Math 8: Quantifiers and More on Sets

Spring 2011; Helena McGahagan

Quantifiers

Many statements in math require *quantifiers*. We often want to assert the existence of some special object: for example, "There is an integer whose square is 1." Other statements apply to a general class of objects: for example, "The square of any integer is non-negative."

DEFINITIONS

The **existential quantifier** is denoted by the symbol $\exists$, and is read "there exists."

The **universal quantifier** is denoted by the symbol $\forall$, and is read "for all."

Exercises:

1. Rewrite the two statements above in symbols and read them aloud.
2. Prove the two statements.
3. Write the negation of the two statements, first in words, and then in symbols. Notice that since the original statements are true, their negations must be false!

If something is not true for all objects, then it is false for a specific object. In other words, the negation of the statement: "$\forall x, P(x)$" will always be "$\exists x$ such that $\overline{P(x)}$." Similarly, the negation of a "there exists" statement will always be a "for all" statement.

Example: Negate the following statement.

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } \forall x \in \mathbb{R}, (|x| < \delta \Rightarrow x^2 < \epsilon)$$

The order of quantifiers matters!! The following two statements look similar, but have completely different meanings – in fact, one is true and one is false:

$\forall a \in \mathbb{Z}, \exists b \in \mathbb{Z}$ such that $a + b = 0$

$\exists b \in \mathbb{Z}$ such that $\forall a \in \mathbb{Z}, a + b = 0$

Sets
Sets are determined by what elements they contain. Therefore, we make the definition that two sets are equal when they contain the same elements: in symbols, $A = B$ means $\forall x, (x \in A \Leftrightarrow x \in B)$. This means that we consider $\{1, 2\}$ and $\{2, 1\}$ and $\{1, 2, 2\}$ and $\{x \in \mathbb{N} : 1 \leq x \leq 2\}$ to all be exactly the same set.

DEFINITIONS Let A and B be two sets.

A is a **subset** of B if every element of A is also an element of B: $A \subseteq B$ if $\forall x, x \in A \Rightarrow x \in B$.

A is a **proper subset** of B (denoted $A \subset B$) if both $A \subseteq B$ and $A \neq B$.

A and B are **disjoint** if $A \cap B = \emptyset$.

Notice that with these definitions, two sets A and B are equal if and only if both $A \subseteq B$ and $B \subseteq A$. A very common way or proving that two sets are equal is to do it in two parts: First, show $A \subseteq B$ and second, show $B \subseteq A$.

Given some sets, we can start putting them together to create new sets. Here are the definitions you'll need:

DEFINITIONS Let A and B be two sets.

The **union** of A and B is the set of all elements that are either in A or in B:
$A \cup B = \{x : x \in A \text{ or } x \in B\}$

The **intersection** of A and B is the set of all elements that are in both A and B:
$A \cap B = \{x : x \in A \text{ and } x \in B\}$.

The **set difference** of A and B is the set of all elements that are in A but not in B:
$A \setminus B = \{x : x \in A \text{ and } x \notin B\}$

The **Cartesian product** of A and B is the set of all *ordered pairs* of the form (a, b) where a is an element of A and b is an element of B: $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$
If A has n elements and B has m elements, how many elements will $A \times B$ have?

The **power set** of a set A is the set that contains all of the subsets of A: $P(A) = \{S : S \subseteq A\}$.
Notice that if A has n elements, the power set of A will have 2^n elements!

Example: Let $A = \{1, 2\}$ and $B = \{1, 3, 5\}$.

1. Find $A \cup B$, $A \cap B$, $B \setminus A$, and $A \times B$.
2. Find $P(A)$ and $P(B)$. (They should have $2^2 = 4$ and $2^3 = 8$ elements, respectively.)