes between $(-\sqrt{1-u^2})$ and $(+\sqrt{1-u^2})$

i.e. R is $0 \le u \le 1$

$$-\sqrt{1-u^2} \le v \le +\sqrt{1-u^2}$$

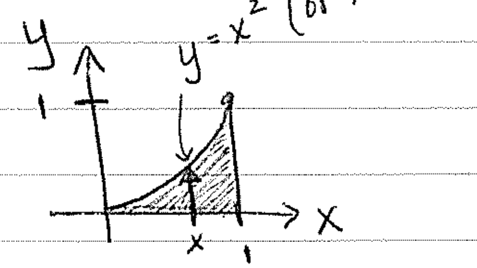
$$\iint_R u\,du\,dv = \int_0^1 \left(\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} u\,dv \right) du$$

$$= \int_0^1 uv \Big|_{v=-\sqrt{1-u^2}}^{\sqrt{1-u^2}} du = \int_0^1 2u\sqrt{1-u^2}\,du$$

$$= \frac{-2}{3}(1-u^2)^{3/2} \Big|_0^1 = 0 + \frac{2}{3} = \underline{\underline{\frac{2}{3}}}$$

Ex. Evaluate $\int_0^1 \int_{\sqrt{y}}^1 \sin(x^3)\,dx\,dy$

The region is $0 \le y \le 1$
$\sqrt{y} \le x \le 1$



Change the order of integration:

For $0 \le x \le 1$, y goes from 0 to x^2.

$$\int_0^1 \int_{\sqrt{y}}^1 \sin(x^3)\,dx\,dy = \int_0^1 \int_0^{x^2} \sin(x^3)\,dy\,dx$$

$$= \int_0^1 \left(\sin(x^3)\, y \Big|_{y=0}^{x^2} \right) dx$$

$$= \int_0^1 x^2 \sin(x^3)\,dx = \frac{-\cos(x^3)}{3} \Big|_0^1 = \underline{\frac{-\cos(1)}{3} + \frac{1}{3}}$$

4.6 Change of variables

Ex. Use polar coordinates to compute

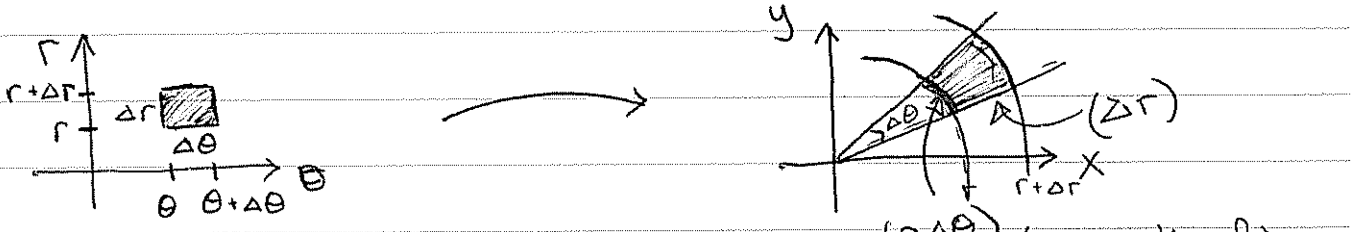
$$\iint_S e^{x^2+y^2}\,dx\,dy$$

where S is the circle $x^2+y^2 \le 4$

$x = r\cos\theta$ so $e^{x^2+y^2}$ becomes e^{r^2}
$y = r\sin\theta$

In the r-θ plane, S is described by
$0 \le r \le 2$
$0 \le \theta \le 2\pi$

How does $dx\,dy$ (the area in the x-y plane) correspond to $dr\,d\theta$ (the area in the r-θ plane)?



$(r\Delta\theta)$ (arclength of circle)

area in r-θ plane: $(\Delta r\,\Delta\theta)$

area in x-y plane:
$\approx (\Delta r)(r\,\Delta\theta)$

As $\Delta r, \Delta\theta \to 0$, we see that $= r\,\Delta r\,\Delta\theta$

$dx\,dy$ corresponds to $\underline{r\,dr\,d\theta}$

Putting everything together,

$$\iint_S e^{x^2+y^2}\,dx\,dy = \underbrace{\int_0^{2\pi}\int_0^2 e^{r^2}\,(r\,dr\,d\theta)}_{\text{everything is in terms of } r \text{ and } \theta!}$$

$$= \int_0^{2\pi} \frac{e^{r^2}}{2}\Big|_{r=0}^{2}\,d\theta$$

$$= \int_0^{2\pi} \left(e^4/2 - 1/2\right) d\theta = \left(e^4/2 - 1/2\right)\theta\Big|_{\theta=0}^{2\pi}$$

$$= 2\pi\left(e^4/2 - 1/2\right) = \underline{\pi\left(e^4 - 1\right)}$$

Notice (area in x-y plane) $\approx$ r (area in $r\theta$ plane)

So r is the ratio of the areas in the x-y and r-θ planes - this should be the Jacobian of the mapping

Recall For polar coordinates $\dfrac{\partial(x,y)}{\partial(r,\theta)} = r$ ✓

Formula for changing variables

$dx\,dy$ is replaced by $\left|\dfrac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta$

absolute value!