Math 8: Functions

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Definitions

- Let S and T be sets. A **function** from S to T is a rule that assigns to each $s \in S$ one element in $t \in T$, denoted $f(s)$. The set S is called the **domain** and the set T is called the **codomain** of f. Notation: We write $f : S \to T$. Also, if $f(s) = t$, we write $s \mapsto t$.

- Let S be any set. The **identity function** on S is the function $\imath_S : S \to S$ defined by $\imath(s) = s$ for all $s \in S$.

- Let $f : S \to T$ be a function. The **image** of f is the set $f(S) = \{f(s) : s \in S\}$. Notice that $f(S) \subseteq T$; i.e., the image is a subset of the codomain.

- We say that two functions f and g are **equal** if they have (1) the same domain, (2) the same codomain, and if (3) for all x in the domain, $f(x) = g(x)$.

- Let $f : S \to T$ be a function. The **graph** of f is the set $\{(x, f(x)) : x \in S\}$. Notice that the graph is a subset of the Cartesian product $S \times T$.

- Let $f : S \to T$ be a function. We say f is **onto** if for all $t \in T$, there exists an $s \in S$ such that $f(s) = t$. In words, the image of f must be the entire codomain.

- Let $f : S \to T$ be a function. We say f is **one-to-one** if for all $s_1, s_2 \in S$, if $f(s_1) = f(s_2)$, then $s_1 = s_2$. (Equivalently, $\forall s_1, s_2 \in S$, if $s_1 \neq s_2$, then $f(s_1) \neq f(s_2)$.) This says that f is one-to-one if every possible output is assumed for only one input.

- Let $f : S \to T$ be a function. We say f is a **bijection** if f is both 1-1 and onto.

- Let $f : S \to T$ and $g : T \to U$ be functions. The **composition** of f and g is the function $g \circ f : S \to U$ defined by $(g \circ f)(s) = g(f(s))$ for all $s \in S$.

- Let $f : S \to T$ and $g : T \to S$ be functions. We say f and g are **inverse functions** if both $g(f(s)) = s$ for all $s \in S$ and $f(g(t)) = t$ for all $t \in T$ (equivalently, if both $g \circ f = \imath_S$ and $f \circ g = \imath_T$).

- Let $f : S \to T$ be a bijection. The **inverse function** of f is the function $f^{-1} : T \to S$ defined by $f^{-1}(t) = s \Leftrightarrow f(s) = t$. (Notice that f^{-1} is a well-defined function because we assume f is one-to-one and onto. Of course, f and f^{-1} are inverse functions.)

- Two sets A and B have the same **cardinality** (or "size") if there exists a bijection from A to B. (In particular, for each n, let $S_n = \{1, 2, ..., n\}$. Then, we can say that A has size n if there is a bijection from A to S_n. But note that our definition of cardinality applies to infinite sets also!)

- A set A is **countable** if there is a bijection from A to $\mathbb{N}$.

For functions whose domains and codomains are of finite size, it can be useful to represent the function by drawing the domain and codomain as circles, listing all of the elements in each set, and showing explicitly where each element in the domain maps to by drawing an arrow (see examples from lecture). Here is a list of some of the other examples we went over in class:

Examples

1. Let S be the set of all humans who have ever lived. Explain why $m : S \to S$ defined by $m(x) =$ "the mother of x" is a function. What is its image? Give two reasons why c, defined by $c(x) =$ "child of x," is *not* a function on S.

2. Define the function $f : \{1, 2, 3, ..., 100\} \to \mathbb{N} \times \mathbb{N}$ by $f(n) = (n^2, n^3)$. Prove that f is 1-1, but not onto.

3. Define $f : \mathbb{N} \times \mathbb{N} \to \mathbb{Z}$ by $f(m, n) = 2m - n$. Prove f is onto. (Fix any $a \in \mathbb{Z}$, then find an element of $\mathbb{N} \times \mathbb{N}$ that returns a. Hint: consider a even and a odd separately.) Show that f is *not* 1-1. (For example, consider $(3, 1)$ and $(4, 3)$.)

4. Let $X = \{a, b, c\}$ and $Y = \{p, q, r, s\}$. Define $f : X \to Y$ by $f(a) = r$, $f(b) = r$, and $f(c) = p$. Draw a picture illustrating this function. What is the image? (Answer: $f(X) = \{p, r\}$.)

5. Consider the following pairs of functions. Are they equal or not?

   (a) Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$, and define $g : \mathbb{N} \to \mathbb{N}$ by $g(n) = n^2$.

   (b) Define $f, g : \mathbb{N} \to \mathbb{N}$ by the formulas: $f(n) = \text{lcm}(2, n)$ and $g(n) = \left(\frac{3 - (-1)^n}{2}\right) n$.

6. Consider $f : S \to \mathbb{R}$ where $S = \{x \in \mathbb{R} : x \geq -4\}$ defined by $f(x) = \sqrt{x + 4}$. What is the image of f? What is the graph of f? Draw a picture of the graph (note that it is a subset of $S \times \mathbb{R}$).

7. Let S be any set. The graph of the identity function $\imath_S$ is the *diagonal* of $S \times S$: $\{(x, x) : x \in S\}$.

8. Draw graphs to help you picture the following functions:

   (a) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ x & \text{if } x \text{ is irrational} \end{cases}$

   (b) $f : \mathbb{Z}_4 \to \mathbb{Z}_4$ defined by $f([x]) = [x^2]$. (What is the image of this function?)

9. Consider the set S of all telephone subscribers (assume each subscriber has only one telephone line.) Let T be the set of all currently active telephone numbers. The telephone directory gives us a way of looking up the values of function $f : S \to T$, $f(s) =$ "s's telephone number." Although it is true that f is a bijection, and therefore has an inverse, given some random number $t \in T$, it is hard to find out the value of $f^{-1}(t)$! (You would have to read through the telephone book until you found the person whose telephone number is t.)

10. Consider $f : \{x \in \mathbb{R} : x \geq 0\} \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$ and $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = 3x - 1$. State whether or not $f \circ g$ or $g \circ f$ is defined (and find the composition if it is).

11. Consider $f, g : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ and $f(x) = 4x - 5$. Find $f \circ g$ and $g \circ f$. Are these functions equal?

12. Let $S = \{1, 2, 3, 4, 5\}$. Define $f : S \to S$ by $1 \mapsto 1$, $2 \mapsto 5$, $3 \mapsto 4$, $4 \mapsto 2$, and $5 \mapsto 4$. Draw a picture of the function f. Also, draw the function $g : S \to S$ defined by $1 \mapsto 2$, $2 \mapsto 3$, $3 \mapsto 4$, $4 \mapsto 1$, and $5 \mapsto 5$. Finally, draw pictures that represent (i) g^{-1}, (ii) $g \circ f$, (iii) $f \circ g$, and (iv) $f \circ g^{-1}$.

13. Let S be the set of all married men, and let T be the set of all people. Consider the "wife of" function $w : S \to T$, the "mother of" function $m : T \to T$, and the "father of" function $f : T \to T$. What do the functions $f \circ m$, $m \circ w$, and $f \circ (m \circ w)$ and $(f \circ m) \circ w$ represent? Explain why the last two are equal functions.

Read Proposition 19.2, which includes a proof of the fact that if $f : S \to T$ and $g : T \to U$ are both bijections, then $g \circ f : S \to U$ is also a bijection. Since a bijection always has an inverse, then knowing f and g have inverses would imply that $(g \circ f)^{-1}$ exists. Can you prove that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$?

Proposition 19.1 discusses what we can tell about sizes of finite sets when we know there exists a function between the sets that is 1-1, onto, or a bijection. For instance, if S and T are finite, and there is some function $f : S \to T$ that is onto, it must be true that $|S| \geq |T|$ (conversely, if there is a function $f : S \to T$ that is 1-1, it must be the case that $|S| \leq |T|$.) This provides the motivation for our definition of cardinality: For any sets (even infinite ones) we define $A \sim B$ (A has the same *cardinality*, or size, as B) if there exists a bijection $f : A \to B$.

Read Propositions 21.2, 21.3, and 21.4 for some results on countable sets. Notice that $\mathbb{Z}$ and $\mathbb{Q}$ are both countable: that is, $\mathbb{N} \sim \mathbb{Z} \sim \mathbb{Q}$. Even though it seems like there must be many more rational numbers than there are natural numbers, the cardinality of these sets is the same because we can define a bijection between the sets. This amounts to finding a way to *list* all of the rational numbers: if we can do this, we can just define the function that takes a natural number n and returns the n^{th} number in the list! See page 182 for a picture and description of how to do this.

Theorem 21.1 proves that $\mathbb{R}$ is an *uncountable* set: that is, there does *not* exist a bijection from $\mathbb{R}$ to $\mathbb{N}$. This means $|\mathbb{N}| < |\mathbb{R}|$, so there is more than one "size" of infinity. There are in fact infinitely many different cardinalities because we can show (see Proposition 21.5) that the power set of any set S has a larger cardinality than S itself: that is, $|S| < |P(S)|$. Then, beginning with $\mathbb{N}$, we have the infinite string of inequalities:

$$|\mathbb{N}| < |P(\mathbb{N})| < |P(P(\mathbb{N}))| < |P(P(P(\mathbb{N})))| < \ldots.$$

(*It is a natural question to ask if there is a size of infinity "in between"* $|\mathbb{N}|$ *and* $|\mathbb{R}|$*. The statement "there is no set whose cardinality is strictly between* $|\mathbb{N}|$ *and* $|\mathbb{R}|$*" is known as the continuum hypothesis. The work of Gödel and others in the mid-1900s proved that it is impossible to either prove or disprove the continuum hypothesis (using the standard set theory axioms)!*)