Let $\alpha = a dx_1 \wedge dx_2$ and $\beta = b dx_1 + c dx_3$ be forms on $\mathbb{R}^4$. Prove that $d(\alpha \wedge \beta) = d\alpha \wedge \beta + \alpha \wedge d\beta$.

If $f : \mathbb{R}^2 \to \mathbb{R}$ is C^2 and $\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial^2 y} = 0$, prove that

$$\iint_D |\nabla f|^2 = \int_{\partial D} f \nabla f \cdot ds.$$

Compute

$$\int_0^\infty \int_0^\infty e^{-xy} \sin x \, dx \, dy$$

Prove that if f, g, f^2, g^2 are all Riemann integrable on $[a, b]$, then

$$\left(\int_a^b f(x) g(x) \right)^2 dx \leq \left(\int_a^b f(x)^2 \, dx \right) \left(\int_a^b g(x)^2 \, dx \right)$$

Consider $R = [0,1] \times [0,1]$ and the function $f : R \to \mathbb{R}$ defined as follows:

$$f(x,y) = \begin{cases} 1 & x \text{ irrational} \\ 2\text{y} & x \text{ rational} \end{cases}$$

(a) Show $\iint_R f(x,y)\,dx\,dy$ does not exist.
(b) What is the iterated integral $\int_0^1 \left(\int_0^1 f(x,y)\,dy \right) dx$?

Define the following function on $[0,1]$: $f(0) = 0$, and for $\frac{1}{2^n} < x \leq \frac{1}{2^{n-1}}$, $f(x) = \frac{1}{2^n}$.
(a) Give two different reasons why the integral $\int_0^1 f(t)\,dt$ exists.
(b) Can you find a formula for $F(x) = \int_0^x f(t)\,dt$?

Recall the definitions of the inner content and the outer content. If A, B are bounded subsets of $\mathbb{R}$, prove that:
(a) $\bar{c}(A \cup B) + \bar{c}(A \cap B) \leq \bar{c}(A) + \bar{c}(B)$
(b) $\underline{c}(A \cup B) + \underline{c}(A \cap B) \geq \underline{c}(A) + \underline{c}(B)$

Assume that f is Riemann integrable on a rectangle $R = [a, b] \times [c, d]$. Show that there exists a point $\mathbf{x_o}$ in the interior of R such that

$$\iint_R f(\mathbf{x})\, d\mathbf{x} = f(\mathbf{x_o})\text{vol}(R)$$

Suppose f is defined on a neighborhood of the triangular region T defined by the vertices $(a, 0), (0, b)$, and $(0, 0)$. Assume that $D_{1,2}f(x, y)$ exists and is continuous on T. Prove that

$$\iint_T D_{1,2}f(x, y)\, dx\, dy = f(0, 0) - f(a, 0) + aD_1f(x_o, y_o)$$

for some point (x_o, y_o) (such that $bx_o + ay_o = ab$).

We can parameterize part of the curve given by the equation $x^3 + y^3 = 3axy$ by

$$\gamma(t) = \left(\frac{3at}{t^3+1}, \frac{3at^2}{t^3+1}\right), \quad t \in [0, \infty)$$

Compute the area of the region this curve encloses.

Find the area of an ellipse $E = \left\{(x, y) : \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 \leq 1\right\}$ in as many different ways as you can think of!

Find the following line integral along a curve γ in $\mathbb{R}^3$ from $(1,1,0)$ to $(2,3,\ln 2)$:

$$\int_\gamma zy^x(\ln y)\,dx + zxy^{x-1}\,dy + (ze^z + y^x)\,dz$$

Compute

$$\int_0^1 \int_y^1 \frac{\sin x}{x}\,dx\,dy$$

Let A be a set such that for every nonempty (a,b) that intersects $[0,1]$, $(a,b)\cap A \neq \emptyset$. If f is Riemann integrable and $f(x)=0$ for all $x \in A$, then $\int_a^b f = 0$.

Prove that if g is a bounded function that is continuous on $[a,b]$ except at the point $c(a < c < b)$, then g is Riemann integrable.