Math 8: Logic

Spring 2011; Helena McGahagan

Consider the following statements:

$P:$ All natural numbers are even. False
$Q: \sqrt{5} > 2$ True
$R:$ There exists an $n \in \mathbb{N}$ such that $n \geq 500$. True

Decide whether each statement is true or false. Then, write the negation of each statement.

$\bar{P}$: There is a natural number that is odd.

$\bar{Q}$: $\sqrt{5} \leq 2$

$\bar{R}$: For all $n \in \mathbb{N}$, $n < 500$.

Label the following statements as either true or false.

P or $\bar{Q}$ False $\qquad$ Q and $\bar{R}$ False

$Q \Rightarrow R$ True $\qquad$ $P \Rightarrow R$ True

2. If P and Q are two statements, write the truth table for the statements $P \Rightarrow Q$. and $Q \Rightarrow P$.

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

Using this, write the truth table for $P \Leftrightarrow Q$. Notice that this statement – "P if and only if Q" – it true only when either both P and Q are true or when both P and Q are false. When $P \Leftrightarrow Q$ is true we often say that P and Q are *equivalent* statements.

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Prove that the contrapositive of $P \Rightarrow Q$ is equivalent to $P \Rightarrow Q$.

To do this, show $P \Rightarrow Q$ and $\overline{Q} \Rightarrow \overline{P}$ have the same truth table.

P	Q	$P \Rightarrow Q$	$\overline{Q}$	$\overline{P}$	$\overline{Q} \Rightarrow \overline{P}$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Using only 'negation', 'and,' and 'or' write a statement that is equivalent to $P \Rightarrow Q$. (Check that the truth table for your statement is exactly the same as the truth table for $P \Rightarrow Q$ – in other words, your statement and "$P \Rightarrow Q$" are either both true or both false!) Can you find an equivalent statement that uses only 'negation' and 'and'?

There are several possible answers:

E.g.

$$(P \Rightarrow Q) \Leftrightarrow (P \text{ and } Q) \text{ or } \overline{P}$$

Also, $(P \Rightarrow Q) \Leftrightarrow (\overline{P} \text{ or } Q)$

Using only 'negation' and 'and', we can write

$$(P \Rightarrow Q) \Leftrightarrow \overline{(P \text{ and } \overline{Q})}$$

3. Consider the following two true statements:

P: If an object is a triangle, then it is a polygon

Q: If an number is even, then it is divisible by two.

What are the converses of these statements? Are they true?

Converse of P: If an object is a polygon, then it is a triangle (FALSE!)

Converse of Q: If a number is divisible by 2, then it is even. (TRUE!)

What are the contrapositives of these statements? Are they true?

The contrapositive of a true statement is always true.

Contrapositive of P: If an object is not a polygon, then it is not a triangle.

Contrapositive of Q: If a number is not divisible by 2, then it is not even.

Math 8: Quantifiers

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1. Consider the following statements:

$$P : \forall x \in \mathbb{R}, \exists y \in \mathbb{R} \text{ such that } f(x) = y$$
$$Q : \exists y \in \mathbb{R} \text{ such that } \forall x \in \mathbb{R}, f(x) = y$$

(a) Write the negation of each statement.

$\overline{P}$: $\exists x \in \mathbb{R}$ such that $\overline{\exists y \in \mathbb{R} \text{ such that } f(x) = y}$

$\overline{P}$: $\exists x \in \mathbb{R}$ such that $\forall y \in \mathbb{R}$, $\overline{f(x) = y}$

$\overline{P}$: $\exists x \in \mathbb{R}$ such that $\forall y \in \mathbb{R}$, $f(x) \neq y$

Going through similar steps, we find

$\overline{Q}$: $\forall y \in \mathbb{R}$, $\exists x \in \mathbb{R}$ such that $f(x) \neq y$.

(b) Whether or not P and Q are true depend on what f is. What do you need to know about about f to conclude P is true? What do you need to know to conclude that Q is true?

If $f : \mathbb{R} \to \mathbb{R}$ is any well-defined function, P is true.

If $f : \mathbb{R} \to \mathbb{R}$ is a constant function, Q is true.

2. (a) Define $A \subseteq B$. Negate this statement

$A \subseteq B$ means "$\forall x$, $x \in A \Rightarrow x \in B$"

The negation is "$\exists x$ such that $x \in A$ and $x \notin B$"

(b) Define $A = B$. Negate this statement.

$A=B$ means "$\forall x,\ x \in A \Leftrightarrow x \in B$"

In other words, "$\forall x, (x \in A \Rightarrow x \in B)$ and $(x \in B \Rightarrow x \in A)$"

The negation is

"$\exists x$ such that $(x \in A$ and $x \notin B)$ or $(x \in B$ and $x \notin A)$"

(c) Negate: $\forall n \in \mathbb{Z}$, $n^2 < e$ or $n^2 > \pi$. Which is true – the original statement or its negation? How can you prove it?

The negation is "$\exists n \in \mathbb{Z}$ such that $n^2 \geq e$ and $n^2 \leq \pi$."

i.e. "$\exists n \in \mathbb{Z}$ such that $e \leq n^2 \leq \pi$"

The original statement is true. Prove it by cases.

Case I $|n| \leq 1$, so $n=0, 1,$ or -1. In this case, $n^2 < e$.

Case II $|n| > 1$ In this case, $n^2 \geq 2^2 > \pi$ ☒

(d) Negate: $\forall n \in \mathbb{Z}$, if $n < 0$, then $n + 5 > 0$. Which is true – the original statement or its negation? How can you prove it?

The negation is "$\exists n \in \mathbb{Z}$ s.t. $n < 0$ and $n+5 \leq 0$"

We can prove the negation by noting $n = -5 \in \mathbb{Z}$ satisfies $n < 0$ and $n+5 = 0 \leq 0$

(Notice $n=-5$ is a counterexample that disproves the original statement.)

3. Write the following statements (from your homework) in symbols, then write their negations.

(a) If n is an integer such that n^2 is even, then n is even.

Statement: $\forall n \in \mathbb{Z},\ 2 | n^2 \Rightarrow 2 | n$

Negation: $\exists n \in \mathbb{Z}$ such that $2 | n^2$ and $2 \nmid n$.

(b) The number 8881 has a prime factor that is at most 89.

Statement: $\exists$ a prime $p \leq 89$ s.t. $p | 8881$; Neg: $\forall$ primes $p \leq 89$, $p \nmid 8881$.

(c) An integer that is equal to three consecutive integers multiplied together is divisible by 3.

Statement: $\forall n \in \mathbb{Z},\ (\exists m \in \mathbb{Z}$ s.t. $n = (m-1)(m)(m+1)) \Rightarrow 3 | n$.

Negation: $\exists n \in \mathbb{Z}$ s.t. $(\exists m \in \mathbb{N}$ s.t. $n = (m-1)(m)(m+1))$ and $3 \nmid n$.

Math 8: Sets

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1. Answer the questions from the lecture notes: Let $A = \{1, 2\}$ and $B = \{1, 3, 5\}$.

(a) Find $A \cup B$, $A \cap B$, $B \setminus A$, $A \setminus B$, and $A \times B$.

$A \cup B = \{1, 2, 3, 5\}$

$A \cap B = \{1\}$

$B \setminus A = \{3, 5\}$

$A \setminus B = \{2\}$

$A \times B = \{(1,1), (1,3), (1,5), (2,1), (2,3), (2,5)\}$

(Notice $A \times B$ has $|A| \cdot |B| = 2 \cdot 3 = 6$ elements.)
Each element is an <u>ordered pair</u>.

(b) Find $P(A)$ and $P(B)$.

$P(A) = \{\phi, \{1\}, \{2\}, \{1,2\}\}$

$P(B) = \{\phi, \{1\}, \{3\}, \{5\}, \{1,3\}, \{1,5\}, \{3,5\}, \{1,3,5\}\}$

(Notice $P(A)$ has $2^{|A|} = 2^2 = 4$ elements
and $P(B)$ has $2^{|B|} = 2^3 = 8$ elements.)

2. Let A and B be given sets.

(a) $P(A \cup B) \supseteq P(A) \cup P(B)$

Assume $S \in P(A) \cup P(B)$. Then by definition of union, either $S \in P(A)$ or $S \in P(B)$

Case I If $S \in P(A)$, then $S \subseteq A$. Since $A \subseteq A \cup B$, this implies $S \subseteq A \cup B$, so $S \in P(A \cup B)$

Case II If $S \in P(B)$, then $S \subseteq B$. Again, $B \subseteq A \cup B$ $\Rightarrow S \subseteq A \cup B \Rightarrow S \in P(A \cup B)$

In either case, $S \in P(A \cup B)$ $\therefore$ $\forall S$, $S \in P(A) \cup P(B) \Rightarrow S \in P(A \cup B)$ $\boxtimes$

(b) Prove that $A = (A \cap B) \cup (A \setminus B)$

First, show $A \subseteq (A \cap B) \cup (A \setminus B)$

Assume $x \in A$. Case I If $x \in B$, then $x \in A \cap B$.

Case II If $x \notin B$, then $x \in A \setminus B$

In either case, $x \in (A \cap B) \cup (A \setminus B)$

Now, show $(A \cap B) \cup (A \setminus B) \subseteq A$.

Assume $x \in (A \cap B) \cup (A \setminus B)$. Then $x \in A \cap B$ or $x \in A \setminus B$.

If $x \in A \cap B$, then $x \in A$. On the other hand, If $x \in A \setminus B$, then $x \in A$ also.

$\boxtimes$

3. Consider a chess tournament in which N people enter. You can represent each person by an integer, and the result of each game can be represented by an ordered pair (n, m), indicating that person n won against person m. (Why is important we used the ordered pairs (n, m), rather than the sets $\{n, m\}$? Can you think of other ways to represent the games using only sets?) Consider the set

$$S = \{(n, m) \in \{1, 2, ..., N\} \times \{1, 2, ..., N\} : n \text{ won a game against } m\}$$

If this is a knockout tournament (so anyone who loses a game is eliminated, until there is only one person left, who won all his or her games), how many elements are in S? $\boxed{N-1}$

N-1 people lose, and there must be one (and only one) ordered pair in which a person who loses appears in the 2nd coordinate.

Math 8: Integers

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1. Let $a, b \in \mathbb{Z}$. Write the definition of the highest common factor of a and b. Then, negate this definition.

$d = \text{hcf}(a,b)$ means "$d \mid a$ and $d \mid b$ and $\forall c \in \mathbb{Z}$ such that $c \mid a$ and $c \mid b$, $c \leq d$"

The negation is "$d \nmid a$ or $d \nmid b$ or $\exists c \in \mathbb{Z}$ such that $c \mid a$ and $c \mid b$ and $c > d$"

2. (a) Let $a, b, d \in \mathbb{Z}$. Consider the equation $a\,s + b\,t = d$. We want to find integer solutions to this equation (i.e., find $s, t \in \mathbb{Z}$). We have a theorem (a consequence of the Euclidean algorithm) that gives a condition on d that guarantees it is possible to find solutions. What is the condition? $d = \text{hcf}(a,b)$

(b) Prove: If $a, b \in \mathbb{Z}$, then any common factor of a and b divides $\text{hcf}(a, b)$.

Let $d = \text{hcf}(a,b)$. Assume c is a common factor of a and b. We know there exist $s, t \in \mathbb{Z}$ such that $d = as + bt$. Since $c \mid a$ and $c \mid b$, we know $c \mid (as + bt)$

$\therefore\ c \mid d$ $\boxtimes$

(c) Consider the equation in part (a) again. If we have a solution (s_o, t_o) (so these numbers satisfy the equation $a\,s_o + b\,t_o = d$), we have infinitely many solutions ... can you find all of them? (There is a trivial case: If $a = 0$, then all integers $s \in \mathbb{Z}$ solve $as + bt_0 = d$.)

Assume $\underline{a \neq 0 \text{ and } b \neq 0}$:

If $\text{hcf}(a,b) = 1$, we can show that, for all n,

$$\begin{cases} s = s_0 + nb \\ t = t_0 - na \end{cases}$$ is a solution of $as + bt = d$.

Otherwise, let $e = \text{hcf}(a,b)$, and $\begin{cases} s = s_0 + n\, b/e \\ t = t_0 - n\, a/e \end{cases}$ is a soln. $\forall n$

(see lecture notes for proofs.)

We must find integers $S \geq 0$ and $t \geq 0$ such that $28s + 34t = 844$

3. A warehouse has two trucks; one can hold 28 boxes, and one can hold 34. If the warehouse only sends out the trucks full, and returns them empty, can it move 844 boxes from the warehouse to a store? How many different ways can this be done?

Using the Euclidean algorithm: $34 = 1 \cdot 28 + 6$, $28 = 4 \cdot 6 + 4$, $6 = 1 \cdot 4 + 2$, $4 = 2 \cdot 2 + 0$

$\Rightarrow hcf(28, 34) = 2$

Then, $\underline{\underline{2}} = 6 - 4 = -28 + 5 \cdot 6 = \underline{5 \cdot 34 - 6 \cdot 28}$ Mutiply by 422 to see that $S_0 = -2532$ and $t_0 = 2110$ solves $884 = t_0 \cdot 34 + S_0 28$.

We know that all solutions of this equation are of the form $\begin{cases} S = -2532 + 17n \\ t = 2110 - 14n \end{cases}$ for some $n \in \mathbb{Z}$

To get positive solutions, we need $17n \geq 2532$ and $14n \leq 2110$

$\Rightarrow 149 \leq n \leq 150$. The only two positive solutions are $(s,t) = \boxed{(1, 24) \text{ OR } (18, 10)}$

↑ Check these work!

4. Gear A turns at two revolutions per minute. Gear A has 32 teeth and is meshed into Gear B which has 120 teeth. How often will both gears be simultaneously back in their starting positions?

$hcf(120, 32) = 8$ $\quad \begin{pmatrix} 120 = 3 \cdot 32 + 24 \\ 32 = 24 + 8 \\ 24 = 3 \cdot 8 + 0 \end{pmatrix}$

then, $\underline{lcm(120, 32)} = \frac{120 \cdot 32}{8} = \underline{15 \cdot 32 = 120 \cdot 4} = 480$

Every 15 times Gear A turns, Gear B has turned 4 times, so both are back in the starting position

Gear A turns 15 times in $\boxed{7.5 \text{ min}}$

5. If $ac|bc$ and $c \neq 0$, prove that $a|b$.

Assume $a, b, c \in \mathbb{Z}$, $c \neq 0$. Also, assume $ac \mid bc$

Then, there exists $q \in \mathbb{Z}$ such that $bc = qac$.

Therefore, $(b - qa)c = 0$ and since $c \neq 0$, this implies $b = qa$. By definition, this means $a \mid b$ ☒

6. Let $a, b, d \in \mathbb{Z}$. Prove that $hcf(ad, bd) = |d| \, hcf(a, b)$. (You may assume $d \neq 0$ and a, b are not both 0.) Let $r = hcf(a, b)$. Then $r \mid a$ and $r \mid b$

Since we also know $|d| \mid d$, $|d| \cdot r \mid da$ and $|d| \cdot r \mid db$

To show $|d| \cdot r$ is the highest common factor of da and db, assume $c \in \mathbb{Z}$ is such that $c \mid da$ & $c \mid db$.

Find s, t such that $r = sa + tb$; then $|d|r = |d|sa + |d|tb$

Since $c \mid |d|a$ and $c \mid |d|b$, we have $c \mid |d| \cdot r$.

Since $|d| \cdot r > 0$, this implies $c \leq |d| \cdot r$ ☒