Math 8: Induction

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1. Let $f(x) = 3x + 5$ and let $g(x) = 3(x - 2) + 8$. What's wrong with the following proof?

Fix any integer $n \geq 1$. Assume that $f(n) = g(n)$. Then, $g(n+1) = 3((n+1) - 2) + 8 = g(n) + 3 = f(n) + 3 = (3(n) + 5) + 3 = 3(n+1) + 5 = f(n+1)$. Therefore, by induction, we have proven $f(n) = g(n)$ for all $n \in \mathbb{N}$.

The base case is not true! $g(1) = 5$ and $f(1) = 8$

(The inductive step has no flaws; it <u>is</u> true that "$f(n) = g(n) \Rightarrow f(n+1) = g(n+1)$")

2. Prove that, for every integer $n \geq 1$,

Let $P(n)$ be the statement "$\displaystyle\sum_{k=1}^{n} k^5 = \frac{n^2(n+1)^2(2n^2+2n-1)}{12}$"

<u>Base case</u>: <u>$n=1$</u> $P(1)$ states $\displaystyle\sum_{k=1}^{1} k^5 = 1^5 = \frac{1^2(2)^2(3)}{12}$ ✓

<u>Inductive step</u>: Fix $n \in \mathbb{N}$. Assume $P(n)$ is true (IH)

Then $$\sum_{k=1}^{n+1} k^5 = \left(\sum_{k=1}^{n} k^5\right) + (n+1)^5$$

$$= \frac{n^2(n+1)^2(2n^2+2n-1)}{12} + (n+1)^5 \quad (\text{by (IH)})$$

$$= \frac{(n+1)^2}{12}\left[(2n^4+2n^3-n^2) + 12(n+1)^3\right]$$

$$= \frac{(n+1)^2}{12}\left[2n^4+2n^3-n^2+12n^3+36n^2+36n+12\right]$$

Check this!

$$= \frac{(n+1)^2}{12}\left[2n^4+14n^3+35n^2+36n+12\right]$$

Then we see that $P(n+1)$ is true.

$$= \frac{(n+1)^2(n+2)^2(2(n+1)^2+2(n+1)-1)}{12}$$

3. Prove that, for any fixed real numbers a and $r \neq 1$ that

$$P(n): \; a + ar + ar^2 + \ldots + ar^{n-1} = a\left(\frac{r^n - 1}{r-1}\right)$$

is true for all $n \in \mathbb{N}$.

Base case: $P(1)$ states $a = a\left(\frac{r^1-1}{r-1}\right)$, which is true.

Inductive Step: Fix $n \in \mathbb{N}$

Assume $P(n)$ is true (IH)

Then, using (IH),

$$(a + ar + \ldots + ar^{n-1}) + ar^{(n+1)-1} = a\left(\frac{r^n-1}{r-1}\right) + ar^n$$

$$= \frac{a(r^n-1) + ar^n(r-1)}{r-1}$$ finding a common denominator

$$= \frac{\cancel{ar^n} - a + ar^{n+1} - \cancel{ar^n}}{r-1} = \frac{a(r^{n+1}-1)}{r-1}$$

$$= a\left(\frac{r^{n+1}-1}{r-1}\right)$$

$\therefore P(n+1)$ is true.

4. Prove that for all $n \geq 1$, $8^n - 3^n$ is divisible by 5.

Let $P(n): 5 \mid (8^n - 3^n)$

Base case $P(1)$: $5 \mid (8-3)$ is true

Inductive step Fix $n \in \mathbb{N}$. Assume $5 \mid (8^n - 3^n)$ (IH)

$$8^{n+1} - 3^{n+1} = 8(8^n - 3^n) + 8 \cdot 3^n - 3^{n+1}$$

$$= 8(8^n - 3^n) + 3^n(8-3)$$

$$= 8(8^n - 3^n) + 3^n \cdot 5$$

Since $5 \mid (8^n - 3^n)$ and $5 \mid 5$, $5 \mid (8^{n+1} - 3^{n+1})$

$\therefore P(n+1)$ is true. By induction $P(n)$ is true for all $n \in \mathbb{N}$ ∎

Math 8: Rational and Irrational Numbers

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1. Prove that $\sqrt{3}$ is irrational.

Assume $\sqrt{3}$ is rational. Then there exist $m, n \in \mathbb{Z}$, with $n > 0$ & $hcf(m,n) = 1$, such that

$$\sqrt{3} = \frac{m}{n}.$$

Then, ⊛ $3n^2 = m^2$

This implies $3 \mid m^2$

There are 3 cases: $m \equiv 0, 1,$ or $2 \mod 3$.

(1) $m \equiv 1 \mod 3$

But this is impossible since then $m^2 \equiv 1 \mod 3$, so $3 \nmid m^2$

(2) $m \equiv 2 \mod 3$

This is also impossible since $m^2 \equiv 4 \equiv 1 \mod 3$

(3) $m \equiv 0 \mod 3$.

The only allowed case is (3), so we know $\underline{3 \mid m}$. This means $\exists K \in \mathbb{Z}$ s.t. $m = 3K$

Using ⊛, $3n^2 = (3K)^2 = 9K^2 \Rightarrow n^2 = 3K^2$

This means $3 \mid n^2$ and therefore $3 \mid n$.

Since $3 \mid n$ and $3 \mid m$, $hcf(m,n) \neq 1 \Rightarrow \Leftarrow$

2. (a) Assume a is rational and b is irrational. What can you say about $a+b$? What about ab?

$a+b$ must be irrational (Prop 2.4(i))

ab is irrational if $a \neq 0$; of course, when $a=0$, $ab=0$ is rational (Prop 2.4(ii))

(b) Come up with examples of a and b, both irrational, such that (i) $a+b$ is rational or (ii) ab is rational.

(i) $\sqrt{2}$ and $-\sqrt{2}$ are irrational; but $\sqrt{2}+(-\sqrt{2})=0 \in \mathbb{Q}$

(ii) $\sqrt{2}$ and $\sqrt{2}$ are irrational, $(\sqrt{2})(\sqrt{2})=2 \in \mathbb{Q}$

(c) If a is rational and $a \neq 0$, what can you say about b if you know that the product ab is rational?

b cannot be irrational. If it were, ab would have to be irrational by Prop. 2.4(ii)

$\therefore$ $b \in \mathbb{Q}$

3. See if you can come up with a proof of the fact that between every pair of real numbers $a < b$, there is an irrational number (without looking back at your book or notes!) Hint: You want to add something small to a (so that the result will still be less than b) and make sure that the result will be irrational. Consider the cases a is rational and a is irrational separately.

If a is rational, notice $a+\frac{\sqrt{2}}{n}$ is irrational for any $n \in \mathbb{N}$ (use Prop. 2.4)

WANT: $a < a+\frac{\sqrt{2}}{n} < b$, so we'll need to take $\frac{\sqrt{2}}{n} < b-a$

or $n > \frac{\sqrt{2}}{b-a}$ (Note: n and $b-a$ are >0)

This is why the proof in the book begins by choosing such an $n \in \mathbb{N}$.

Then, you just need to find a number that works when a is irrational

In this case, note $a+\frac{1}{n}$ is irrational. Prove $a < a+\frac{1}{n} < b$.