Math 8: Sets, Logic, and Proofs

Spring 2011; Helena McGahagan

Set theory is one of the foundations of mathematics. In nearly every area of math, the main definitions refer to sets: for example, you've seen that a vector space is a "set" with certain properties. We could discuss set theory very rigorously, listing the axioms we must assume in order to define and work with sets (if you want to read more about this, one place to start is http://plato.stanford.edu/entries/set-theory/). However, in the interest of getting things done, most mathematicians take the intuitive view that we know what a set is when we see one! A set is essentially any collection of objects you can describe (although we have to avoid self-referential descriptions: try thinking about "the set of all sets that do not contain themselves as an element" to see why. In a set theory course, you'd learn that the axioms prevent this from being considered a description of a legitimate set.)

Definitions

- **set**: Any well-defined collection of objects.
- **elements**: The objects in a set. We write "$x \in S$" to mean "x is an element contained in the set S."
- **empty set**: The set containing no elements. We denote it by the symbol $\emptyset$.

We use curly braces to define a set: whatever is contained inside are the elements. For this reason, the empty set $\emptyset$ is sometimes written as $\{\}$. There are two main ways to describe a set:

1. List the elements.

Examples: $\{3, 7, \{1\}\}$ or $\{1, 2, 3, ..., 100\}$

2. Use a rule.

Examples: $\{x : x \in \mathbb{R} \text{ and } 1 < x < 2\}$ or $\{x : x \in \mathbb{N} \text{ and } 1 < x < 2\}$
(What other ways could you write these two sets?)

We will look at operations we can perform on sets soon. First, it will be useful to define some of the language we use in proofs and logic.

Definitions

- **statement**: A mathematical sentence that is either true or false (but not both). Example: $x \in S$ is a statement since either x is an element of S or it isn't!

- **axiom**: A statement in math that is assumed to be true, without requiring proof. (For example, "the empty set exists" is one of the basic axioms of set theory.)
- **and, or, implies**: Words that can be used to join statements into a longer statement. We'll explain these further with truth tables. Notice we denote "P implies Q" with the symbols $P \Rightarrow Q$ (this can also be read "if P, then Q.")
- **if and only if**: The statement $P \Leftrightarrow Q$ is true only when both $P \Rightarrow Q$ *and* $Q \Rightarrow P$ are true. In other words, $P \Leftrightarrow Q$ means that P and Q are equivalent statements. Example: The statement "$x > 5$ if and only if $x - 2 > 3$" is true.
- **negation**: The negation of a statement P is denoted $\bar{P}$ (you'll often see the notation $\neg P$ or $\sim P$ in other books.) $\bar{P}$ is true if and only if P is false.
- **converse**: The converse of $P \Rightarrow Q$ is $Q \Rightarrow P$.
- **contrapositive**: The contrapositive of $P \Rightarrow Q$ is $\bar{Q} \Rightarrow \bar{P}$.

Truth Tables

We can describe when $\bar{P}$ is true with the following truth table:

P	$\bar{P}$
T	F
F	T

In other words, we can look up whether $\bar{P}$ is true once we know whether P is T or F. We can do the same for more complicated statements, like "P and Q":

P	Q	P and Q
T	T	T
T	F	F
F	T	F
F	F	F

P	Q	P or Q
T	T	T
T	F	T
F	T	T
F	F	F

We read the statement $P \Rightarrow Q$ as either "P implies Q" or "If P, then Q." The truth table for P implies Q is a little trickier! Let P be the statement "It is raining," and let Q be the statement "It is cloudy." Now, think about the statement "If it is raining, then it is cloudy." Clearly, this is supposed to be true if P and Q are both true. Fill in the rest of the truth table below. The only case that is tricky is when P is false and Q is true – that is, when it is not raining, and it is not cloudy.

P	Q	$P \Rightarrow Q$
T	T	T
T	F	
F	T	
F	F	

Notice that P implies Q is *always* true when P is false. Many theorems are of the form $P \Rightarrow Q$: If you want to prove such a theorem, you only need to worry about what happens when P is true! The outline of such a proof would look like "Assume P is true. (... logical arguments ...) Therefore, Q is true." This proof guarantees that the statement $P \Rightarrow Q$ is true.

A related idea is that $P \Rightarrow Q$ is *only* false when P is true and Q is false. To show that $P \Rightarrow Q$ is true it is enough to show that this case is impossible! Here's an outline of such a proof: "Assume P is true. Also, assume Q is false. (... logical arguments ...) Conclude that something impossible is true!" This proof really shows that the only case when $P \Rightarrow Q$ is false does not occur; therefore, $P \Rightarrow Q$ must be true!

Methods of Proof

- **Direct**: Starting with any basic axioms or facts known to be true, use a series of implications to conclude that the desired result is true.
- **Proof by contradiction**: Assume that the desired result is *false*. If this leads to an impossible conclusion, then the result must be true instead!
- **Counterexample**: An example that proves a statement is false.

Examples:

Prove that if $x^2 - 2 < 2$, then $x < 4$.
Prove that the converse of this is false.
Prove that there are infinitely many prime numbers.
Prove that if r is irrational, then $\frac{1}{r}$ is irrational.