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A C^0 linear finite element method for two fourth-order eigenvalue problems

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In this article, we construct a C^0 linear finite element method for two fourth-order eigenvalue problems: the biharmonic and the transmission eigenvalue problems. The basic idea of our construction is to use gradient recovery operator to compute the higher-order derivatives of a C^0 piecewise linear function, which do not exist in the classical sense. For the biharmonic eigenvalue problem, the optimal convergence rates of eigenvalue/eigenfunction approximation are theoretically derived and numerically verified. For the transmission eigenvalue problem, the optimal convergence rate of the eigenvalues is verified by two numerical examples: one for constant refraction index and the other for variable refraction index. Compared with existing schemes in the literature, the proposed scheme is straightforward and simpler, and computationally less expensive to achieve the same order of accuracy.

Keywords: biharmonic eigenvalue; transmission eigenvalue; gradient recovery; superconvergence; linear finite element.

1. Introduction

This article is concerned with the numerical approximation of two fourth-order eigenvalue problems. The biharmonic eigenvalue problem describes the eigenmodes of a vibrating homogeneous isotropic plate with constant thickness, and the transmission eigenvalue problem simulates the eigenmodes of the inverse scattering of acoustic waves. Recently, these two eigenvalue problems have attracted much attention of researchers from both theoretical and computational fields (see Ciarlet, 1978; Canuto, 1978, 1981; Rannacher, 1979; Cakoni *et al.*, 2007, 2009; Brenner & Scott, 2008; Colton *et al.*, 2010; Ji & Sun, 2013; Ji *et al.*, 2014 for an incomplete list of references).

Biharmonic eigenvalue problems are numerically solved by conforming, nonconforming and mixed/hybrid finite element methods. The conforming finite element method requires a C^1 space (Ciarlet, 1978; Brenner & Scott, 2008) so that its basis functions contain at least quintic polynomials in two dimensions, which is rather expensive. Alternatively, Rannacher (1979) considered eigenvalue approximation for fourth-order self-adjoint eigenvalue problems by nonconforming finite elements. The disadvantage of the nonconforming method lies in the delicate design of the finite element space in order to guarantee convergence. Canuto (1978, 1981) and Ishihara (1978) considered the mixed/hybrid element approximation for the biharmonic eigenvalue problem and derived error estimates for the eigenpairs. Further, Mercier *et al.* (1981) developed an abstract analysis for the approximate eigenpairs using mixed/hybrid finite element methods based on the general theory of compact operators (see also Chatelin, 1983 and Babuska & Osborn, 1991). Then the techniques accelerating for the convergence of mixed finite element approximations for the eigenpairs of the biharmonic operator and $2m$-order self-adjoint eigenvalue problems have been proposed in Andreeva *et al.* (2005) and Racheva & Andreev (2002), respectively. An extension to the superconvergence of the Hermite bicubic element for the biharmonic eigenvalue problem has been carried out in Wu (2001).

Transmission eigenvalue problems are often solved by reformulating them as fourth-order eigenvalue problems. The transmission eigenvalues usually provide qualitative information about the material properties of the scattering object from far-field data (Cakoni *et al.*, 2007, 2009). Fast and accurate computation of transmission eigenvalues is desired in practice (Colton *et al.*, 2010; An & Shen, 2013; Ji & Sun, 2013; Cakoni *et al.*, 2014; Ji *et al.*, 2014; Yang *et al.*, 2015). It is worth mentioning that, in An & Shen (2013), the authors proposed an efficient spectral element method for computing transmission eigenvalues in radially stratified media. However, for the transmission problem in an arbitrary domain, fast finite element methods are still under developed. The complication of conforming FEMs and the inconsistency of mixed FEMs limit the applications of both methods.

In this article, we introduce a new idea developed in a recent article (Guo *et al.*, 2015) to solve these two eigenvalue problems. The variational equation of a fourth-order problem involves the second derivative of the discrete solution, which is impossible to obtain from a direct calculation of a C^0 linear element whose gradient is piecewise constant (w.r.t. the underlying mesh) and discontinuous across each element. To overcome this difficulty, we use the gradient recovery operator G_h to 'lift' discontinuous piecewise constant Dv_h into a continuous piecewise linear function G_hv_h (see e.g., Zienkiewicz & Zhu, 1992; Zhang & Naga, 2005; Zhang, 2007 for the details of different recovery operators). In other words, we apply the special *difference operator* DG_h to the standard Ritz–Galerkin method to construct our finite element schemes.

Our method is straightforward: it does not require the complicated construction of C^1 finite element basis functions for conforming/nonconforming method nor complicated penalty terms for the discontinuous Galerkin method. On the other hand, the fact that the recovery operator G_h can be defined on a general unstructured grid implies that the method is valid for problems on arbitrary domains and meshes. Moreover, our method has only function value unknowns at nodal points instead of both function value and derivative unknowns, so its computational complexity is much lower than existing conforming and nonconforming methods in the literature.

It is worth mentioning that gradient recovery operators were used to discretize and solve biharmonic equations by Lamichhane (2011, 2014). However, to guarantee the stability and/or *optimal convergence* *orders* of their corresponding schemes, some additional conditions were enforced on gradient recovery operators. Since popular gradient recovery operators such as superconvergent patch recovery (SPR) and polynomial preserving recovery (PPR) do not satisfy these additional conditions, the application of their method is very limited, and this might be one reason why no numerical example is provided in Lamichhane (2011, 2014). In our article, we use popular gradient recovery operators to discretize high-order partial differential equations. This makes our scheme more practical.

Although the construction of our algorithm is simple, the numerical eigenvalues astonishingly converge to the exact ones with optimal rate. This fact has been observed in our numerical experiments for both biharmonic and transmission eigenvalue problems. In addition, a theoretical proof of this optimal convergence has been provided for biharmonic eigenvalue problems.

The remaining parts of this article are organized as follows. Section 2 introduces a gradient recovery operator and is devoted to the discretization of the biharmonic eigenvalue problem. We present a recovery-based linear finite element method and derive error estimates for the eigenmodes in various norms. Section 3 applies the new scheme to a transmission eigenvalue problem. Some numerical experiments are presented in Section 4. Finally, we make some concluding remarks in Section 5.

Throughout the article, the letter C denotes a generic positive constant, which may be different at different occurrences. For convenience, the symbol $\lesssim$ will be used: $x \lesssim y$ means $x \le Cy$ for some constants C independent of the mesh size. Then $x \sim y$ means both $x \lesssim y$ and $y \lesssim x$ hold.

2. Biharmonic Eigenvalue problem

In this section, we consider the following biharmonic eigenvalue value problem

$$\Delta^2 u = \lambda u \quad \text{in } \Omega, \tag{2.1}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{2.2}$$

$$\partial_{\mathbf{n}} u = 0 \quad \text{on } \partial\Omega, \tag{2.3}$$

where Ω is bounded Lipschitz continuous domain in $\mathbb{R}^2$ and $\mathbf{n}$ is the unit outward normal vector on the boundary $\partial\Omega$. The corresponding weak form is Find $(\lambda, u) \in \mathbb{R} \times V$ such that $\|u\|_0 = 1$ and

$$a(u, v) := \int_\Omega D^2 u : D^2 v = \lambda(u, v) \quad \forall v \in V, \tag{2.4}$$

where the space

$$V := H_0^2(\Omega) = \{v \in H^2(\Omega) \mid v = \partial_{\mathbf{n}} v = 0, \text{on } \partial\Omega\},$$

and the Frobenius product ':' for two matrixes $B_k = (b_{ij}^k), k = 1, 2$ is defined as

$$B_1 : B_2 = \sum_{i,j=1}^{2} b_{ij}^1 b_{ij}^2.$$

If the boundary condition (2.3) is replaced by $\partial_{\mathbf{nn}}^2 u = 0$, the weak form becomes: Find $(\lambda, u) \in \mathbb{R} \times \big(H^2(\Omega) \cap H_0^1(\Omega)\big)$ such that $\|u\|_0 = 1$ and

$$a(u, v) = \lambda(u, v) \quad \forall v \in H^2(\Omega) \cap H_0^1(\Omega). \tag{2.5}$$

Let $\mathcal{T}_h$ be a *triangulation* of the domain Ω with simplicial grids having the mesh-size h. We denote the set of vertices and edges of $\mathcal{T}_h$ by $\mathcal{N}_h$ and $\mathcal{E}_h$, respectively. Let V_h be the standard P_1 finite element space corresponding to $\mathcal{T}_h$ of Ω. We define

$$V_h^0 = \{v_h \in V_h : v_h = 0 \text{ on } \partial\Omega\}$$

and

$$V_h^{00} = \{v_h \in V_h^0 : (G_h v_h) \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\},$$

where $G_h : V_h \longrightarrow V_h \times V_h$ is a weighted averaging, SPR or PPR gradient recovery operator (Zienkiewicz & Zhu, 1992; Naga & Zhang, 2005). In Xu & Zhang (2004) and Zhang & Naga (2005), the following properties of the gradient recovery operators G_h have been proved

$$\|G_h v_h\|_0 \lesssim |v_h|_1 \quad \forall v_h \in V_h, \tag{2.6}$$

$$\|\nabla u - G_h u_I\|_0 \lesssim h^2 |u|_{3,\infty} \quad \forall u \in W^{3,\infty}(\Omega). \tag{2.7}$$

When the mesh is uniform, the following discrete Poincaré inequality is established in Guo *et al.* (2015)

$$\|v_h\|_i \lesssim \|G_h v_h\|_i \quad \forall v_h \in V_h^0, i = 0, 1. \tag{2.8}$$

REMARK 2.1 Weighted averaging, SPR or PPR becomes the same gradient recovery operator on uniform meshes. Equation (2.7) holds when the mesh form an $O(h^{1+\alpha})$ parallelogram for weighted averaging and SPR gradient operator. Equation (2.7) is always true for PPR gradient recovery operator due to polynomial preserving property.

REMARK 2.2 For a general function $v_h \in V_h$, $G_h v_h = 0$ does not imply that $v_h = 0$. However, for functions in V_h^0, the discrete Poincaré inequality $\|v_h\|_0 \lesssim \|G_h v_h\|_0$ implies the *coercivity* of G_h.

The discrete eigenvalue problem for (2.4) seeks eigenpairs $(\lambda_h, u_h) \in \mathbb{R} \times V_h^{00}$ with $\|u_h\|_0 = 1$ such that

$$a_h(u_h, v_h) = \lambda_h(u_h, v_h) \quad \forall v_h \in V_h^{00}, \tag{2.9}$$

where the discrete bilinear form

$$a_h(u_h, v_h) := \int_\Omega DG_h u_h : DG_h v_h.$$

Similarly, the discrete eigenvalue problem for (2.5) seeks eigenpairs $(\lambda_h, u_h) \in \mathbb{R} \times V_h^0$ with $\|u_h\|_0 = 1$ such that

$$a_h(u_h, v) = \lambda_h(u_h, v) \quad \forall v \in V_h^0. \tag{2.10}$$

In the rest of the article, we will only analyse the problem (2.4) and its discretization (2.9), since similar results can be obtained for (2.5) and its discretization counterpart (2.10) by the same reasoning.

It is known from the spectral theory (Chatelin, 1983) that the inverse of a compact self-adjoint operator has countably many eigenvalues, which are real and positive with $+\infty$ as its unique accumulation point. Therefore, we can suppose that the eigenvalues of (2.4) are enumerated as

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots$$

and denote by $(u_1, u_2, u_3, \ldots)$ some L^2-orthonormal system of corresponding eigenfunctions. For any $j \in \mathbb{N}$, the eigenspace corresponding to λ_j is defined as

$$V_{\lambda_i} := \{u \in H_0^2(\Omega) \mid (\lambda_i, u) \text{ satisfies } (2.4)\} = \text{span}\{u_k \mid k \in \mathbb{N} \text{ and } \lambda_k = \lambda_i\}.$$

Apparently, the space V_{λ_i} has a finite dimension. On the other hand, the fact that $\|v\|_0 \lesssim |G_h v|_1, \forall v \in V_h^0$ (see Guo *et al.*, 2015) implies that the discrete eigenvalues for (2.9) can be enumerated as

$$0 < \lambda_{h,1} \leq \lambda_{h,2} \leq \lambda_{h,3} \leq \ldots$$

with corresponding L^2-orthonormal eigenfunctions $(u_{h,1}, u_{h,2}, u_{h,3}, \ldots)$. The discrete eigenspace corresponding to $\lambda_{h,i}$ defined as

$$V_{\lambda_{h,i}} := \{u \in V_h^{00} \mid (\lambda_{h,i}, u_h) \text{ satisfies } (2.9)\} = \text{span}\{u_{h,k} \mid k \in \mathbb{N} \text{ and } \lambda_{h,k} = \lambda_{h,i}\}$$

is also of finite dimension.

Given $f \in V$, let $u \in V$ denote the unique solution to the linear problem

$$a(u, v) = (f, v) \quad \forall v \in V. \tag{2.11}$$

This defines a mapping $T : V \to V, f \mapsto u = Tf$, which is a self-adjoint operator since there holds the following equality: for any $u, v \in L^2$,

$$(Tu, v) = (v, Tu) = a(Tv, Tu) = a(Tu, Tv) = (u, Tv).$$

With this operator, (2.4) has an equivalent formulation

$$Tu = \lambda^{-1} u = \mu u,$$

where $\mu := \lambda^{-1}$ and $V_\mu := V_\lambda$.

Similarly, define the discrete operator $T_h : L^2 \to V_h^{00} \subset H^1$ by letting

$$a_h(T_h f, v) = (f, v) \quad \forall v \in V_h^{00}. \tag{2.12}$$

The discrete eigenvalue problem (2.9) has an equivalent formulation

$$T_h u_h = \lambda_h^{-1} u_h = \mu_h u_h,$$

where $\mu_h := \lambda_h^{-1}$ and $V_{\mu_h} := V_{\lambda_h}$. The mapping T_h is also self-adjoint, since we also have that: for any $u, v \in L^2$

$$(T_h u, v) = (v, T_h u) = a_h(T_h v, T_h u) = a_h(T_h u, T_h v) = (u, T_h v).$$

The following error estimates for the discretization of source problem has been shown in Guo *et al.* (2015):

LEMMA 2.3 For any given $f \in L^2(\Omega)$, let $u = Tf$ and $u_h = T_hf$ be the solution of (2.11) and (2.12), respectively. If $u \in H^5(\Omega)$ then

$$\begin{aligned} \|u - u_h\|_0 &\lesssim h^2\|u\|_5, & \|u - u_h\|_1 &\lesssim h\|u\|_5, \\ \|Du - G_hu_h\|_0 &\lesssim h^2\|u\|_5, & \|Du - G_hu_h\|_1 &\lesssim h\|u\|_5. \end{aligned} \tag{2.10}$$

REMARK 2.4 The following weak estimate

$$\left|\int_\Omega \nabla v \cdot (G_hv_h - \nabla v_h)\right| \lesssim h\|v\|_2\|G_hv_h\|_1 \tag{2.13}$$

plays an important role to show the consistency of the scheme. With the consistency and coercivity mentioned in Remark 2.1, we can obtain the above convergence results (see Guo *et al.*, 2015 for details). Note that for a general gradient operator G_h, a direct estimate for $\|G_hv_h - \nabla v_h\|_0$ seems to be a very difficult task.

From the first estimate of (2.13), we obtain that

$$\|u - u_h\|_0 \lesssim h^2\|f\|_1.$$

On the other hand, it is easy to deduce from (2.11) that $\|u\|_0 \lesssim \|f\|_{-1}$. Moreover, the facts that $\|v_h\|_0 \lesssim \|DG_hv_h\|_1$ and u_h is the solution (2.12) imply $\|u_h\|_0 \lesssim \|f\|_{-1}$. That is,

$$\|u - u_h\|_0 \lesssim \|f\|_{-1}.$$

Then by the interpolating theory (Brenner & Scott, 2008), there exists some $s > 0$ such that

$$\|Tf - T_hf\|_0 = \|u - u_h\|_0 \lesssim h^s\|f\|_0.$$

It immediately follows that

$$\|T - T_h\|_0 \longrightarrow 0 \quad \text{as } h \to 0. \tag{2.11}$$

Consequently (see Yang, 2012, Theorem 1.4.5),

$$\mu_{h,i} \to \mu_i \quad \text{and} \quad \lambda_{h,i} \to \lambda_i.$$

For simplicity, we drop off now the subscript i from $\mu_{h,i}, \lambda_{h,i}, u_{h,i}$ and μ_i, λ_i, u_i. Since the eigenvalue μ is isolated, there exists a constant $d(\mu) > 0$, such that

$$\min_{\mu_j \neq \mu} |\mu_h - \mu_j| \geq d(\mu)$$

provided h is sufficiently small. By Yang (2012, Theorem 1.4.6) and (2.11), we have the following result.

LEMMA 2.5 Let (μ_h, u_h) with $\|u_h\|_0 = 1$ be the ith eigenpair of T_h and μ the ith eigenvalue of T. Then $\mu_h \to \mu$, and there exists $u \in V_\mu$ with $\|u\|_0 = 1$ such that

$$\mu - \mu_h = \frac{1}{(u, u_h)}(Tu_h - T_h u_h, u), \tag{2.12}$$

$$\|u_h - u\|_0 \leq \frac{\|Tu_h - T_h u_h\|_0}{d(\mu)}\left(1 + \frac{\|Tu_h - T_h u_h\|_0^2}{d(\mu)^2}\right)^{\frac{1}{2}}. \tag{2.13}$$

We are now ready to estimate the error of the discrete eigenpairs.

THEOREM 2.6 Let (λ_h, u_h) with $\|u_h\|_0 = 1$ be the ith eigenpair of (2.9) and λ the ith eigenvalue of (2.5). Then $\lambda_h \to \lambda$ as $h \to 0$, and there exists $u \in V_\lambda$ with $\|u\|_0 = 1$ such that

$$\lambda_h - \lambda = \frac{\lambda\lambda_h}{(u, u_h)}((T - T_h)u, u) + R_1, \tag{2.14}$$

$$\|u_h - u\|_0 \lesssim \|(T - T_h)u\|_0, \tag{2.15}$$

$$\|G_h u_h - Du\|_0 \lesssim \lambda\|D(Tu) - G_h(T_h u)\|_0 + \|(T - T_h)u\|_0, \tag{2.16}$$

$$D^2 u - DG_h u_h = \lambda(D^2(Tu) - DG_h(T_h u)) + R_2, \tag{2.17}$$

where $|R_1| \lesssim \|(T - T_h)u\|_0^2$, $\|R_2\|_0 \lesssim \|(T - T_h)u\|_0$.

Proof. By (2.13),

$$\begin{aligned}\|u_h - u\|_0 &\lesssim \|(T - T_h)u_h\|_0 \\ &\lesssim \|(T - T_h)u\|_0 + \|T - T_h\|_0\|u_h - u\|_0.\end{aligned}$$

Noticing $\|T - T_h\|_0 \to 0$ $(h \to 0)$, we obtain (2.15).

Next we show (2.14). By the properties of T, T_h, we have

$$\begin{aligned}(Tu_h - T_h u_h, u) &= (Tu_h, u) - (\lambda_h^{-1} u_h, u) \\ &= (u_h, Tu) - (\lambda_h^{-1} u_h, u) \\ &= (\lambda^{-1} - \lambda_h^{-1})(u_h, u).\end{aligned}$$

It follows that

$$\begin{aligned}\lambda_h - \lambda &= \frac{\lambda\lambda_h}{(u, u_h)}((T - T_h)u_h, u) \\ &= \frac{\lambda\lambda_h}{(u, u_h)}\big(((T - T_h)u, u) + ((T - T_h)(u_h - u), u)\big) \\ &= \frac{\lambda\lambda_h}{(u, u_h)}((T - T_h)u, u) + R_1.\end{aligned}$$

By the facts that T, T_h are both self-adjoint, $\lambda_h \to \lambda$ and (2.15), we have

$$\begin{aligned}|R_1| &= \left|\frac{\lambda\lambda_h}{(u,u_h)}\big((T-T_h)(u_h-u),u\big)\right| \\ &\lesssim |(u_h-u,(T-T_h)u)| \\ &\lesssim \|u_h-u\|_0\|((T-T_h)u)\|_0 \\ &\lesssim \|(T-T_h)u\|_0^2,\end{aligned}$$

which implies that (2.14) holds. Note that in the above estimate, we have used the fact $(u_h, u) \to 1$, which is deduced from (2.15) and $\|u\|_0 = 1$.

Next we show (2.17). From (2.9) and the definition of T_h, it follows that

$$\begin{aligned}a_h(u_h-\lambda T_h u, u_h-\lambda T_h u) &= (\lambda_h u_h-\lambda u, u_h-\lambda T_h u) \\ &\le \|\lambda_h u_h-\lambda u\|_0\|u_h-\lambda T_h u\|_0 \\ &\le (\|\lambda_h u_h-\lambda u\|_0+\|u_h-\lambda T_h u\|_0)^2.\end{aligned}$$

Together with (2.14) and (2.15), we obtain

$$\|DG_h(u_h-\lambda T_h u)\|_0 \lesssim \|(T-T_h)u\|_0. \tag{2.18}$$

On the other hand, using the fact that $u = \lambda Tu$, we have

$$\begin{aligned}D(Du-G_hu_h) &= -DG_h(u_h-\lambda T_h u)+D^2u-\lambda DG_hT_hu \\ &= R_2+\lambda(D^2(Tu)-DG_h(T_hu)).\end{aligned}$$

By (2.18),

$$\|R_2\|_0 \lesssim \|(T-T_h)u\|_0,$$

which validates (2.17).

Finally, by (2.18) and the Poincaré inequality,

$$\|\lambda G_hT_hu-G_hu_h\|_0 \lesssim \|DG_h(u_h-\lambda T_h u)\|_0 \lesssim \|(T-T_h)u\|_0.$$

Then

$$\begin{aligned}\|Du-G_hu_h\|_0 &\le \|D(\lambda Tu)-\lambda G_hT_hu\|_0+\|\lambda G_hT_hu-G_hu_h\|_0 \\ &\lesssim \lambda\|D(Tu)-G_hT_hu\|_0+\|(T-T_h)u\|_0,\end{aligned}$$

which is the desired (2.16). □

Theorem 2.6 transfers the error estimates of eigenvalue problem into the ones of the corresponding source problem. As an immediate consequence of this theorem and Lemma 2.1, we have the following results.

THEOREM 2.7 Let (λ_h, u_h) with $\|u_h\|_0 = 1$ be the ith eigenpair of (2.9) and λ the ith eigenvalue of (2.5). Then $\lambda_h \to \lambda$ as $h \to 0$, and there exists $u \in V_\lambda$ with $\|u\|_0 = 1$ such that

$$|\lambda - \lambda_h| \lesssim h^2\|u\|_5,$$
$$\|u_h - u\|_0 \lesssim h^2\|u\|_5,$$
$$\|G_h u_h - Du\|_0 \lesssim h^2\|u\|_5,$$
$$\|D^2u - DG_h u_h\|_0 \lesssim h\|u\|_5.$$

3. Transmission eigenvalue problem

This section is dedicated to an application of gradient recovery operator on the numerical solution of transmission eigenvalue problem.

3.1 *Variational form*

Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and n a real value function in $L^\infty(\Omega)$ with $n > 1$. We seek a complex quantity $k \in \mathbb{C}$ and a nontrivial pair of functions $(v, w) \in L^2(\Omega) \times L^2(\Omega)$ such that $v - w \in H^2(\Omega)$ and

$$\begin{array}{ll} \Delta w + k^2 n(x) w = 0 & \text{in } \ \Omega, \\ \Delta v + k^2 v = 0 & \text{in } \ \Omega, \\ w = v & \text{on } \partial\Omega, \\ \frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} & \text{on } \partial\Omega, \end{array} \tag{3.1}$$

where ν is the unit out normal vector of boundary $\partial\Omega$. As in Cakoni *et al.* (2014), we first rewrite (3.1) as a fourth-order eigenvalue problem. Let $u = w - v \in H_0^2(\Omega)$, (3.1) implies that

$$(\Delta + k^2 n)\frac{1}{n-1}(\Delta + k^2)u = 0 \quad \text{in } \Omega. \tag{3.2}$$

The weak form is to find an eigenpair $(k, u) \in \mathbb{C} \times H_0^2(\Omega)$ with $u \neq 0$ such that

$$\int_\Omega \frac{1}{n-1}(\Delta u + k^2 u)(\Delta \bar{v} + k^2 n \bar{v})\,\mathrm{d}x = 0 \quad \forall v \in H_0^2(\Omega). \tag{3.3}$$

Expanding (3.3) gives

$$(\Delta u, \Delta v)_{n-1} + k^2(u, \Delta v)_{n-1} + k^2(\Delta u, nv)_{n-1} + k^4(nu, v)_{n-1} = 0, \tag{3.4}$$

where the weighted inner product $(\cdot,\cdot)_{n-1}$ is defined as

$$(u, v)_{n-1} = \int_\Omega \frac{1}{n-1} u\bar{v}\,\mathrm{d}x.$$

The Poincaré inequality implies that $k = 0$ is not an eigenvalue since otherwise we have $\Delta u = 0$, which will lead to $u = 0$. Note that (3.4) is a quadratic eigenvalue problem, which is more difficult than a linear one. Letting $\phi \in H_0^1(\Omega)$ be the weak solution of the following elliptic equation

$$\Delta\phi = k^2 \frac{n}{n-1} u \quad \text{in } \Omega, \tag{3.5}$$

the quadratic eigenvalue problem (3.4) can be reformulated as: Find the eigenpair $(u, \phi, k) \in \mathbf{H} \times \mathbb{C}$ such that (see e.g., Cakoni *et al.*, 2014)

$$\begin{aligned} &(\Delta u, \Delta v)_{n-1} = -k^2((u, \Delta v)_{n-1} + (\Delta u, nv)_{n-1} - (\nabla\phi, \nabla v)) \quad \forall v \in H_0^2(\Omega), \\ &(\nabla\phi, \nabla\psi) = -k^2(nu, \psi)_{n-1} \quad \forall \psi \in H_0^1(\Omega), \end{aligned} \tag{3.6}$$

where $\mathbf{H} = H_0^2(\Omega) \times H_0^1(\Omega)$. Note that (3.6) is a linear eigenvalue problem.

REMARK 3.1 If index of refraction $n(x)$ is smooth, the quadratic eigenvalue problem (3.4) is also equivalent to the following fourth-order eigenvalue problem: find $(u, v) \in \mathbf{H} \times \mathbb{C}$ such that

$$\begin{aligned} &(\Delta u, \Delta v)_{n-1} = k^2 \left[\left(\nabla \left(\frac{nu}{n-1} \right), \nabla v \right)_{n-1} + \left(\nabla u, \nabla \left(\frac{nv}{n-1} \right) \right) + (\nabla\phi, \nabla v) \right], \\ &(\nabla\phi, \nabla\psi) = -k^2(nu, \psi)_{n-1}, \end{aligned} \tag{3.7}$$

for any $v \in H_0^2(\Omega)$ and $\psi \in H_0^1(\Omega)$. Comparing the two weak forms (3.6) and (3.7), the formulation (3.6) is more generally since it works for nonsmooth index of refraction function n.

For $s = 0, 1, 2$, we define the space $\mathbf{H}^s = H^s(\Omega) \times H^{s-1}(\Omega)$ with the norm $\|(u, w)\|_s = \|u\|_s + \|w\|_{s-1}$. The Hilbert space $\mathbf{H} = H_0^2(\Omega) \times H_0^1(\Omega)$ is a subspace of $\mathbf{H}^2$ with the norm $\|\cdot\| = \|\cdot\|_2$.

Define the bilinear form

$$A((u, \phi), (v, \psi)) := (\Delta u, \Delta v)_{n-1} + (\nabla\phi, \nabla\psi) \tag{3.8}$$

on $\mathbf{H} \times \mathbf{H}$, and the bilinear form

$$B((u, \phi), (v, \psi)) = -((u, \Delta v)_{n-1} + (\Delta u, nv)_{n-1} - (\nabla\phi, \nabla v)) - (nu, \psi)_{n-1}, \tag{3.9}$$

on $\mathbf{H} \times \mathbf{H}$. The transmission eigenvalue problem (3.6) is to seek $(u, \phi, k) \in \mathbf{H} \times \mathbb{C}$ such that

$$A((u, \phi), (v, \psi)) = k^2 B((u, \phi), (v, \psi)) \quad \forall (v, \psi) \in H_0^2(\Omega) \times H_0^1(\Omega). \tag{3.10}$$

One can easily verify that for any given $(f, g) \in \mathbf{H}^2, B((f, g), (v, z))$ is a continuous linear form on $\mathbf{H}^2$:

$$|B((f, g), (v, z))| \lesssim \|(f, g)\|_1 \|(v, z)\|_2, \forall (v, z) \in \mathbf{H}^2.$$

Defining a linear operator $T : H_0^2(\Omega) \times H_0^1(\Omega) \to H_0^2(\Omega) \times H_0^1(\Omega)$ as the solution operator of the following variational problems

$$A(T(u, \phi), (v, \psi)) = B((u, \phi), (v, \psi)) \quad \forall (v, \psi) \in H_0^2(\Omega) \times H_0^1(\Omega), \tag{3.11}$$

the variational form (3.10) can be rewritten as: find $(u,\phi) \in H_0^2(\Omega) \times H_0^1(\Omega)$ and $\lambda \in \mathbb{C}$

$$T(u,\phi) = \lambda(u,\phi), \tag{3.12}$$

where $\lambda = \frac{1}{k^2}$. As in Yang *et al.* (2015), it is easy to show that T is compact from $\mathbf{H}^2$ to $\mathbf{H}^2$ and from $\mathbf{H}^1$ to $\mathbf{H}^1$ when $n \in W^{1,\infty}(\Omega)$ and T is compact from $\mathbf{H}^0$ to $\mathbf{H}^0$ when $n \in W^{2,\infty}(\Omega)$.

3.2 *Recovery-based linear finite elements discretization*

We define the discrete counter part of bilinear forms A and B, for all $u_h, v_h \in V_h^{00}$ and $\phi_h, \psi_h \in V_h^0$, respectively, by

$$A_h((u_h,\phi_h),(v_h,\psi_h)) = (\mathrm{div} G_h u_h, \mathrm{div} G_h v_h)_{n-1} + (\nabla\phi_h, \nabla\psi_h) \tag{3.13}$$

and

$$B_h((u_h,\phi_h),(v_h,\psi_h)) = -\Big(\frac{nu_h}{n-1}, \mathrm{div} G_h v_h\Big) - \Big(\mathrm{div} G_h u_h, \frac{nv_h}{n-1}\Big) + (\nabla\phi_h, \nabla v_h) - \Big(\frac{nu_h}{n-1}, \psi_h\Big), \tag{3.14}$$

where G_h is the gradient recovery operator mentioned in the previous section. The C^0 linear finite element approximation of (3.10) is to find $(u_h, \phi_h, k_h) \in V_h^{00} \times V_h^0 \times \mathbb{C}$ such that

$$A_h((u_h,\phi_h),(v_h,\psi_h)) = k_h^2 B_h((u_h,\phi_h),(v_h,\psi_h)) \quad \forall (v_h,\phi_h) \in V_h^{00} \times V_h^0. \tag{3.15}$$

Let the mapping $T_h : H_0^1(\Omega) \times H_0^1(\Omega) \to V_h^{00} \times V_h^0$ satisfy the discrete variational problem

$$A_h(T_h(u,\phi),(v_h,\psi_h)) = B_h((u_I,\phi),(v_h,\psi_h)) \quad \forall (v_h,\psi_h) \in V_h^{00} \times V_h^0, \tag{3.16}$$

where u_I is the interpolation of u in V_h, the discrete problem (3.15) can be rewritten in the operator form: Find $(u_h, \phi_h, \lambda_h) \in V_h^{00} \times V_h^0 \times \mathbb{C}$ such that

$$T_h(u_h,\phi_h) = \lambda_h(u_h,\phi_h), \tag{3.17}$$

where $\lambda_h = \frac{1}{k_h^2}$.

REMARK 3.2 Numerical examples in the next section indicate that the approximated transmission eigenvalue converges to the exact one at rate of $O(h^2)$. In addition, the numerical scheme produces a lower bound for the exact transmission eigenvalue. Note that all existing methods in the literature provide upper bounds.

REMARK 3.3 Since $V_h \not\subseteq H_0^2(\Omega)$, the scheme (3.15) is a nonconforming method.

REMARK 3.4 There are several alternative discrete schemes of the bilinear form $B(\cdot,\cdot)$. For example, one may choose the form as

$$B_h^1((u_h,\phi_h),(v_h,\psi_h)) = \Big(\nabla\Big(\frac{nu_h}{n-1}\Big), G_h v_h\Big) + \Big(G_h u_h, \nabla\Big(\frac{nv_h}{n-1}\Big)\Big) + (\nabla\phi_h, \nabla v_h) - \Big(\frac{nu_h}{n-1}, \psi_h\Big). \tag{3.18}$$

Compared to this form, the form (3.14) has an advantage that it can be defined for nonsmooth index of refraction n.

One may also choose

$$B_h^2((u_h,\phi_h),(v_h,\psi_h)) = \Big(\nabla\Big(\frac{nu_h}{n-1}\Big),\nabla v_h\Big) + \Big(\nabla u_h,\nabla\Big(\frac{nv_h}{n-1}\Big)\Big) + (\nabla\phi_h,\nabla v_h) - \Big(\frac{nu_h}{n-1},\psi_h\Big). \quad (3.19)$$

Here, we would like to point out that several of our numerical experiments show that this scheme does not have an optimal convergence order, even if n is sufficiently smooth, e.g., $n \in W^{2,\infty}$.

4. Numerical experiments

In this section, we provide several numerical examples to demonstrate the effectiveness and convergence rates of our methods. The first two examples are designed for biharmonic eigenvalue problems and the other two are for transmission eigenvalue problems.

In the following tables, all convergence rates are listed with respect to the degree of freedom (Dof). Noticing Dof $\approx h^{-2}$ for a two-dimensional grid, the corresponding convergent rates with respect to the mesh size h are double of what we present in the tables.

Example 1: Simply supported biharmonic eigenvalue problem.

Consider the following biharmonic eigenvalue problem with simply supported plate boundary condition

$$\begin{cases} \Delta^2 u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \partial_{nn} u = 0 & \text{on } \partial\Omega, \end{cases} \quad (4.1)$$

where $\Omega = (0,1)\times(0,1)$. The eigenvalues of (4.1) are $\lambda_{k,\ell} = (k^2+\ell^2)^2\pi^4$ and the corresponding eigenfunctions are $u_{k,\ell} = 2\sin(k\pi x)\sin(\ell\pi y)$ with $k,\ell = 1,2,\ldots$. In this example, we focus on numerical computation of the first three eigenvalues: $\lambda_1 = 4\pi^4$ and $\lambda_2 = \lambda_3 = 25\pi^4$. Here we use the following notation:

$$e := \|u_i - u_{i,h}\|_{0,\Omega}$$

and

$$D^2 e := |Du_i - G_h u_{i,h}|_{1,\Omega}.$$

Table 1 lists the numerical errors of the three smallest eigenvalues and their corresponding eigenvalue functions on regular pattern uniform triangular meshes. We see that the numerical eigenvalue $\lambda_{i,h}$ approximates the exact eigenvalue λ_i at rate of $O(h^2)$. In addition, $O(h^2)$ convergence of eigenfunction approximation in the L_2 norm and $O(h)$ convergence in the discrete H^2 norm can be observed. All those observations consist with our theoretical results. An interesting phenomenon is that $\lambda_{i,h}$ approximates the exact eigenvalue from below; see Column 4 in Table 1. We want to remark that lower bound of eigenvalue is very important in practice, and many efforts have been made to obtain eigenvalue approximation from below. The readers are referred to Armentano & Durán (2004), Guo *et al.* (2016), Yang *et al.* (2010) and Zhang *et al.* (2007) for other ways to approximate eigenvalue from below. Our numerical experiments

TABLE 1 *Numerical results of* (4.1) *on uniform mesh*

i	Dof	λ_i	$\lambda_{i,h} - \lambda_i$	Order	e	Order	D^2e	Order
1	289	382.503	−7.13e+00	—	4.31e−03	—	1.87e+00	—
1	1089	387.810	−1.83e+00	1.03	9.83e−04	1.11	9.33e−01	0.52
1	4225	389.175	−4.61e−01	1.02	2.38e−04	1.05	4.63e−01	0.52
1	16641	389.521	−1.16e−01	1.01	5.87e−05	1.02	2.30e−01	0.51
1	66049	389.607	−2.90e−02	1.00	1.46e−05	1.01	1.15e−01	0.51
1	263169	389.629	−7.22e−03	1.00	3.66e−06	1.00	5.71e−02	0.50
2	289	2328.173	−1.07e+02	—	1.43e−02	—	7.12e+00	—
2	1089	2406.649	−2.86e+01	1.00	2.69e−03	1.26	3.49e+00	0.54
2	4225	2427.890	−7.34e+00	1.00	6.08e−04	1.10	1.72e+00	0.52
2	16641	2433.371	−1.86e+00	1.00	1.48e−04	1.03	8.51e−01	0.51
2	66049	2434.761	−4.67e−01	1.00	3.67e−05	1.01	4.24e−01	0.51
2	263169	2435.110	−1.17e−01	1.00	9.15e−06	1.00	2.11e−01	0.50
3	289	2347.191	−8.80e+01	—	1.13e−02	—	6.44e+00	—
3	1089	2411.798	−2.34e+01	1.00	2.19e−03	1.23	3.17e+00	0.53
3	4225	2429.200	−6.03e+00	1.00	5.02e−04	1.09	1.56e+00	0.52
3	16641	2433.700	−1.53e+00	1.00	1.23e−04	1.03	7.74e−01	0.51
3	66049	2434.843	−3.84e−01	1.00	3.05e−05	1.01	3.85e−01	0.51
3	263169	2435.131	−9.63e−02	1.00	7.63e−06	1.00	1.92e−01	0.50

are also performed on chevron, Criss-cross and Unionjack pattern uniform meshes. The numerical results are similar, and hence they are not reported here.

We also tested our schemes on unstructured meshes. The first level coarse mesh is generated by EasyMesh (Niceno, 1997) and the five following levels of meshes are obtained by regular refinement. Table 2 presents the eigenpair approximation errors. As predicted in Theorem 2.7, we observe $|\lambda_i - \lambda_{i,h}|$, e and D^2e decay at rate $O(h^2)$, $O(h^2)$ and $O(h)$, respectively. Furthermore, $\lambda_{i,h}$ is smaller than λ_i.

Example 2: Clamped plate biharmonic eigenvalue problem.

We consider the biharmonic eigenvalue problem with clamped plate boundary condition

$$\begin{cases} \Delta^2 u = \lambda u & \text{in } \Omega, \\ u = \partial_n u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.2}$$

where $\Omega = (0,1) \times (0,1)$.

We focus on the numerical computation of the smallest eigenpair, which is unknown *a priori*. To compute the eigenvalue approximation error, we use $\lambda_1 = 1,294.93393$ as reference eigenvalue value (Chen & Lin, 2007). Also we use the follow relative errors

$$\hat{e} := \|u_{i,2h} - u_{i,h}\|_{0,\Omega}$$

and

$$D^2\hat{e} := |Gu_{i,2h} - G_h u_{i,h}|_{1,\Omega},$$

TABLE 2 *Numerical results of* (4.1) *on Delaunay mesh*

i	Dof	λ_i	$\lambda_{i,h}-\lambda_i$	Order	e	Order	D^2e	Order
1	139	370.309	−1.93e+01	—	8.73e−03	—	2.50e+00	—
1	513	384.857	−4.78e+00	1.07	1.95e−03	1.15	1.29e+00	0.51
1	1969	388.512	−1.12e+00	1.08	4.30e−04	1.12	6.12e−01	0.56
1	7713	389.361	−2.76e−01	1.03	1.02e−04	1.05	3.03e−01	0.52
1	30529	389.568	−6.85e−02	1.01	2.49e−05	1.03	1.51e−01	0.51
1	121473	389.619	−1.71e−02	1.01	6.12e−06	1.01	7.55e−02	0.50
2	139	2207.716	−2.28e+02	—	3.38e-02	—	9.43e+00	—
2	513	2374.042	−6.12e+01	1.01	7.00e−03	1.20	4.67e+00	0.54
2	1969	2420.180	−1.50e+01	1.04	1.23e−03	1.29	2.23e+00	0.55
2	7713	2431.486	−3.74e+00	1.02	2.74e−04	1.10	1.09e+00	0.52
2	30529	2434.293	−9.34e−01	1.01	6.55e−05	1.04	5.41e−01	0.51
2	121473	2434.994	−2.33e−01	1.00	1.60e−05	1.02	2.69e−01	0.51
3	139	2213.645	−2.22e+02	—	2.91e−02	—	9.27e+00	—
3	513	2374.249	−6.10e+01	0.99	6.84e−03	1.11	4.77e+00	0.51
3	1969	2420.601	−1.46e+01	1.06	1.22e−03	1.28	2.25e+00	0.56
3	7713	2431.627	−3.60e+00	1.03	2.73e−04	1.09	1.10e+00	0.52
3	30529	2434.332	−8.95e−01	1.01	6.54e−05	1.04	5.45e−01	0.51
3	121473	2435.004	−2.23e−01	1.01	1.59e−05	1.02	2.71e−01	0.51

TABLE 3 *Numerical Result of Clamped biharmonic eigenvalue problem on uniform mesh*

Dof	λ_1	$\lambda_{1,h}-\lambda_1$	Order	$\hat{e}$	Order	$D^2\hat{e}$	Order
1089	1226.035	−6.89e+01	—	2.66e−01	—	3.00e+01	—
4225	1265.917	−2.90e+01	0.64	2.24e−02	1.83	7.09e+00	1.06
16641	1287.059	−7.88e+00	0.95	4.68e−03	1.14	2.61e+00	0.73
66049	1292.903	−2.03e+00	0.98	1.36e−03	0.90	1.25e+00	0.53
263169	1294.419	−5.15e−01	0.99	3.59e−04	0.96	6.18e−01	0.51

to access the approximation quality of eigenfunction. Tables 3 and 4 show the results of eigenvalue/eigenfunction approximation on regular pattern uniform and Delaunay meshes, respectively. As in numerical experiments for biharmonic eigenvalue with simply supported plate, we observe that the numerical eigenvalue $\lambda_{1,h}$ approximates the exact one with order $O(h^2)$. Also, we observe that $\hat{e}$ decays at rate of $O(h^2)$ and $D^2\hat{e}$ decays at rate of $O(h)$. It indicates that $u_{i,h}$ converges to the exact eigenfunction at rate of $O(h^2)$ in the L^2 norm and at rate of $O(h)$ in the recovered H_2 norm, which consists with our theoretical results. Moreover, we observe that the numerical eigenvalues are always smaller than the exact one.

Example 3: Transmission eigenvalue problem with constant index of refraction.

We consider the transmission eigenvalue problem on the circular disk centered at $(0,0)$ with radius $\frac{1}{2}$. The index of refraction function $n(x)$ is a constant 16. Table 5 lists the six smallest transmission

TABLE 4 *Numerical Result of Clamped biharmonic eigenvalue problem on Delaunay mesh*

Dof	λ_1	$\lambda_{1,h} - \lambda_1$	Order	$\hat{e}$	Order	$D^2\hat{e}$	Order
1969	1231.796	−6.31e+01	—	4.69e−02	—	1.00e+01	—
7713	1278.029	−1.69e+01	0.97	9.27e−03	1.19	3.78e+00	0.72
30529	1290.579	−4.35e+00	0.99	2.60e−03	0.92	1.79e+00	0.54
121473	1293.829	−1.10e+00	0.99	6.95e−04	0.96	8.61e−01	0.53
484609	1294.655	−2.79e−01	1.00	1.80e−04	0.98	4.20e−01	0.52

TABLE 5 *Discrete eigenvalues of transmission eigenvalue problem on circular disk*

j	Dof	λ_{1,h_j}	λ_{2,h_j}	λ_{3,h_j}	λ_{4,h_j}	λ_{5,h_j}	λ_{6,h_j}
1	360	1.980655	2.583499	2.590153	3.174488	3.186215	3.694087
2	1375	1.986065	2.604242	2.604443	3.207476	3.207625	3.730442
3	5373	1.987482	2.610793	2.610838	3.222034	3.222034	3.737946
4	21241	1.987857	2.612387	2.612399	3.225490	3.225490	3.740130
5	84465	1.987956	2.612789	2.612792	3.226353	3.226353	3.740713

eigenvalues computed according to (3.15). Our numerical results match well with previous numerical results in the literature (Colton *et al.*, 2010; An & Shen, 2013; Ji *et al.*, 2014). Define the relative error as

$$\mathrm{err}_i = \frac{|\lambda_{i,h_j} - \lambda_{i,h_{j+1}}|}{\lambda_{i,h_{j+1}}},$$

where λ_{i,h_j} is the ith smallest transmission eigenvalue on the jth level mesh with $i = 1, 2, 3, 4, 5, 6$ and $j = 1, 2, 3, 4, 5$. As plotted in Fig. 1, the relative errors converge at rate of $O(h^2)$. It is worth mentioning that most numerical methods in the literature produce upper discrete eigenvalues for transmission problems, while our method generates lower discrete eigenvalues.

Example 4: Transmission eigenvalue problem with variable index of refraction.
We consider a general case that the index of refraction is a general variable function. Specifically, we take $n(x) = 8 + x - y, (x, y) \in \Omega = (0, 1)^2$ as in Ji *et al.* (2014). We list in Table 6 the approximation results of the five smallest transmission eigenvalues which are computed by our scheme (3.15). We observe that the discrete eigenvalues increase when the degree of freedom gets bigger. Namely, the discrete eigenvalues computed by our scheme converges from below to the exact eigenvalues. To illustrate the convergence rates, we depict in Fig. 2 the relative error in terms of the degree of freedom. As in Example 3, we use $\mathrm{err_i}, i = 1, \ldots, 5$ to denote the relative error of the ith eigenvalue. We observe that all $\mathrm{err_i}, i = 1, \ldots, 5$ converge with second order, which are consistent with the results in Ji *et al.* (2014). We would like to mention that our computational cost is much lower than the corresponding ones in Ji *et al.* (2014).

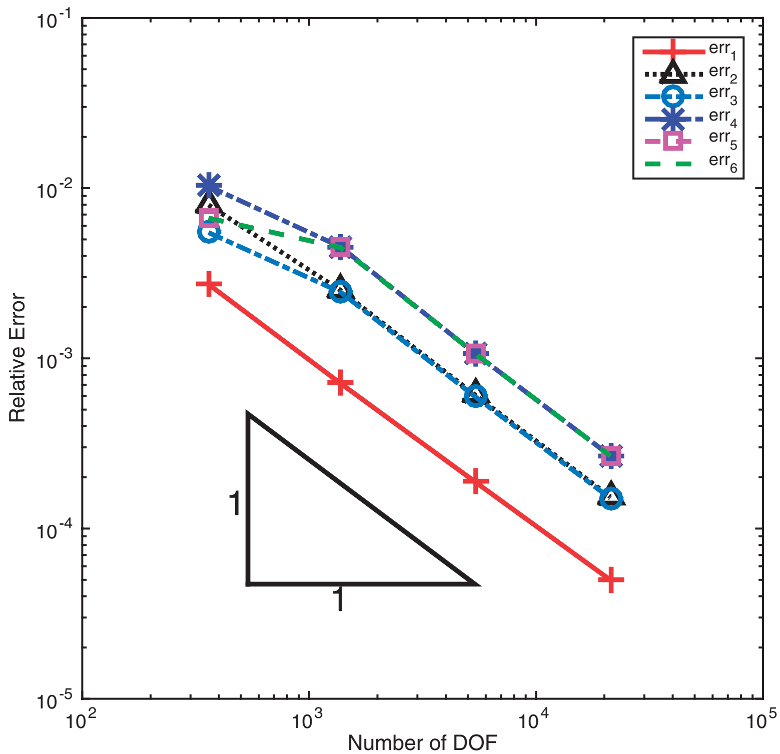

FIG. 1. Convergence rate of the transmission eigenvalue problem on circular disk.

TABLE 6 *Numerical results for the transmission eigenvalue problem on unit square*

j	Dof	λ_{1,h_j}	λ_{2,h_j}	λ_{3,h_j}	λ_{4,h_j}	λ_{5,h_j}
1	289	2.798184	3.479101	3.482872	4.040773	4.359484
2	1089	2.814542	3.516790	3.520860	4.080405	4.462083
3	4225	2.820122	3.532973	3.534310	4.107758	4.491470
4	16641	2.821651	3.537235	3.537801	4.115179	4.499126
5	66049	2.822052	3.538328	3.538691	4.117093	4.501074

5. Conclusion

In this article, a straightforward C^0 linear finite element method is developed for both biharmonic and transmission eigenvalue problems. The method circumvents the complicated construction of C^1 conforming elements and uses only values at element vertices as degrees of freedom, and hence is much simpler and more efficient than nonconforming finite elements. Although we observed the optimal convergence rate for both cases numerically, we provide only theoretical justification for the biharmonic eigenvalue

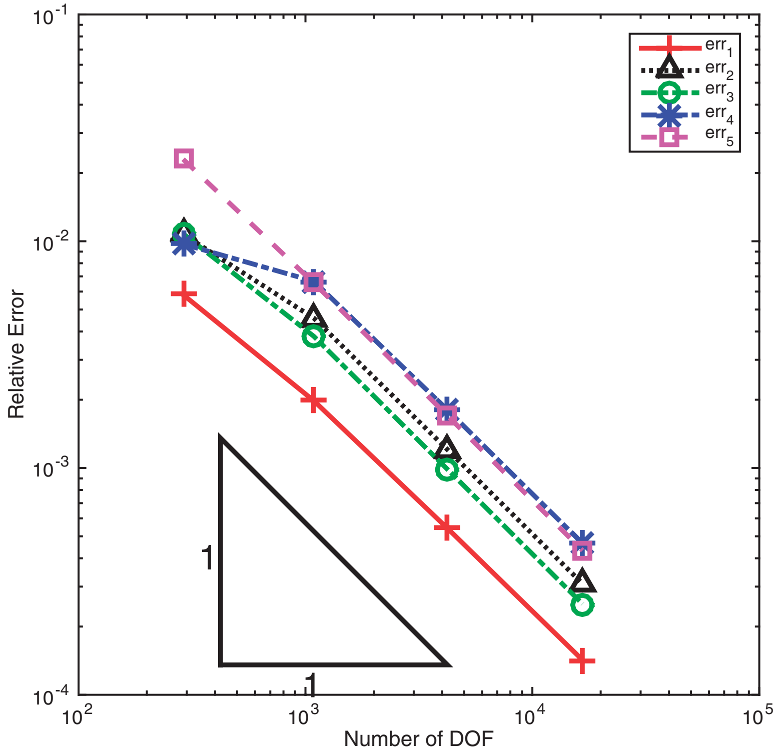

FIG. 2. Convergence rates of the transmission eigenvalue problem on the unit square.

problem. An interesting observation from our numerical experiments is that discrete eigenvalues based on proposed method converge to the exact eigenvalues from below for biharmonic eigenvalue problems as well as transmission eigenvalue problems. A theoretical proof of this phenomenon is one of our ongoing research projects.

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