Math 8: Equivalence Relations

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Definitions

We can think of algebra as the study of a set of numbers and the operations and relations on that set. Operations are things that, like addition and multiplication, allow us to input two numbers in order to find a new number. Relations are things that allow us to *compare* two elements of a set. For example, "greater than" is a relation on the set of integers. To define a relation, we must define exactly which pairs of elements are related: a convenient way to do this is to define a relation as a set of ordered pairs.

Definitions

- Let S be a set. A set R is a **relation** on S if $R \subseteq S \times S$.
- If R is a relation on S and $a, b \in S$, we write $a \sim b$ if the ordered pair $(a, b) \in \mathbb{R}$ (and we write $a \nsim b$ if $(a, b) \notin \mathbb{R}$.)
- Let S be a set and let $\sim$ be a relation on the set S.
  - The relation $\sim$ is **reflexive** if $a \sim a$ for all $a \in S$.
  - The relation $\sim$ is **symmetric** if for all $a, b \in S$, $a \sim b \Rightarrow b \sim a$.
  - The relation $\sim$ is **transitive** if for all $a, b, c \in S$, ($a \sim b$ and $b \sim c$)$\Rightarrow a \sim c$.
- A relation $\sim$ on a set S is called an **equivalence relation** if $\sim$ is reflexive, symmetric, and transitive.

Examples:

1. Let $S = \mathbb{R}$ and define the relation $\sim$ by (for any $a, b \in S$) $a \sim b \Leftrightarrow a > b$. (This the the "greater than" relation.) A similar example is to let $T = \{1, 2, 3, 4\}$; using the greater than relation, can you write down the set R that defines this relation?
2. $S = \mathbb{Z}$; $a \sim b \Leftrightarrow a \mid b$ (This is the "divisibility" relation.)
3. $S = P(\mathbb{Z})$; $A \sim B \Leftrightarrow A \subseteq B$.
4. Let S be any set; we can define the "relation of equality" $a \sim b \Leftrightarrow a = b$.
5. Fix an integer $m \geq 2$. Let $S = \mathbb{Z}$ and define the "relation of congruence mod m" by $a \sim b \Leftrightarrow a \equiv b \mod m$

6. Let S be the set of all triangles in the plane. Define $T_1 \sim T_2 \Leftrightarrow T_1$ is similar to T_2.

7. Let S be the set of all Americans. Define $a \sim b \Leftrightarrow a$ is the brother of b.

8. Let S be the set of all Americans. Define $a \sim b \Leftrightarrow a$ has the same last name as b.

We can consider whether each of these relations is reflexive, symmetric, or transitive. For example, we prove the relation in 1 is transitive: "Let $a, b, c \in \mathbb{R}$ be given. Assume $a \sim b$ and $b \sim c$. This means $a > b$ and $b > c$. By the rules for inequalities, this implies $a > c$, so $a \sim c$." This is a little pedantic, since we are using the fact that inequalities is transitive (this is where our idea for the definition of transitive comes from!) but it shows in general how to prove that the definition of transitivity is satisfied for a relation. Remember if you are proving a statement of the form "for every a, b, c, the statement $P(a, b, c)$ is true," the first line of the proof is always to fix arbitrary elements a, b, c; then, try to prove $P(a, b, c)$.

Convince yourself that the relations in 1 and 7 are not reflexive or symmetric and that the relations in 2 and 3 are not symmetric. However, the relations in 4, 5, 6, and 8 are all reflexive, symmetric, and transitive, so they are all equivalence relations. In some sense an equivalence relation $\sim$ should be something that acts like an equals sign! So, if $\sim$ is an equivalence relation on the integers, things like $a + b \sim b + c$ should mean that $a \sim c$. Congruence modulo m is a very important example of an equivalence relation, and as we've seen in Chapters 13 and 14, we can indeed usually treat $\equiv \mod m$ as if it were simply a regular equal sign.

Definitions

- Let S be a set and let $\sim$ be an equivalence relation on S. For any $a \in S$, the **equivalence class of $\sim$ containing a** is the set $\mathrm{cl}(a) = \{s \in S : s \sim a\}$. We call any element $x \in \mathrm{cl}(a)$ a **representative** of the equivalence class $\mathrm{cl}(a)$.

- Let S be a set. A collection of subsets $\{S_i\}$ (the number of these might be finite or infinite) is called a **partition** of S if $\cup S_i = S$ and $S_i \cap S_j = \emptyset$ for ever $i \neq j$. (In words, these conditions say that the union of all the S_i's is equal to the original set, and the S_i's are *pairwise disjoint*. You can find this definition in your book on page 151; notice it's equivalent to the definition given earlier on page 134.)

Examples

- For the relation in 3, what are the equivalence classes of the relation of equality? Notice that nothing is equal to anything else in the set other than itself. Therefore, for every $a \in S$, $\mathrm{cl}(a) = \{a\}$. There are exactly as many distinct equivalence classes as there are elements in the set S.

- For the relation in 8, consider a particular American named John Smith. The set $\mathrm{cl}(\textit{John Smith})$ is the set of everyone in America with last name Smith. We would have, for example, that both John Smith and his daughter June Smith would be *representatives* of this equivalence class. There are many different equivalence classes of this relation; almost all will have more than one element in the class, so there will be fewer equivalence classes than there are elements in the set S.

- For the relation in 5, what are all of the equivalence classes? If we consider all the things related to 0, we'll find cl(0) has infinitely many integers in it (all the integers divisible by m). Similarly, cl(1) will contain everything that has remainder 1 (modulo m). Writing these sets down:

$$\begin{aligned}
\text{cl}(0) &= \{..., -2m, -m, 0, m, 2m, ...\} \\
\text{cl}(1) &= \{..., -2m+1, -m+1, 1, m+1, 2m+1, ...\} \\
\text{cl}(2) &= \{..., -2m+2, -m+2, 2, m+2, 2m+2, ...\} \\
&\vdots \\
\text{cl}(m-1) &= \{..., -m-1, -1, m-1, 2m-1, 3m-1, ...\}
\end{aligned}$$

  Notice that if we wrote down cl(m), we would just be getting the exact same set as cl(0)! This makes sense since m is a representative of cl(0); anything equal to m (modulo m) will also be equal to 0 (modulo m). Similarly, we find that $\text{cl}(-1) = \text{cl}(m-1)$, and so on. Therefore, there are only m distinct equivalence class (each with infinitely many elements) for the relation of congruence mod m. Each class corresponds to a particular remainder – notice that these sets actually give us a way to rigorously define the elements in the system $\mathbb{Z}_m$.

The equivalence classes of congruence mod m illustrate a very important property of equivalence classes: Every integer appears in one (and only one!) of the classes, and if we look at two equivalence classes, they must either be the exact same set or they must be disjoint. That is, the equivalence classes form a *partition* of the set. See Proposition 18.1 in your book; we can prove this proposition by showing the following:

Proposition Let S be a set and let $\sim$ be an equivalence relation on S. For all $a, b \in S$,

(i) $a \in \text{cl}(a)$

(ii) $\text{cl}(a) = \text{cl}(b)$ if and only if $a \sim b$.

(iii) $\text{cl}(a) \cap \text{cl}(b) = \emptyset$ if and only if $a \not\sim b$

Proof: Let $a, b \in S$ be given.

(i) We know that $\text{cl}(a) = \{s \in S : s \sim a\}$. Since $\sim$ is reflexive, we know that $a \sim a$. This means, by definition, that $a \in \text{cl}(a)$.

(ii) First, we prove the forward direction ($\Rightarrow$). Assume that $\text{cl}(a) = \text{cl}(b)$. Then, from part (i), $a \in \text{cl}(a) = \text{cl}(b)$. Since $a \in \text{cl}(b)$, by definition, $a \sim b$. On the other hand, to prove ($\Leftarrow$), assume that $a \sim b$. We want to show that the two sets cl(a) and cl(b) are equal:

First, let $x \in \text{cl}(a)$. Therefore, $x \sim a$. Since we have both $x \sim a$ and $a \sim b$ (by assumption), then by the transitivity of the relation, we know $x \sim b$; i.e., $x \in \text{cl}(b)$.

Similarly, if $x \in \text{cl}(b)$, then $x \sim b$ and $b \sim a$ (using the symmetry of the relation), and therefore, by transitivity $x \sim a$, so $x \in \text{cl}(a)$.

This shows that both $\mathrm{cl}(a) \subseteq \mathrm{cl}(b)$ and $\mathrm{cl}(b) \subseteq \mathrm{cl}(a)$. Therefore, $\mathrm{cl}(a) = \mathrm{cl}(b)$.

(iii) Again, we must prove an if and only if statement. For the proof of ($\Rightarrow$), suppose that $\mathrm{cl}(a) \cap \mathrm{cl}(b) = \emptyset$. Since by (i), $a \in \mathrm{cl}(a)$, we know $a \notin \mathrm{cl}(b)$; therefore, $a \not\sim b$. To prove ($\Leftarrow$), suppose $a \not\sim b$. For a contradiction, assume that $\mathrm{cl}(a) \cap \mathrm{cl}(b) \neq \emptyset$. Then, there exists an $x \in S$ such that $x \in \mathrm{cl}(a)$ and $x \in \mathrm{cl}(b)$. This means that we know $x \sim a$ and $x \sim b$. By symmetry, we also know $a \sim x$. Finally, using transitivity, we find that $a \sim b$ is true, which is a contradiction. $\square$