Quiz 2

Let

\[ A = \begin{bmatrix}
0 & 0 & 4 \\
0 & -3 & -2 \\
-3 & 9 & -6
\end{bmatrix} \]

Do the columns of \( A \) span \( \mathbb{R}^3 \)? (\( \mathbb{R}^3 \) is the set of all vectors with 3 components).

You do not need to do much computation, so justify your answer clearly and thoroughly.

_Hint: Recall that the columns of a \( n \times m \) matrix \( A \) span \( \mathbb{R}^n \) if and only if for every vector \( b \) in \( \mathbb{R}^n \), the equation \( Ax = b \) is consistent._

Let us make use of the hint (aka Theorem 4, Ch 1.4). Let us see that \( Ax = b \) is consistent, no matter what we choose for \( b \).

Choose \( b \) to be the vector

\[ \begin{bmatrix}
b_1 \\
b_2 \\
b_3
\end{bmatrix} \]

Then the system \( Ax = b \) corresponds to the matrix

\[ \begin{bmatrix}
0 & 0 & 4 & | & b_1 \\
0 & -3 & -2 & | & b_2 \\
-3 & 9 & -6 & | & b_3
\end{bmatrix} \]

Switch R1 and R3:

\[ \begin{bmatrix}
-3 & 9 & -6 & | & b_3 \\
0 & -3 & -2 & | & b_2 \\
0 & 0 & 4 & | & b_1
\end{bmatrix} \]

The matrix is now in row echelon form. Furthermore, it does not have any pivot in the last (augmented) column. Therefore it is consistent, no matter what we choose for \( b_1, b_2, b_3 \). (Alternatively, none of the rows have the form \([0 \ 0 \ 0 \ | \ b]\) where \( b \neq 0 \).)