Math 8: Rational Numbers

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Intuitively, we think of rational numbers as fractions – given two integers m and n ($n \neq 0$), we would like to say that $\frac{m}{n}$ is a rational number. The problem with this is that many of these fractions should actually be the *same* rational number, so we can't simply define the rational numbers as the set of all $\{(m,n) : m, n \in \mathbb{Z} \text{ and } n \neq 0\}$ (Notice this set is the Cartesian product $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$.) If we did this, we would be saying $(1,2)$ and $(-3,-6)$ are different rational numbers (since they are different elements of this set). We can fix this by defining the following equivalence relation on $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$:

$$\text{For all } (a,b) \text{ and } (c,d) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}), \text{define } (a,b) \sim (c,d) \Leftrightarrow ad = bc.$$

We can now define a **rational number** to be any equivalence class of this relation. The set of rational numbers is denoted by the symbol $\mathbb{Q}$ (therefore, the set of *irrational* numbers can be written as $\mathbb{R} \setminus \mathbb{Q}$). Then, we have that $(1,2)$ and $(-3,-6)$ belong to the same equivalence class and are therefore *representatives* of the same rational number. To use our usual notation, we allow ourselves to write $\frac{1}{2}$ instead of $(1,2)$, and we write $\frac{1}{2} = \frac{-3}{-6}$ since these fractions represent the same equivalence class (i.e., they represent the same rational number).

We should check that all our usual operations on rational numbers can be defined and work the way we expect. For example, we need to define how to add rational numbers: Given two of the equivalence classes, what answer do we get when we add them together? The easiest thing to do is to take a representative from each equivalence class and define the addition of the ordered pairs:

$$(a,b) + (c,d) = (ad + bc, bd).$$

Our answer should be the rational number represented by $\frac{ad+bc}{bd}$. But is this *well-defined*? Our definition of addition starts off by choosing representatives (a,b) and (c,d) for each rational number – what would have happened if you had taken *different* representatives (a_1, b_1) and (c_1, d_1) of the same two rational numbers? The addition is only well-defined if the choice does not make a difference: that is, in either case, the answer we get by adding should represent the same rational number. You can skip the proof below since we didn't do it in lecture, but it's included here in case you want to see the details:

Assume $(a,b) \sim (a_1,b_1)$ and $(c,d) \sim (c_1,d_1)$. This means (i) $ab_1 = ba_1$ and (ii) $cd_1 = dc_1$. We want to show that $(a,b) + (c,d) \sim (a_1,b_1) + (c_1,d_1)$. In other words, whichever pair of representatives we choose, their addition represents the same rational number. Using the definition of addition, we see that we want to show that $(ad+bc, bd) \sim (a_1d_1 + b_1c_1, b_1d_1)$. By definition of the equivalence relation, this means we need to show $(ad+bc)b_1d_1 = bd(a_1d_1+b_1c_1)$. Compute, and use (i) and (ii) when necessary:

$$\begin{aligned}(ad+bc)b_1d_1 &= (ab_1)(dd_1) + (bb_1)(cd_1) = (ba_1)(dd_1) + (bb_1)(dc_1) = bd(a_1d_1) + bd(b_1c_1) \\ &= bd(a_1d_1 + b_1c_1).\end{aligned}$$

So we have that the two answers represent the same equivalence class, and we therefore get a unique answer when we add two rational numbers.

In class, we proved the following:

- Every non-zero rational number can be *uniquely* written as a fraction $\frac{m}{n}$ such that $n > 0$ and $\text{hcf}(m, n) = 1$. (In other words, a non-zero rational number can be written in *lowest terms*.)
- Proposition 2.3: $\sqrt{2}$ is not rational. (More precisely, there is no rational number $\frac{m}{n}$ such that $\frac{m^2}{n^2} = 2$.)
- Proposition 2.4: Consider $a \in \mathbb{Q}$ and $b \in \mathbb{R} \setminus \mathbb{Q}$. We are guaranteed that the sum $a + b$ is irrational, and as long as we know that $a \neq 0$, then we are also guaranteed that the product ab is irrational. (*Notice that if a, b are both irrational there are examples where their sum/product is not irrational! E.g., if we multiply the two irrational numbers $\sqrt{2}$ and $\frac{1}{\sqrt{2}}$ together, we get the rational number 1. Although we can prove the familiar facts that π and e are irrational numbers, it is unknown whether or not the numbers $\pi + e$ and πe are both irrational!*)
- Proposition 2.1: Between any two rational numbers there is another rational number.
- Proposition 2.5: Between any two real numbers there is an irrational number.
- Generalizing Propositions 2.1 and 2.5, you proved on your homework that "Between any two real numbers, there is both a rational and an irrational number."

Since the first item in the list is not proven in the book, we include a proof here:

PROPOSITION Every non-zero rational number r can be *uniquely* written as a fraction $\frac{m}{n}$ such that $n > 0$ and $\text{hcf}(m, n) = 1$.

Proof: Given $r \in \mathbb{Q}$ ($r \neq 0$), we know we can represent it as a fraction $\frac{p}{q}$ for some $p, q \in \mathbb{Z}$ ($p, q \neq 0$). Without loss of generality, we may assume $q > 0$. (If $q < 0$, we can use the representation $\frac{-p}{-q}$ instead.) Let $d = \text{hcf}(p, q)$. Then, we know we can write $p = md$ and $q = nd$ for some $m, n \in \mathbb{Z}$ with $\text{hcf}(m, n) = 1$ and also $n > 0$ (since $q > 0$). Then, $r = \frac{p}{q} = \frac{md}{nd} = \frac{m}{n}$. Now, to show uniqueness, assume that there are two such representations: that is, assume

$$r = \frac{m_1}{n_1} = \frac{m_2}{n_2} \text{ for some } m_1, n_1, m_2, n_2 \in \mathbb{Z} : n_1 > 0, n_2 > 0, \text{hcf}(m_1, n_1) = 1, \text{hcf}(m_2, n_2) = 1.$$

Since $\frac{m_1}{n_1} = \frac{m_2}{n_2}$ represent the same natural number, we know that (i) $m_1 n_2 = m_2 n_1$. This implies both $n_2 \mid m_2 n_1$ and $n_1 \mid m_1 n_2$. However, since we know that n_1 and m_1 are coprime and that n_2 and m_2 are coprime, this implies $n_2 \mid n_1$ and $n_1 \mid n_2$. Since n_1 and n_2 are both positive, we must have the inequalities $n_2 \leq n_1$ and $n_1 \leq n_2$. This is only possible if $n_1 = n_2$. Then, using (i), $m_1 n_2 = m_2 n_2 \Rightarrow (m_1 - m_2) n_2 = 0 \Rightarrow m_1 = m_2$ since $n_2 \neq 0$. Hence, both $n_1 = n_2$ and $m_1 = m_2$, and therefore, the representation of r as a fraction written in lowest terms is unique. $\square$