Activity 10: Practice Final

These problems focus on the new material only. You should also go over the previous midterms and practice midterms to prepare.

Question. Is the following set of vectors orthogonal?

\[
\begin{bmatrix}
2 \\
-7 \\
-1
\end{bmatrix},
\begin{bmatrix}
-6 \\
-3 \\
9
\end{bmatrix},
\begin{bmatrix}
3 \\
1 \\
-1
\end{bmatrix}
\]

Question. Find a least squares solution of \(Ax = b\) by constructing the normal equations for \(\hat{x}\) and solving for \(\hat{x}\) with

\[
A = \begin{bmatrix}
1 & 3 \\
1 & -1 \\
1 & 1
\end{bmatrix}
\quad \text{and} \quad
b = \begin{bmatrix}
5 \\
1 \\
0
\end{bmatrix}.
\]
Question. Diagonalize the following matrix $M$ by finding $D$, $P$ and $P^{-1}$ so that $M = PDP^{-1}$:

$$M = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

Question. Use your answer to the previous problem to find a formula for $M^n$ where $n$ is an arbitrary integer.
Question. If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation and \{v_1, v_2, v_3\} is linearly dependent in $\mathbb{R}^n$. Why is the set \{T(v_1), T(v_2), T(v_3)\} linearly dependent in $\mathbb{R}^m$?

Question. Find the inverse of \[
\begin{bmatrix}
8 & 6 \\
5 & 4
\end{bmatrix}
\] and use it to solve the system:

\[
\begin{align*}
8x_1 + 6x_2 &= 2 \\
5x_1 + 4x_2 &= -1
\end{align*}
\]

(You must use the inverse. Do not RREF the system of equations.)
Question. Let $W$ be the first and third quadrants in the plane, i.e. $W = \{(x,y)|xy \geq 0\}$.

- Is $W$ closed under scalar multiplication? Why or why not?
- Find two vectors, $u$ and $v$ in $W$, so that $u + v$ is not in $W$.
- Is $W$ a subspace of $\mathbb{R}^2$?

Question. Let $W$ be the set of all vectors of the form

$$\begin{bmatrix}
2b + 3c \\
-b \\
2c
\end{bmatrix}$$

where $b$ and $c$ are arbitrary.

Find vectors $u$ and $v$ so that $W = \text{Span}\{u, v\}$. Why does this show that $W$ is a subspace of $\mathbb{R}^3$?
Question. Find Col $A$ and Nul $A$:

$$
A = \begin{bmatrix}
1 & -4 & 0 & 2 & 0 \\
0 & 0 & 1 & -5 & 0 \\
0 & 0 & 0 & 0 & 2
\end{bmatrix}.
$$

Question. Find $A$ such that the given set is Col $A$:

$$
\left\{ \begin{bmatrix}
2s + t \\
r - s + 2t \\
3r + s \\
2r - s - t
\end{bmatrix} : r, s, t \text{ real} \right\}
$$
Question. Is the following set a basis for $\mathbb{R}^3$? Fully justify your answer.

$$
\begin{bmatrix}
1 \\
0 \\
-3
\end{bmatrix},
\begin{bmatrix}
3 \\
1 \\
-4
\end{bmatrix},
\begin{bmatrix}
-2 \\
-1 \\
1
\end{bmatrix}
$$

Question. Find the vector $x$ determined by the given coordinate vector $[x]_B = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$ and the basis $B = \left\{ \begin{bmatrix} 3 \\ -5 \end{bmatrix}, \begin{bmatrix} -4 \\ 6 \end{bmatrix} \right\}$
Question. Find the coordinate vector $[x]_B$ of $x = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ relative to the basis $B = \left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \end{bmatrix} \right\}$.

Question. $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation that first reflects across the line $x = y$ and then rotates by 90 degrees clockwise. Find the standard matrix for $T$, and it’s eigenvalues and eigenvectors.
Question. Find the eigenvalues and eigenvectors for the matrix
\[
\begin{bmatrix}
3 & 1 & 1 \\
0 & 5 & 0 \\
-2 & 0 & 0
\end{bmatrix}
\]

Question. Define linear independence, span, basis, eigenvalue, eigenvector, orthogonal and orthonormal.

Question. • Find a unit vector in the direction of \( \mathbf{u} = \begin{bmatrix} -6 \\ 4 \\ -3 \end{bmatrix} \).

• Find a unit vector orthogonal to \( \mathbf{u} \).