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Superconvergent Two-Grid Methods for Elliptic Eigenvalue Problems

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Abstract Some numerical algorithms for elliptic eigenvalue problems are proposed, analyzed, and numerically tested. The methods combine advantages of the two-grid algorithm (Xu and Zhou in Math Comput 70(233):17–25, 2001), the two-space method (Racheva and Andreev in Comput Methods Appl Math 2:171–185, 2002), the shifted inverse power method (Hu and Cheng in Math Comput 80:1287–1301, 2011; Yang and Bi in SIAM J Numer Anal 49:1602–1624, 2011), and the polynomial preserving recovery enhancing technique (Naga et al. in SIAM J Sci Comput 28:1289–1300, 2006). Our new algorithms compare favorably with some existing methods and enjoy superconvergence property.

Keywords Eigenvalue problems · Two-grid method · Gradient recovery · Superconvergence · Polynomial preserving · Adaptive

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1 Introduction

A tremendous variety of science and engineering applications, e.g. the buckling of columns and shells and the vibration of elastic bodies, contain models of eigenvalue problems of partial differential equations. A recent survey article [17] of SIAM Review listed 515 references on theory and application of the Laplacian eigenvalue problems. As one of the most popular numerical methods, finite element method has attracted considerable attention in numerical solution of eigenvalue problems. A priori error estimates for the finite element approximation of eigenvalue problems have been investigated by many authors, see e.g., Babuška and Osborn [5,6], Chatelin [9], Strang and Fix [38], and references cited therein.

To reduce the computational cost of eigenvalue problems, Xu and Zhou introduced a two-grid discretization scheme [42]. Later on, similar ideas were applied to non self-adjoint eigenvalue problems [23] and semilinear elliptic eigenvalue problems [11]. Furthermore, it also has been generalized to three-scale discretization [16] and multilevel discretization [25]. Recently, a new shifted-inverse power method based two-grid scheme was proposed in [22,43].

To improve accuracy of eigenvalue approximation, many methods have been proposed. In [37], Shen and Zhou introduced a defect correction scheme based on averaging recovery, like a global L^2 projection and a Clément-type operator. In [34], Naga, Zhang, and Zhou used Polynomial Preserving Recovery to enhance eigenvalue approximation. In [40], Wu and Zhang further showed polynomial preserving recovery can also enhance eigenvalue approximation on adaptive meshes. The idea was further studied in [15,31]. Alternatively, Racheva and Andreev proposed a two-space method to achieve better eigenvalue approximation [36] and it was also applied to biharmonic eigenvalue problem [1].

In this paper, we propose some fast and efficient solvers for elliptic eigenvalue problems. We combine ideas of the two-grid method, two-space method, shifted-inverse power method, and PPR enhancement to design our new algorithms. The first purpose is to introduce two superconvergent two-grid methods for eigenvalue problems. Our first algorithm is a combination of the shifted-inverse power based two-grid scheme [22,43] and polynomial preserving recovery enhancing technique [34]. The second algorithm can be viewed as a combination of the two-grid scheme [22,43] and the two-space method [1,36]. It can be thought as a special hp method. The new proposed methods enjoy all advantages of the above methods: low computational cost and superconvergence.

Solutions of practical problems are often suffered from low regularity. Adaptive finite element method(AFEM) is a fundamental tool to overcome such difficulty. In the context of adaptive finite element method for elliptic eigenvalue problems, residual type a posteriori error estimators are analyzed in [14,20,21,28,39] and recovery type a posteriori error estimators are investigated by [27,29,40]. For all adaptive methods mentioned above, an algebraic eigenvalue problem has to be solved during every iteration, which is very time consuming. This cost dominates the computational cost of AFEM and usually is ignored. To reduce computational cost, Mehrmann and Miedlar [30] introduced a new adaptive method which only requires an inexact solution of algebraic eigenvalue equation on each iteration by only performing a few iterations of Krylov subspace solver. Recently, Li and Yang [24] proposed an adaptive finite element method based on multi-scale discretization for eigenvalue problems and Xie [41] introduced a type of adaptive finite element method based on the multilevel correction scheme. Both methods only solve an eigenvalue problem on the coarsest mesh and solve boundary value problems on adaptive refined meshes.

The second purpose of this paper is to propose two multilevel adaptive methods. Using our methods, solving an eigenvalue problem by AFEM will not be more difficult than solving a boundary value problem by AFEM. The most important feature which distinguishes them from the methods in [24,41] is that superconvergence of eigenfunction approximation and ultraconvergence (two order higher) of eigenvalue approximation can be numerically observed.

The rest of this paper is organized as follows. In Sect. 2, we introduce finite element discretization of elliptic eigenvalue problem and polynomial preserving recovery. Section 3 is devoted to presenting two superconvergent two-grid methods and their error estimates. In Sect. 4, we propose two multilevel adaptive methods. In Sect. 5, we use some numerical examples to demonstrate efficiency of our new methods. Finally, conclusive remarks are made in Sect. 6.

2 Preliminary

In this section, we first introduce the model eigenvalue problem and its conforming finite element discretization. Then, we give a simple description of polynomial preserving recovery for linear element.

2.1 A PDE Eigenvalue Problem and its Finite Element Discretization

Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain with Lipschitz continuous boundary $\partial\Omega$. Throughout this article, we shall use the standard notation for Sobolev spaces and their associated norms, seminorms, and inner products as in [8,12]. For a subdomain G of Ω, $W^{k,p}(G)$ denotes the Sobolev space with norm $\|\cdot\|_{k,p,G}$ and the seminorm $|\cdot|_{k,p,G}$. When $p = 2$, we denote simply $H^m(G) = W^{m,2}(G)$ and the subscript is omitted in corresponding norms and seminorms. In this article, the letter C, with or without subscript, denotes a generic constant which is independent of mesh size h and may not be the same at each occurrence. To simplify notation, we denote $X \leq CY$ by $X \lesssim Y$.

Consider the following second order self adjoint elliptic eigenvalue problem:

$$\begin{cases} -\nabla(\mathcal{D}\nabla u) + cu = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega; \end{cases} \tag{2.1}$$

where $\mathcal{D}$ is a 2×2 symmetric positive definite matrix and $c \in L^\infty(\Omega)$. Define a bilinear form $a(\cdot, \cdot) : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ by

$$a(u, v) = \int_\Omega (\mathcal{D}\nabla u \cdot \nabla v + cuv) dx.$$

Without loss of generality, we may assume that $c \geq 0$. It is easy to see that

$$a(u, v) \leq C\|u\|_{1,\Omega}\|v\|_{1,\Omega}, \quad \forall u, v \in H_0^1(\Omega),$$

and

$$a(u, u) \geq \alpha\|u\|_{1,\Omega}^2, \quad \forall u \in H_0^1(\Omega).$$

for some $\alpha > 0$. Define $\|\cdot\|_{a,\Omega} = \sqrt{a(\cdot, \cdot)}$. Then $\|\cdot\|_{a,\Omega}$ and $\|\cdot\|_{1,\Omega}$ are two equivalent norms in $H_0^1(\Omega)$.

The variational formulation of (2.1) reads as: Find $(\lambda, u) \in \mathbb{R} \times H_0^1(\Omega)$ with $u \neq 0$ such that

$$a(u, v) = \lambda(u, v), \quad \forall v \in H_0^1(\Omega). \tag{2.2}$$

It is well known that (2.2) has a countable sequence of real eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \to \infty$ and corresponding eigenfunctions $u_1, u_2, u_3, \cdots$ which can be assumed to satisfy $a(u_i, u_j) = \lambda_i(u_i, u_j) = \delta_{ij}$. In the sequence $\{\lambda_j\}$, the λ_i are repeated according to geometric multiplicity.

Let $\mathcal{T}_h$ be a conforming triangulation of the domain Ω into triangles T with diameter h_T less than or equal to h. Furthermore, assume $\mathcal{T}_h$ is shape regular [12]. Let $r \in \{1, 2\}$ and define the continuous finite element space of order r as

$$S^{h,r} = \left\{ v \in C(\bar{\Omega}) : v|_T \in \mathbb{P}_r(T), \forall T \in \mathcal{T}_h \right\} \subset H^1(\Omega),$$

where $\mathbb{P}_r(T)$ is the space of polynomials of degree less than or equal to r over T. In addition, let $S_0^{h,r} = S^{h,r} \cap H_0^1(\Omega)$. In most cases, we shall use linear finite element space and hence denote $S^{h,1}$ and $S_0^{h,1}$ by S^h and S_0^h to simplify notation. The finite element discretization of (2.1) is : Find $(\lambda_h, u_h) \in \mathbb{R} \times S_0^{h,r}$ with $u_h \neq 0$ such that

$$a(u_h, v_h) = \lambda_h(u_h, v_h), \quad \forall v_h \in S_0^{h,r}. \tag{2.3}$$

Similarly, (2.3) has a finite sequence of eigenvalues $0 < \lambda_{1,h} \leq \lambda_{2,h} \leq \cdots \leq \lambda_{n_h,h}$ and corresponding eigenfunctions $u_{1,h}, u_{2,h}, \cdots, u_{n_h,h}$ which can be chosen to satisfy $a(u_{i,h}, u_{j,h}) = \lambda_{i,h}(u_{i,h}, u_{j,h}) = \delta_{ij}$ with $i, j = 1, 2, \cdots, n_h$ and $n_h = \dim S_0^{h,r}$.

Suppose that the algebraic multiplicity of λ_i is equal to q, i.e. $\lambda_i = \lambda_{i+1} = \cdots = \lambda_{i+q-1}$. Let $M(\lambda_i)$ be the space spanned by all eigenfunctions corresponding to λ_i and $M_h(\lambda_i)$ be the direct sum of eigenspaces corresponding to all eigenvalue $\lambda_{i,h}$ that convergences to λ_i. Also, let $\hat{M}(\lambda_i) = \{v : v \in M(\lambda_i), \|v\|_a = 1\}$ and $\hat{M}_h(\lambda_i) = \{v : v \in M_h(\lambda_i), \|v\|_a = 1\}$.

In addition, we introduce two linear operators $T : L^2(\Omega) \to H_0^1(\Omega)$ and $T_{h,r} : L^2(\Omega) \to S_0^{h,r}(\Omega)$ such that

$$a(Tf, v) = (f, v), \quad \forall v \in H_0^1(\Omega). \tag{2.4}$$

and

$$a(T_{h,r}f, v) = (f, v), \quad \forall v \in S_0^{h,r}(\Omega). \tag{2.5}$$

Let

$$\eta_a(h) = \sup_{f \in H_0^1(\Omega), \|f\|_a = 1} \inf_{v \in S^{h,r}} \|Tf - v\|_a, \tag{2.6}$$

$$\delta_h(\lambda_i) = \sup_{w \in \hat{M}(\lambda_i)} \inf_{v \in S^{h,r}} \|w - v\|_a, \tag{2.7}$$

$$\rho_\Omega = \sup_{f \in L_2(\Omega), \|f\|_0 = 1} \inf_{v \in S^{h,r}} \|Tf - v\|_a. \tag{2.8}$$

From standard approximation theory [8,12], we can see that

$$\delta_h(\lambda_k) \lesssim \eta_a(h) \lesssim \rho_\Omega \lesssim h^r, \tag{2.9}$$

provided that $Tf \in H^{r+1}(\Omega)$ and $\hat{M}(\lambda_i) \subset H^{r+1}(\Omega)$.

For the conforming finite element discretization (2.3), the following result has been established by many authors [6,42].

Theorem 2.1 *Suppose $M(\lambda_i) \subset H_0^1(\Omega) \cap H^{r+1}(\Omega)$. Let $\lambda_{i,h}$ and λ_i be the ith eigenvalue of* (2.3) *and* (2.2)*, respectively. Then*

$$\lambda_i \leq \lambda_{i,h} \leq \lambda_i + Ch^{2r}. \tag{2.10}$$

For any eigenfunction $u_{i,h}$ corresponding to $\lambda_{i,h}$ satisfying $\|u_{i,h}\|_{a,\Omega} = 1$, there exists $u_i \in M(\lambda_i)$ such that

$$\|u_i - u_{i,h}\|_{a,\Omega} \leq Ch^r. \tag{2.11}$$

Before ending this subsection, we present an important identity [6] of eigenvalue and eigenfunction approximation.

Lemma 2.2 *Let* (λ, u) *be the solution of* (2.2). *Then for any* $w \in H_0^1(\Omega)\backslash\{0\}$, *there holds*

$$\frac{a(w,w)}{(w,w)} - \lambda = \frac{a(w-u,w-u)}{(w,w)} - \lambda\frac{(w-u,w-u)}{(w,w)}. \tag{2.12}$$

This identity shall play an important role in our superconvergence analysis.

2.2 Polynomial Preserving Recovery

Polynomial Preserving Recovery (PPR) [32,33,45] is an important alternative of the famous Superconvergent Patch Recovery proposed by Zienkiewicz and Zhu [47]. Let $G_h : S^h \to S^h \times S^h$ be the PPR operator and u_h be a function in S^h. For any vertex z on $\mathcal{T}_h$, construct a patch of elements $\mathcal{K}_z$ containing at least six vertices around z. Select all vertices in $\mathcal{K}_z$ as sampling points and fit a quadratic polynomial $p_z \in \mathbb{P}_2(\mathcal{K}_z)$ in least square sense at those sampling points. Then the recovered gradient at z is defined as

$$(G_h u_h)(z) = \nabla p_z(z).$$

$G_h u_h$ on the whole domain is obtained by interpolation. In [33,45], Zhang and Naga proved that G_h enjoys the following properties

(1) $\|\nabla u - G_h u_I\| \lesssim h^2 |u|_{3,\Omega}$, where u_I is the linear interpolation of u in S_h.
(2) $\|G_h v_h\|_{0,\Omega} \lesssim \|\nabla v_h\|_{0,\Omega}, \forall v_h \in S_h$.

According to [34], two adjacent triangles (sharing a common edge) form an $\mathcal{O}(h^{1+\alpha})$ ($\alpha > 0$) approximate parallelogram if the lengths of any two opposite edges differ by only $\mathcal{O}(h^{1+\alpha})$.

Definition 2.3 The triangulation $\mathcal{T}_h$ is said to satisfy Condition α if any two adjacent triangles form an $\mathcal{O}(h^{1+\alpha})$ parallelogram.

Using the same methods [37,45], we can prove the following superconvergence result:

Theorem 2.4 *Suppose* $M(\lambda_i) \subset H_0^1(\Omega) \cap W^{3,\infty}(\Omega)$ *and* $\mathcal{T}_h$ *satisfies Condition* α. *Let* G_h *be the polynomial preserving recovery operator. Then for any eigenfunction of* (2.3) *corresponding to* $\lambda_{i,h}$, *there exists an eigenfunction* $u_i \in M(\lambda_i)$ *corresponding to* λ_i *such that*

$$\left\|\mathcal{D}^{\frac{1}{2}}\nabla u_i - \mathcal{D}^{\frac{1}{2}}G_h u_{i,h}\right\|_{0,\Omega} \lesssim h^{1+\beta}\|u_i\|_{3,\infty,\Omega}, \quad \beta = \min(\alpha,1). \tag{2.13}$$

As pointed out in [34], $\alpha = \infty$ if $\mathcal{T}_h$ is generated using regular refinement. Fortunately, the fine grid $\mathcal{T}_h$ is always a regular refinement of some coarse grid $\mathcal{T}_H$ for two-grid method. When we introduce two-grid methods in Sect. 3, we only perform gradient recovery on fine grid $\mathcal{T}_h$. Thus we assume $\alpha = \infty$ and hence $\beta = 1$ in Sect. 3.

3 Superconvergent Two-Grid Methods

In the literature, two-grid methods [22,42,43] were proposed to reduce the cost of eigenvalue computations. To further improve the accuracy, two different approaches: gradient recovery enhancement [31,34,37] and two-space methods [1,36] can be used. Individually, those tools are useful in certain circumstances. Combining them properly, we are able to design much effective and superconvergence algorithms, which we shall describe below.

3.1 Gradient Recovery Enhanced Shifted Inverse Power Two-Grid Scheme

In this scheme, we first use the shifted inverse power based two-grid scheme [22,43] and then apply the gradient recovery enhancing technique [34].

Algorithm 1

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1. Solve the eigenvalue problem on a coarse grid $\mathcal{T}_H$: Find $(\lambda_{i,H}, u_{i,H}) \in \mathbb{R} \times S_0^H$ and $\|u_{i,H}\|_a = 1$ satisfying

$$a(u_{i,H}, v_H) = \lambda_{i,H}(u_{i,H}, v_H), \quad \forall v_H \in S_0^H. \tag{3.1}$$

2. Solve a source problem on the fine grid $\mathcal{T}_h$: Find $u_h^i \in S_0^h$ such that

$$a\left(u_h^i, v_h\right) - \lambda_{i,H}\left(u_h^i, v_h\right) = (u_{i,H}, v_h), \quad \forall v_h \in S_0^h, \tag{3.2}$$

and set $u^{i,h} = \dfrac{u_h^i}{\|u_h^i\|_a}$.

3. Apply the gradient recovery operator G_h on $u^{i,h}$ to get $G_h u^{i,h}$.
4. Set

$$\lambda^{i,h} = \frac{a\left(u^{i,h}, u^{i,h}\right)}{\left(u^{i,h}, u^{i,h}\right)} - \frac{\left\| \mathcal{D}^{\frac{1}{2}} \nabla u^{i,h} - \mathcal{D}^{\frac{1}{2}} G_h u^{i,h} \right\|_{0,\Omega}^2}{\left(u^{i,h}, u^{i,h}\right)}. \tag{3.3}$$

In the proof of our main superconvergence result, we need the following Lemma.

Lemma 3.1 *Suppose that $M(\lambda_i) \subset H_0^1(\Omega) \cap H^2(\Omega)$. Let $(\lambda^{i,h}, u^{i,h})$ be an approximate eigenpair of (2.2) obtained by Algorithm 1 and let H be properly small. Then*

$$dist\left(u^{i,h}, M_h(\lambda_i)\right) \lesssim H^4 + h^2, \tag{3.4}$$

where $dist(u^{i,h}, M_h(\lambda_i)) = \inf\limits_{v \in M_h(\lambda_i)} \|u^{i,h} - v\|_{a,\Omega}$.

Proof Let $u_0 = \lambda_{i,H} T_{h,1} u_{i,H} / \|\lambda_{i,H} T_{h,1} u_{i,H}\|_a$. According to formula (4.4) in [43], we know that

$$dist\left(u_0, \hat{M}(\lambda_i)\right) \leq C\left(\delta_H^2(\lambda_i) + \eta_a(H)\delta_H(\lambda_i) + \delta_h(\lambda_i)\right) \lesssim H^2 + h, \tag{3.5}$$

where we have used (2.9). Formula (4.7) in [43] implies

$$
\begin{aligned}
\text{dist}\left(u^{i,h}, M_h(\lambda_i)\right) &\lesssim (\lambda_{i,H} - \lambda_i)\,\text{dist}\left(u_0, \hat{M}(\lambda_i)\right) + \delta_H(\lambda_i)^2 \delta_h(\lambda_i) \\
&\lesssim H^2(H^2 + h) + H^2 h \\
&\lesssim H^4 + H^2 h \\
&\lesssim H^4 + h^2,
\end{aligned}
$$

where we have used (2.9), (3.5) and Young's inequality. It completes our proof. $\qquad\square$

Based on the above Lemma, we can establish the superconvergence result for eigenfunctions.

Theorem 3.2 *Suppose that $M(\lambda_i) \subset H_0^1(\Omega) \cap W^{3,\infty}(\Omega)$. Let $(\lambda^{i,h}, u^{i,h})$ be an approximate eigenpair of* (2.2) *obtained by Algorithm* 1 *and let H be properly small. Then there exists $u_i \in M(\lambda_i)$ such that*

$$
\left\| \mathcal{D}^{\frac{1}{2}} G_h u^{i,h} - \mathcal{D}^{\frac{1}{2}} \nabla u_i \right\|_{0,\Omega} \lesssim (H^4 + h^2). \tag{3.6}
$$

Proof Let the eigenfunctions $\{u_{j,h}\}_{j=i}^{i+q-1}$ be an orthonormal basis of $M_h(\lambda_i)$. Note that

$$
\text{dist}\left(u^{i,h}, M_h(\lambda_i)\right) = \left\| u^{i,h} - \sum_{j=i}^{i+q-1} a\left(u^{i,h}, u_{j,h}\right) u_{j,h} \right\|_{a,\Omega}.
$$

Let $\tilde{u}_h = \sum_{j=i}^{i+q-1} a(u^{i,h}, u_{j,h}) u_{j,h}$. According to Theorem 2.4, there exist $\{\tilde{u}_j\}_{j=i}^{i+q-1} \subset M(\lambda_i)$ such that

$$
\|\mathcal{D}^{\frac{1}{2}} G_h u_{j,h} - \mathcal{D}^{\frac{1}{2}} \nabla \tilde{u}_j\|_{0,\Omega} \lesssim h^2. \tag{3.7}
$$

Let $u_i = \sum_{j=i}^{i+q-1} a(u^{i,h}, u_{j,h}) \tilde{u}_j$; then $u_i \in M(\lambda_i)$. Using (3.7), we can derive that

$$
\begin{aligned}
&\left\| \mathcal{D}^{\frac{1}{2}} G_h \tilde{u}^h - \mathcal{D}^{\frac{1}{2}} \nabla u_i \right\|_{0,\Omega} \\
&= \left\| \sum_{j=i}^{i+q-1} a\left(u^{i,h}, u_{j,h}\right) \left(\mathcal{D}^{\frac{1}{2}} G_h u_{j,h} - \mathcal{D}^{\frac{1}{2}} \nabla \tilde{u}_j\right) \right\|_{0,\Omega} \\
&\lesssim \left(\sum_{j=i}^{i+q-1} \left\| \left(\mathcal{D}^{\frac{1}{2}} G_h u_{j,h} - \mathcal{D}^{\frac{1}{2}} \nabla \tilde{u}_j\right) \right\|_{0,\Omega}^2 \right)^{\frac{1}{2}} \lesssim h^2.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
&\left\| \mathcal{D}^{\frac{1}{2}} G_h u^{i,h} - \mathcal{D}^{\frac{1}{2}} \nabla u_i \right\|_{0,\Omega} \\
&\leq \left\| \mathcal{D}^{\frac{1}{2}} G_h \left(u^{i,h} - \tilde{u}_h\right) \right\|_{0,\Omega} + \left\| \mathcal{D}^{\frac{1}{2}} G_h \tilde{u}_h - \mathcal{D}^{\frac{1}{2}} \nabla u_i \right\|_{0,\Omega} \\
&\lesssim \left\| G_h \left(u^{i,h} - \tilde{u}_h\right) \right\|_{0,\Omega} + h^2 \\
&\lesssim \left\| \nabla \left(u^{i,h} - \tilde{u}_h\right) \right\|_{0,\Omega} + h^2
\end{aligned}
$$

$$\lesssim \left\| u^{i,h} - \tilde{u}_h \right\|_{a,\Omega} + h^2$$
$$\lesssim (H^4 + h^2) + h^2$$
$$\lesssim H^4 + h^2;$$

where we use Lemma 3.1 to bound $\|u^{i,h} - \tilde{u}_h\|_{a,\Omega}$. $\square$

The following Lemma is needed in the proof of a superconvergence property of our eigenvalue approximation.

Lemma 3.3 *Suppose that* $M(\lambda_i) \subset H_0^1(\Omega) \cap W^{3,\infty}(\Omega)$. *Let* $(\lambda^{i,h}, u^{i,h})$ *be an approximate eigenpair of* (2.2) *obtained by Algorithm* 1 *and let H be properly small. Then*

$$\left\| \mathcal{D}^{\frac{1}{2}} G_h u^{i,h} - \mathcal{D}^{\frac{1}{2}} \nabla u^{i,h} \right\|_{0,\Omega} \lesssim (H^2 + h). \tag{3.8}$$

Proof Let $\tilde{u}_h$ be defined as in Theorem 3.2. Then we have

$$\left\| \mathcal{D}^{\frac{1}{2}} G_h u^{i,h} - \mathcal{D}^{\frac{1}{2}} \nabla u^{i,h} \right\|_{0,\Omega}$$
$$\leq \left\| \mathcal{D}^{\frac{1}{2}} G_h u^{i,h} - \mathcal{D}^{\frac{1}{2}} G_h \tilde{u}_h \right\|_{0,\Omega} + \left\| \mathcal{D}^{\frac{1}{2}} G_h \tilde{u}_h - \mathcal{D}^{\frac{1}{2}} \nabla \tilde{u}_h \right\|_{0,\Omega} + \left\| \mathcal{D}^{\frac{1}{2}} \nabla \tilde{u}_h - \mathcal{D}^{\frac{1}{2}} \nabla u^{i,h} \right\|_{0,\Omega}$$
$$\lesssim \left\| G_h u^{i,h} - G_h \tilde{u}_h \right\|_{0,\Omega} + \left\| \mathcal{D}^{\frac{1}{2}} G_h \tilde{u}_h - \mathcal{D}^{\frac{1}{2}} \nabla \tilde{u}_h \right\|_{0,\Omega} + \left\| \mathcal{D}^{\frac{1}{2}} \nabla \tilde{u}_h - \mathcal{D}^{\frac{1}{2}} \nabla u^{i,h} \right\|_{0,\Omega}$$
$$\lesssim \left\| \nabla u_{i,h} - \nabla \tilde{u}_h \right\|_{0,\Omega} + \left\| \mathcal{D}^{\frac{1}{2}} G_h \tilde{u}_h - \mathcal{D}^{\frac{1}{2}} \nabla \tilde{u}_h \right\|_{0,\Omega}$$
$$\lesssim \left\| u_{i,h} - \tilde{u}_h \right\|_{a,\Omega} + \left\| \mathcal{D}^{\frac{1}{2}} G_h \tilde{u}_h - \mathcal{D}^{\frac{1}{2}} \nabla \tilde{u}_h \right\|_{0,\Omega}$$
$$\lesssim (H^4 + h^2) + h$$
$$\lesssim (H^2 + h).$$

Here we use the fact that $\| \cdot \|_{a,\Omega}$ and $\| \cdot \|_{1,\Omega}$ are two equivalent norms on $H_0^1(\Omega)$. $\square$

Now we are in a perfect position to prove our main superconvergence result for eigenvalue approximation.

Theorem 3.4 *Suppose that* $M(\lambda_i) \subset H_0^1(\Omega) \cap W^{3,\infty}(\Omega)$. *Let* $(\lambda^{i,h}, u^{i,h})$ *be an approximate eigenpair of* (2.2) *obtained by Algorithm* 1 *and let H be properly small.*

$$|\lambda^{i,h} - \lambda_i| \lesssim H^6 + h^3. \tag{3.9}$$

Proof It follows from (2.12) and (3.3) that

$$\lambda^{i,h} - \lambda_i$$
$$= \frac{a\left(u^{i,h}, u^{i,h}\right)}{\left(u^{i,h}, u^{i,h}\right)} - \frac{\left\| \mathcal{D}^{\frac{1}{2}} \nabla u^{i,h} - \mathcal{D}^{\frac{1}{2}} G_h u^{i,h} \right\|_{0,\Omega}^2}{\left(u^{i,h}, u^{i,h}\right)} - \lambda_i$$
$$= \frac{a\left(u^{i,h} - u_i, u^{i,h} - u_i\right)}{\left(u^{i,h}, u^{i,h}\right)} - \frac{\left\| \mathcal{D}^{\frac{1}{2}} \nabla u^{i,h} - \mathcal{D}^{\frac{1}{2}} G_h u^{i,h} \right\|_{0,\Omega}^2}{\left(u^{i,h}, u^{i,h}\right)} - \frac{\lambda_i\left(u^{i,h} - u_i, u^{i,h} - u_i\right)}{\left(u^{i,h}, u^{i,h}\right)}$$
$$= \frac{\left(\mathcal{D}^{\frac{1}{2}}\left(u^{i,h} - u_i\right), \mathcal{D}^{\frac{1}{2}}\left(u^{i,h} - u_i\right)\right)}{\left(u^{i,h}, u^{i,h}\right)} - \frac{\left\| \mathcal{D}^{\frac{1}{2}} \nabla u^{i,h} - \mathcal{D}^{\frac{1}{2}} G_h u^{i,h} \right\|_{0,\Omega}^2}{\left(u^{i,h}, u^{i,h}\right)}$$

$$+ \frac{\left(c\left(u^{i,h} - u_i\right), u^{i,h} - u_i\right) - \lambda_i \left(u^{i,h} - u_i, u^{i,h} - u_i\right)}{\left(u^{i,h}, u^{i,h}\right)}$$

$$= \frac{\left\| \mathcal{D}^{\frac{1}{2}} \nabla G_h u^{i,h} - \mathcal{D}^{\frac{1}{2}} \nabla u_i \right\|_{0,\Omega}^2}{\left(u^{i,h}, u^{i,h}\right)} + \frac{2\left(\mathcal{D}^{\frac{1}{2}} G_h u^{i,h} - \mathcal{D}^{\frac{1}{2}} \nabla u_i, \mathcal{D}^{\frac{1}{2}} \nabla u^{i,h} - \mathcal{D}^{\frac{1}{2}} G_h u^{i,h}\right)}{\left(u^{i,h}, u^{i,h}\right)}$$

$$+ \frac{\left(c\left(u^{i,h} - u_i\right), u^{i,h} - u_i\right) - \lambda_i \left(u^{i,h} - u_i, u^{i,h} - u_i\right)}{\left(u^{i,h}, u^{i,h}\right)}.$$

From Theorem 4.1 in [43], we know that $\|u^{i,h} - u_i\|_{0,\Omega} \lesssim (H^4 + h^2)$ and hence the last term in the above equation is bounded by $\mathcal{O}((H^4 + h^2)^2)$. Theorem 3.2 implies that the first term is also bounded by $\mathcal{O}((H^4 + h^2)^2)$. Using the Hölder inequality, we obtain

$$\left| \left(\mathcal{D}^{\frac{1}{2}} G_h u^{i,h} - \mathcal{D}^{\frac{1}{2}} \nabla u_i, \mathcal{D}^{\frac{1}{2}} \nabla u^{i,h} - \mathcal{D}^{\frac{1}{2}} G_h u^{i,h} \right) \right|$$

$$\leq \left\| \mathcal{D}^{\frac{1}{2}} G_h u^{i,h} - \mathcal{D}^{\frac{1}{2}} \nabla u_i \right\|_{0,\Omega} \left\| \mathcal{D}^{\frac{1}{2}} \nabla u^{i,h} - \mathcal{D}^{\frac{1}{2}} G_h u^{i,h)} \right\|_{0,\Omega}$$

$$\lesssim \left(H^4 + h^2 \right) \left(H^2 + h \right) \lesssim H^6 + h^3 \tag{3.10}$$

and hence

$$|\lambda^{i,h} - \lambda_i| \lesssim H^6 + h^3.$$

This completes our proof. $\square$

Taking $H = \mathcal{O}(\sqrt{h})$, Theorems 3.2 and 3.4 imply that we can get $\mathcal{O}(h^2)$ superconvergence and $\mathcal{O}(h^3)$ superconvergence for eigenfunction and eigenvalue approximation, respectively.

Remark 3.1 Using the Hölder inequality to estimate (3.10) does not take into account the cancellation in the integral. Similar as [34], numerical experiments show that the actual bound is

$$\left| \left(\mathcal{D}^{\frac{1}{2}} G_h u^{i,h} - \mathcal{D}^{\frac{1}{2}} \nabla u_i, \mathcal{D}^{\frac{1}{2}} \nabla u^{i,h} - \mathcal{D}^{\frac{1}{2}} G_h u^{i,h} \right) \right| \lesssim \left(H^4 + h^2 \right)^2,$$

which indicates that we have "double"-order gain by applying recovery.

Remark 3.2 Algorithm 1 is a combination of the shifted inverse power two-grid method [22,43] and gradient recovery enhancement [34]. It inherits all excellent properties of both methods: low computational cost and superconvergence. We will demonstrate in our numerical tests that Algorithm 1 outperform shifted inverse power two-grid method in [22,43].

Remark 3.3 If we first use classical two-grid methods as in [42] and then apply gradient recovery, we can prove $\|\mathcal{D}^{\frac{1}{2}} G_h u^{i,h} - \mathcal{D}^{\frac{1}{2}} \nabla u_i\|_{0,\Omega} \lesssim (H^2 + h^2)$ and $|\lambda^{i,h} - \lambda_i| \lesssim H^3 + h^3$. It means we can only get optimal convergence rate instead of superconvergent convergence rate when $H = \mathcal{O}(\sqrt{h})$.

3.2 Higher Order Space Based Superconvergent Two-Grid Scheme

Our second scheme can be viewed as a combination of the two-grid scheme proposed by Hu and Cheng [22] or Yang and Bi [43] and the two-space method introduced by Racheva and Andreev [36].

Note that we use linear finite element space S_0^H on coarse grid $\mathcal{T}_H$ and quadratic finite element space $S_0^{h,2}$ on fine grid $\mathcal{T}_h$. Compared with the two-grid scheme [22,43], the main

---

Algorithm 2

1. Solve an eigenvalue problem on a coarse grid $\mathcal{T}_H$: Find $(\lambda_{i,H}, u_{i,H}) \in \mathbb{R} \times S_0^H$ and $\|u_{i,H}\|_a = 1$ satisfying

$$a(u_{i,H}, v_H) = \lambda_{i,H}(u_{i,H}, v_H), \quad \forall v_H \in S_0^H. \tag{3.11}$$

2. Solve a source problem on the fine grid $\mathcal{T}_h$: Find $u^{i,h} \in S_0^{h,2}$ such that

$$a\left(u^{i,h}, v_h\right) - \lambda_{i,H}\left(u^{i,h}, v_h\right) = (u_{i,H}, v_h), \quad \forall v_h \in S_0^{h,2}. \tag{3.12}$$

3. Compute the Rayleigh quotient

$$\lambda^{i,h} = \frac{a\left(u^{i,h}, u^{i,h}\right)}{\left(u^{i,h}, u^{i,h}\right)}. \tag{3.13}$$

---

difference is that Algorithm 2 uses linear element on coarse grid $\mathcal{T}_H$ and quadratic element on fine grid $\mathcal{T}_h$ while the two-grid uses linear element on both coarse grid $\mathcal{T}_H$ and $\mathcal{T}_h$. Compared with the two-space method [36], the main difference is that Algorithm 2 uses a coarse grid $\mathcal{T}_H$ and a fine grid $\mathcal{T}_h$ whereas the two-space method only uses a grid $\mathcal{T}_h$. Algorithm 2 shares the advantages of both methods: low computational cost and high accuracy. Thus, we would expect Algorithm 2 performs much better than both methods.

For Algorithm 2, we have the following Theorem:

Theorem 3.5 *Suppose that $M(\lambda_i) \subset H_0^1(\Omega) \cap H^3(\Omega)$. Let $(\lambda^{i,h}, u^{i,h})$ be an approximate eigenpair of (2.2) by Algorithm 1 and let H be properly small. Then there exists $u_i \in M(\lambda_i)$ such that*

$$\left|u^{i,h} - u_i\right|_{a,\Omega} \lesssim \left(H^4 + h^2\right); \tag{3.14}$$

$$\lambda^{i,h} - \lambda_i \lesssim \left(H^8 + h^4\right). \tag{3.15}$$

Proof By Theorem 4.1 in [43], we have

$$\left|u^{i,h} - u_i\right|_{a,\Omega} \lesssim \eta_a(H)\delta_H^3(\lambda_i) + \delta_h(\lambda_i); \tag{3.16}$$

and

$$\lambda^{i,h} - \lambda_i \lesssim \eta_a^2(H)\delta_H^6(\lambda_i) + \delta_h^2(\lambda_i). \tag{3.17}$$

Since we use linear element on $\mathcal{T}_H$ and quadratic element on $\mathcal{T}_h$, it follows from the interpolation error estimate [8,12] that

$$\eta_a(H) \lesssim H, \quad \delta_H(\lambda_i) \lesssim H, \quad \delta_h(\lambda_i) \lesssim h^2.$$

Substituting the above three estimate into (3.16) and (3.17), we get (3.14) and (3.15). $\square$

Comparing Algorithms 1 with 2, the main difference is that Algorithm 1 solves a source problem on fine grid $\mathcal{T}_h$ using linear element and then performs gradient recovery while Algorithm 2 solves a source problem on fine grid $\mathcal{T}_h$ using quadratic element. Both Algorithms 1 and 2 lead to $\mathcal{O}(h^2)$ superconvergence for eigenfunction approximation and $\mathcal{O}(h^4)$ ultraconvergence for eigenvalue approximation by taking $H = \mathcal{O}(\sqrt{h})$. The message we would like to deliver here is that polynomial preserving recovery plays a similar role as quadratic element, but with much lower computational cost.

Remark 3.4 In order to get higher order convergence, we require higher regularity, such as $M(\lambda_i) \subset H_0^1(\Omega) \cap W^{3,\infty}(\Omega)$ for Algorithm 1 and $M(\lambda_i) \subset H_0^1(\Omega) \cap H^3(\Omega)$ for Algorithm 2, in the proof. However, we can use Algorithms 1 and 2 to improve approximation accuracy even with low regularity.

Algorithm 3 Given a tolerance $\epsilon > 0$ and a parameter $0 \leq \theta < 1$.

1. Generate an initial mesh $\mathcal{T}_{h_0}$.
2. Solve (2.3) on $\mathcal{T}_{h_0}$ to get a discrete eigenpair $(\bar{\lambda}^{h_0}, u^{h_0})$.
3. Set $\ell = 0$.
4. Compute $\eta(u^{h_\ell}, T)$ and $\eta(u^{h_\ell}, \Omega)$, then let

$$\lambda^{h_\ell} = \bar{\lambda}^{h_\ell} - \eta(u^{h_\ell}, \Omega)^2.$$

5. If $\eta(u^{h_\ell}, \Omega)^2 < \epsilon$, stop; else go to 6.
6. Choose a minimal subset of elements $\widehat{\mathcal{T}}_{h_\ell} \subset \mathcal{T}_{h_\ell}$ such that

$$\sum_{T \in \widehat{\mathcal{T}}_{h_\ell}} \eta^2(u_h, T) \geq \theta \eta^2(u_h, \Omega);$$

then refine the elements in $\widehat{\mathcal{T}}_{h_\ell}$ and necessary elements to get a new conforming mesh $\mathcal{T}_{h_{\ell+1}}$.

7. Find $u \in S_0^{h_{\ell+1}}$ such that

$$a(u, v) = \lambda_{h_\ell} \left(u^{h_\ell}, v \right), \quad v \in S_0^{h_{\ell+1}},$$

and set $u^{h_{\ell+1}} = \frac{u}{\|u\|_{0,\Omega}}$. Define

$$\bar{\lambda}^{h_{\ell+1}} = \frac{a \left(u^{h_{\ell+1}}, u^{h_{\ell+1}} \right)}{b \left(u^{h_{\ell+1}}, u^{h_{\ell+1}} \right)}. \tag{4.1}$$

8. Let $\ell = \ell + 1$ and go to 4.

---

4 Multilevel Adaptive Methods

In this section, we incorporate two-grid methods and gradient recovery enhancing technique into the framework of adaptive finite element method and propose two multilevel adaptive methods. Both methods only need to solve an eigenvalue problem on initial mesh and solve an associated boundary value problem on adaptive refined mesh during every iteration.

Let u_h be a finite element solution in S^h and G_h be PPR recovery operator. Define a local a posteriori error estimator on the element T as

$$\eta(u_h, T) = \left\| \mathcal{D}^{\frac{1}{2}} G_h u_h - \mathcal{D}^{\frac{1}{2}} \nabla u_h \right\|_{0,T}, \tag{4.3}$$

and a global error estimator as

$$\eta(u_h, \Omega) = \left(\sum_{T \in \mathcal{T}_h} \eta(u_h, T) \right)^{\frac{1}{2}}. \tag{4.4}$$

Algorithm 4 Given a tolerance $\epsilon > 0$ and a parameter $0 \leq \theta < 1$.

1. Generate an initial mesh $\mathcal{T}_{h_0}$.
2. Solve (2.3) on $\mathcal{T}_{h_0}$ to get a discrete eigenpair $(\bar{\lambda}^{h_0}, u^{h_0})$.
3. Set $\ell = 0$.
4. Compute $\eta(u^{h_\ell}, T)$ and $\eta(u^{h_\ell}, \Omega)$, then let

$$\lambda^{h_\ell} = \bar{\lambda}^{h_\ell} - \eta(u^{h_\ell}, \Omega)^2.$$

5. If $\eta(u^{h_\ell}, \Omega)^2 < \epsilon$, stop; else go to 6.
6. Choose a minimal subset of elements $\widehat{\mathcal{T}}_{h_\ell} \subset \mathcal{T}_{h_\ell}$ such that

$$\sum_{T \in \widehat{\mathcal{T}}_{h_\ell}} \eta^2(u_h, T) \geq \theta \eta^2(u_h, \Omega);$$

then refine the elements in $\widehat{\mathcal{T}}_{h_\ell}$ and necessary elements to get a new conforming mesh $\mathcal{T}_{h_{\ell+1}}$.
7. Find $u \in S_0^{h_{\ell+1}}$ such that

$$a(u, v) - \lambda_{h_\ell}(u, v) = \left(u^{h_\ell}, v\right), \quad v \in S_0^{h_{\ell+1}}, \tag{4.2}$$

and set $u^{h_{\ell+1}} = \frac{u}{\|u\|_{0,\Omega}}$. Define

$$\bar{\lambda}^{h_{\ell+1}} = \frac{a\left(u^{h_{\ell+1}}, u^{h_{\ell+1}}\right)}{b\left(u^{h_{\ell+1}}, u^{h_{\ell+1}}\right)}.$$

8. Let $\ell = \ell + 1$ and go to 4.

Given a tolerance ϵ and a parameter θ, we describe our multilevel adaptive methods in Algorithms 3 and 4. Here we use Dörfler marking strategy [13] in step 6.

Note that the only difference between Algorithms 3 and 4 is that they solve different boundary value problems in step 7. Algorithm 3 solves boundary value problem (4.1) like two-grid scheme in [42] while Algorithm 4 solves boundary value problem (4.2) similar to two-grid scheme in [22,43]. Boundary value problem (4.2) would lead to a near singular linear system. Numerical results of both methods are almost the same as indicated by examples in the next section.

Compared with methods in [24,41], Algorithms 3 and 4 use recovery based a posteriori error estimator. The propose of gradient recovery in the above two algorithms is twofold. The first one is to provide an asymptotically exact a posteriori error estimator. The other is to greatly improve the accuracy of eigenvalue and eigenfunction approximations. Super-convergence result $\mathcal{O}(N^{-1})$ and ultraconvergence $\mathcal{O}(N^{-2})$ are numerically observed for eigenfunction and eigenvalue approximation respectively. However, methods in [24,41] can only numerically give asymptotically optimal results. We want to emphasize that the new algorithms can get superconvergence or ultraconvergence results with no more or even less computational cost compared to the methods proposed in [24,41].

5 Numerical Experiment

In this section, we present several numerical examples to demonstrate the effectiveness and superconconvergence of the proposed algorithms and verify our theoretical results. All algorithms are implemented using finite element package iFEM developed by Chen [10].
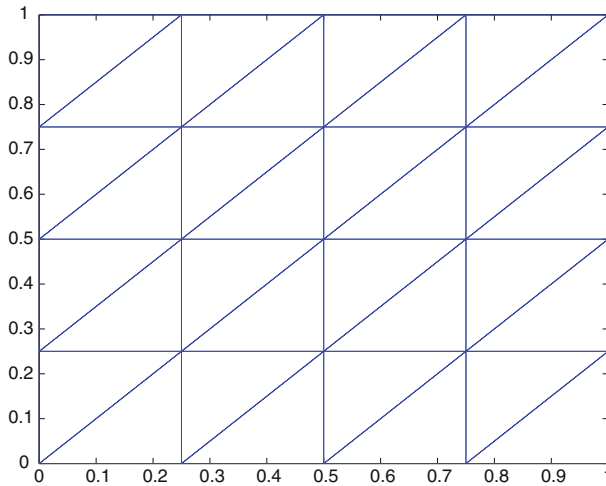
Fig. 1 Uniform mesh for Example 1

Table 1 Eigenpair errors of Algorithm 1 for Example 1 on Uniform Mesh

i	H	h	$\lambda^{i,\mathrm{A1}} - \lambda_i$	Order	$\|G_h u^{i,\mathrm{A1}} - \nabla u_i\|_{0,\Omega}$	Order
1	4	16	$-5.40\mathrm{e}{-03}$	–	$7.059395\mathrm{e}{-02}$	–
1	8	64	$-2.19\mathrm{e}{-05}$	3.97	$4.387700\mathrm{e}{-03}$	2.00
1	16	256	$-8.59\mathrm{e}{-08}$	4.00	$2.734342\mathrm{e}{-04}$	2.00
1	32	1024	$-3.36\mathrm{e}{-10}$	4.00	$1.707544\mathrm{e}{-05}$	2.00
2	4	16	$-3.65\mathrm{e}{-02}$	–	$2.383387\mathrm{e}{-01}$	–
2	8	64	$-1.24\mathrm{e}{-04}$	4.10	$1.619009\mathrm{e}{-02}$	1.94
2	16	256	$-4.40\mathrm{e}{-07}$	4.07	$1.072544\mathrm{e}{-03}$	1.96
2	32	1024	$-1.66\mathrm{e}{-09}$	4.02	$6.819323\mathrm{e}{-05}$	1.99
3	4	16	$-3.63\mathrm{e}{-02}$	–	$4.414434\mathrm{e}{-01}$	–
3	8	64	$-2.19\mathrm{e}{-04}$	3.69	$2.707125\mathrm{e}{-02}$	2.01
3	16	256	$-8.23\mathrm{e}{-07}$	4.03	$1.697971\mathrm{e}{-03}$	2.00
3	32	1024	$-3.15\mathrm{e}{-09}$	4.01	$1.063125\mathrm{e}{-04}$	2.00

The first example is designed to demonstrate superconvergence property of Algorithm 1 and 2 and make some comparison with the two-grid scheme in [22,43]. Let the ith eigenpairs obtained by Algorithms 1 and 2 be denoted by $(\lambda^{i,\mathrm{A1}}, u^{i,\mathrm{A1}})$ and $(\lambda^{i,\mathrm{A2}}, u^{i,\mathrm{A2}})$. Also, let $(\lambda^{i,\mathrm{TG}}, u^{i,\mathrm{TG}})$ be the ith eigenpair produced by the shift inverse based two-grid scheme in [22,43].

The presentation of other examples are to illustrate the effectiveness and superconvergence of Algorithm 3 and 4. In these examples, we focus on the first eigenpair. Let $\bar{\lambda}_{\mathrm{A3}}$ and λ_{A3} be the eigenvalue generated by Algorithm 3 without and with gradient recovery enhancing, respectively. Define $\bar{\lambda}_{\mathrm{A4}}$, λ_{A4}, u_{A3}, and u_{A4} in a similar way.

Table 2 Eigenpair errors of Algorithm 2 for Example 1 on Uniform Mesh

i	H	h	$\lambda^{i,A2} - \lambda_i$	Order	$\|\nabla u^{i,A2} - \nabla u_i\|_{0,\Omega}$	Order
1	4	16	9.32e−04	–	3.344371e−02	–
1	8	64	3.56e−06	4.02	2.076378e−03	2.00
1	16	256	1.41e−08	3.99	1.308168e−04	1.99
1	32	1024	5.52e−11	4.00	8.198527e−06	2.00
2	4	16	5.11e−02	–	2.766129e−01	–
2	8	64	1.95e−04	4.02	1.741376e−02	1.99
2	16	256	8.22e−07	3.95	1.133600e−03	1.97
2	32	1024	3.29e−09	3.98	7.181289e−05	1.99
3	4	16	2.26e−01	–	5.777043e−01	–
3	8	64	5.38e−04	4.36	2.888882e−02	2.16
3	16	256	2.04e−06	4.02	1.789339e−03	2.01
3	32	1024	7.97e−09	4.00	1.119668e−04	2.00

Table 3 Eigenpair errors of shift-inverse Two-grid scheme for Example 1 on Uniform Mesh

i	H	h	$\lambda^{i,TG} - \lambda_i$	Order	$\|\nabla u^{i,TG} - \nabla u_i\|_{0,\Omega}$	Order
1	4	16	1.91e−01	–	4.374696e−01	–
1	8	64	1.19e−02	2.00	1.090666e−01	1.00
1	16	256	7.43e−04	2.00	2.726154e−02	1.00
1	32	1024	4.64e−05	2.00	6.815303e−03	1.00
2	4	16	8.51e−01	–	9.264564e−01	–
2	8	64	5.13e−02	2.03	2.265764e−01	1.02
2	16	256	3.20e−03	2.00	5.653313e−02	1.00
2	32	1024	2.00e−04	2.00	1.413126e−02	1.00
3	4	16	1.43e+00	–	1.212868e+00	–
3	8	64	8.02e−02	2.08	2.834861e−01	1.05
3	16	256	4.98e−03	2.00	7.058802e−02	1.00
3	32	1024	3.11e−04	2.00	1.764238e−02	1.00

Example 1 Consider the following Laplace eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u, \text{ in } \Omega, \\ u = 0, \qquad \text{on } \partial\Omega; \end{cases} \tag{5.1}$$

where $\Omega = (0, 1) \times (0, 1)$. The eigenvalue of (5.1) are $\lambda_{k,l} = (k^2 + l^2)\pi^2$ and the corresponding eigenfunctions are $u_{k,l} = \sin(k\pi)\sin(l\pi)$ with $k, l = 1, 2, \cdots$. It is easy to see the first three eigenvalues are $\lambda_1 = 2\pi^2$ and $\lambda_2 = \lambda_3 = 5\pi^2$.

First, uniform mesh as in Fig. 1 is considered. The fine meshes $\mathcal{T}_h$ are of sizes $h = 2^{-j}$ ($j = 4, 6, 8, 10$) and the corresponding coarse meshes $\mathcal{T}_H$ of size $H = \sqrt{h}$. Table 1 lists the numerical results for Algorithm 1. $\|G_h u^{i,A1} - \nabla u_i\|_{0,\Omega}$ ($i = 1, 2, 3$) superconverges at rate of $\mathcal{O}(h^2)$ which consists with our theoretical analysis. However, $|\lambda^{i,A1} - \lambda_i|$ ($i = 1, 2$,

Table 4 Comparison of three Algorithms for Example 1 on Uniform mesh

| i	H	h	$\lambda^{i,A1} - \lambda_i$	$\lambda^{i,A2} - \lambda_i$	$\lambda^{i,TG} - \lambda_i$
1	1/2	1/16	3.69e−01	4.69e−01	6.11e−01
1	1/4	1/256	6.41e−04	6.50e−04	1.39e−03

Table 5 Eigenpair errors of Algorithm 1 for Example 1 on Delaunay Mesh

i	N_H	N_h	$\lambda^{i,A1} - \lambda_i$	Order	$\|G_h u^{i,A1} - \nabla u_i\|_{0,\Omega}$	Order
1	31	385	−3.56e−03	–	5.338236e−02	–
1	105	5761	−1.06e−05	2.15	2.835582e−03	1.08
1	385	90625	−3.69e−08	2.05	1.686396e−04	1.02
1	1473	1443841	−1.37e−10	2.02	1.049196e−05	1.00
2	31	385	−4.05e−02	–	1.913200e−01	–
2	105	5761	−1.33e−04	2.11	1.078447e−02	1.06
2	385	90625	−4.80e−07	2.04	6.507215e−04	1.02
2	1473	1443841	−1.82e−09	2.01	4.050826e−05	1.00
3	31	385	−4.69e−02	–	2.021704e−01	–
3	105	5761	−1.66e−04	2.09	1.167256e−02	1.05
3	385	90625	−6.12e−07	2.03	7.128723e−04	1.01
3	1473	1443841	−2.33e−09	2.01	4.459081e−05	1.00

Table 6 Eigenpair errors of Algorithm 2 for Example 1 on Delaunay Mesh

i	N_H	N_h	$\lambda^{i,A2} - \lambda_i$	Order	$\|\nabla u^{i,A2} - \nabla u_i\|_{0,\Omega}$	Order
1	31	385	8.49e−05	–	9.258930e−03	–
1	105	5761	3.23e−07	2.06	5.705799e−04	1.03
1	385	90625	1.26e−09	2.01	3.555028e−05	1.01
1	1473	1443841	5.14e−12	1.99	2.220103e−06	1.00
2	31	385	2.63e−03	–	5.673797e−02	–
2	105	5761	7.13e−06	2.18	2.973429e−03	1.09
2	385	90625	2.59e−08	2.04	1.785969e−04	1.02
2	1473	1443841	9.66e−11	2.02	1.104344e−05	1.01
3	31	385	3.55e−03	–	6.475644e−02	–
3	105	5761	7.73e−06	2.27	2.963878e−03	1.14
3	385	90625	2.78e−08	2.04	1.776282e−04	1.02
3	1473	1443841	1.05e−10	2.01	1.100772e−05	1.00

3) ultraconverges at rate of $\mathcal{O}(h^4)$ which is better than the results predicted by Theorem 3.4. In particular, it verifies the statement in Remark 3.1. One important thing we want to point out is that we numerically observe that λ_{A1} obtained by Algorithm 1 approximates the exact eigenvalue from below; see column 3 in Table 1. Similar phenomenon was observed in [15] where they use a local high-order interpolation recovery. We want to remark that lower

Table 7 Eigenpair errors of shift-inverse Two-grid scheme for Example 1 on Delaunay Mesh

i	N_H	N_h	$\lambda^{i,\mathrm{TG}} - \lambda_i$	Order	$\|\nabla u^{i,\mathrm{TG}} - \nabla u_i\|_{0,\Omega}$	Order
1	31	385	8.20e−02	–	2.865548e−01	–
1	105	5761	5.13e−03	1.02	7.159847e−02	0.51
1	385	90625	3.20e−04	1.01	1.789928e−02	0.50
1	1473	1443841	2.00e−05	1.00	4.474820e−03	0.50
2	31	385	4.80e−01	–	6.940436e−01	–
2	105	5761	2.99e−02	1.03	1.730144e−01	0.51
2	385	90625	1.87e−03	1.01	4.324668e−02	0.50
2	1473	1443841	1.17e−04	1.00	1.081155e−02	0.50
3	31	385	5.45e−01	–	7.396147e−01	–
3	105	5761	3.39e−02	1.03	1.842669e−01	0.51
3	385	90625	2.12e−03	1.01	4.606308e−02	0.50
3	1473	1443841	1.33e−04	1.00	1.151577e−02	0.50
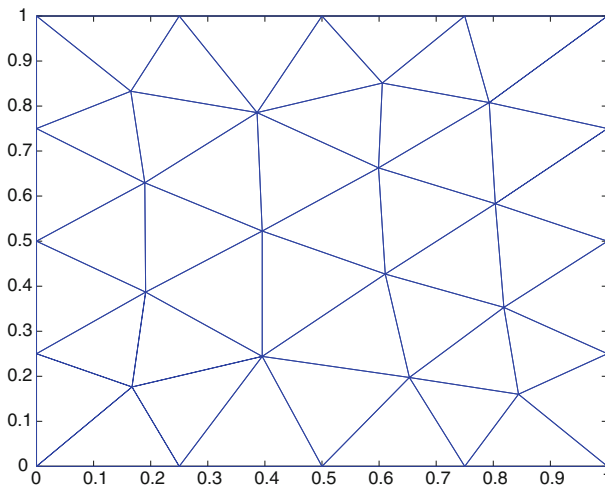
Fig. 2 Delaunay mesh for Example 1

bound of eigenvalue is very important in practice and there are many efforts are made to obtain eigenvalue approximation from below. The readers are referred to [3,26,44,46] for other ways to approximate eigenvalue from below. In Table 2, we report the numerical result of Algorithm 2. As expected, $\mathcal{O}(h^4)$ convergence of eigenvalue approximation and $\mathcal{O}(h^2)$ convergence of eigenfunction approximation are observed which validate our Theorem 3.5. The shift-inverse power method based two-grid scheme in [22,43] is then considered, the result being displayed in Table 3. $\lambda^{i,\mathrm{TG}}$ approximates λ_i (i = 1, 2, 3) at a rate $\mathcal{O}(h^2)$ and $\|u^{i,\mathrm{TG}} - u_i\|_{a,\Omega}$ (i=1, 2, 3) converges at a rate of $\mathcal{O}(h)$.

Comparing Tables 1 and 2 to 3, huge advantages of Algorithms 1 and 2 are demonstrated. For instance, on the fine grid with size $h = 1/1024$ and corresponding coarse grid with size $H = 1/32$, the approximate first eigenvalues produced by Algorithms 1 and 2 are exact up
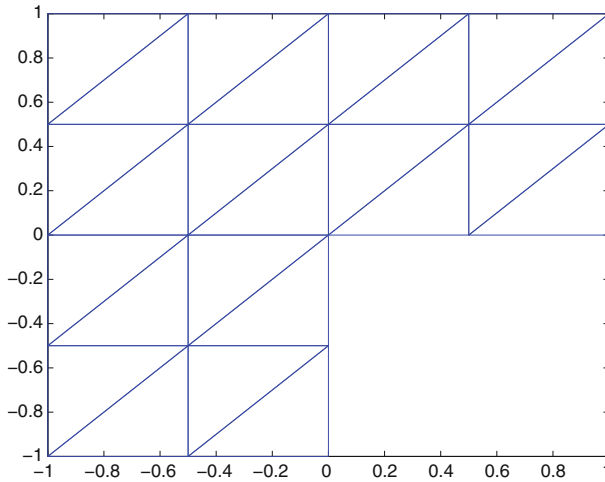
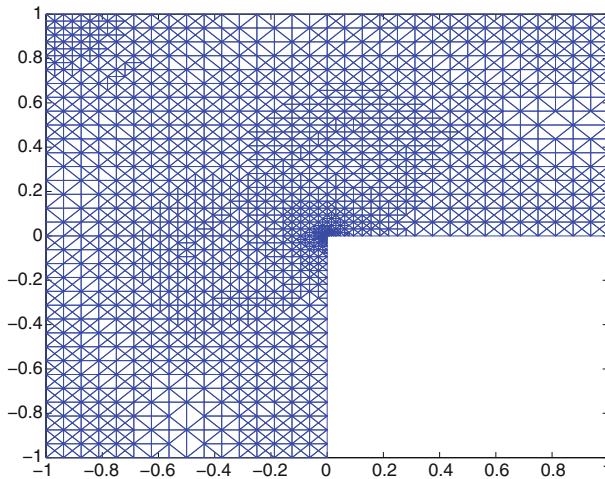**Fig. 3** Initial mesh for Example 2

**Fig. 4** Adaptive mesh for Example 2

to 10 digits while one can only trust the first five digits of the first eigenvalue generated by the two-grid scheme in [22,43].

Then we consider the case $H = \mathcal{O}(\sqrt[4]{h})$ for the first eigenvalue. We use the fine meshes of mesh size $h = 2^{-j}$ with $j = 4, 8$ and corresponding coarse meshes satisfying $H = \sqrt[4]{h}$. The numerical results are showed in Table 4. We can see that the two proposed Algorithms give better approximate eigenvalues. Thus Algorithms 1 and 2 outperform the two-grid scheme even in the case $H = \sqrt[4]{h}$. One interesting thing that we want to mention is that $\lambda^{i,A1}$ approximates λ_i from above in this case, see column 4 in Table 4.

Now, we turn to unstructured meshes. First we generate a coarse mesh $\mathcal{T}_H$ and repeat regular refinement on $\mathcal{T}_H$ until $H = \mathcal{O}(\sqrt{h})$ to get the corresponding fine mesh $\mathcal{T}_h$. The first level coarse mesh as in Fig. 2 is generated by EasyMesh [35] and the other three level coarse mesh as in Fig. 2 are generated by regular refinement. The numerical results are provided in
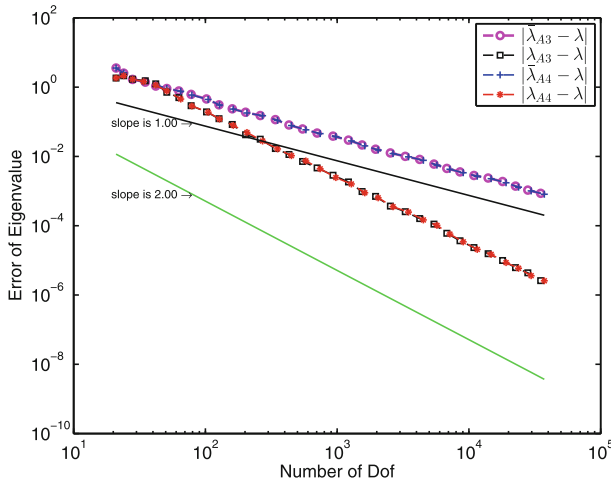
Fig. 5 Eigenvalue approximation error for Example 2
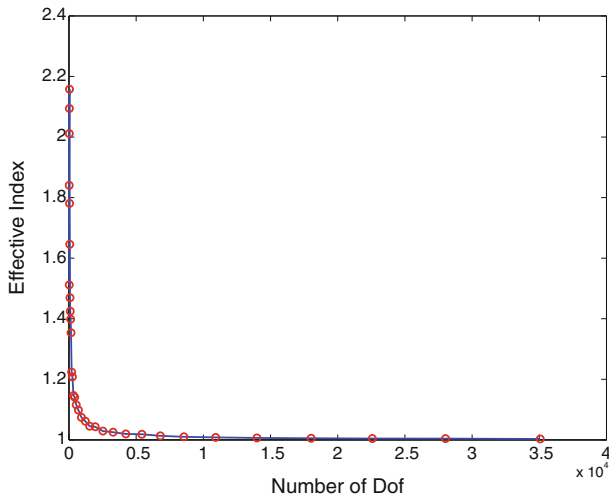
Fig. 6 Effective index of Algorithm 3 for Example 2

Tables 5, 6 and 7. Note that N_H and N_h denote the number of vertices on coarse mesh $\mathcal{T}_H$ and fine mesh $\mathcal{T}_h$, respectively. Concerning the convergence of eigenvalue, Algorithms 1 and 2 ultraconverge at rate $\mathcal{O}(h^4)$ while the two-grid scheme converges at rate $\mathcal{O}(h^2)$. Note that in Tables 5, 6 and 7, $N_H \approx H^{-2}$ and $N_h \approx h^{-2}$. Therefore, convergent rates for H and h "double" the rates for N_H and N_h, respectively. As for eigenfunction, $\|G_h u^{i,\text{A1}} - \nabla u_i\|_{0,\Omega}$ and $\|\nabla u^{i,\text{A2}} - \nabla u_i\|_{0,\Omega}$ are about $\mathcal{O}(h^2)$ while $\|\nabla u^{i,\text{TG}} - \nabla u_i\|_{0,\Omega} \approx \mathcal{O}(h)$ (Fig. 2).

Example 2 In the previous example, the eigenfunctions u are analytic. Here we consider Laplace eigenvalue value problem on the L-shaped domain $\Omega = (-1, 1) \times (-1, 1)/[0, 1) \times (-1, 0]$. The first eigenfunction has a singularity at the origin. To capture this singularity, multilevel adaptive Algorithms 3 and 4 are used with $\theta = 0.4$. Since the first exact eigenvalue
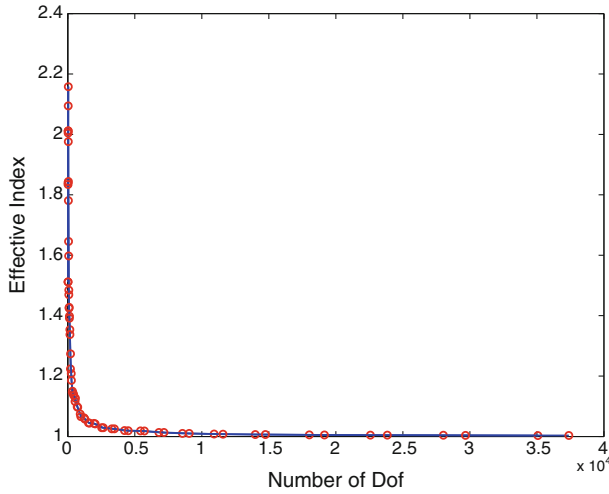
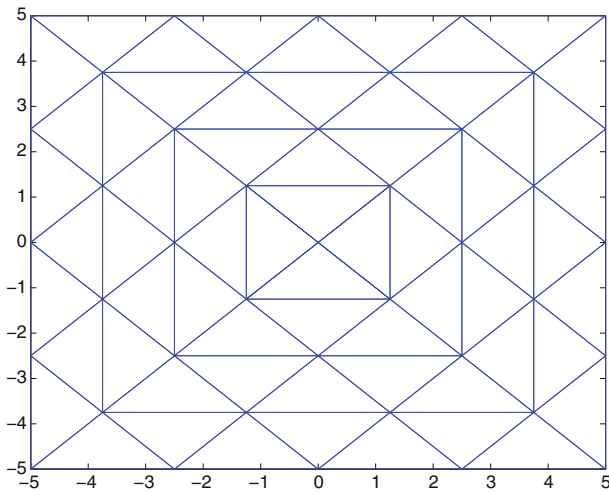**Fig. 7** Effective index of Algorithm 4 for Example 2



Fig. 8 Initial mesh for Example 3

is not available, we choose an approximation $\lambda = 9.6397238440219$ obtained by Betcke and Trefethen in [7], which is correct up to 14 digits.

Figure 3 shows the initial uniform mesh while Fig. 4 is the mesh after 18 adaptive iterations. Figure 5 reports numerical results of the first eigenvalue approximation. It indicates clearly $\bar{\lambda}_{A3}$ and $\bar{\lambda}_{A4}$ approximate λ at a rate of $\mathcal{O}(N^{-1})$ while λ_{A3} and λ_{A4} approximate λ at a rate of $\mathcal{O}(N^{-2})$. The numerical results for Algorithms 3 and 4 are almost the same.

In the context of adaptive finite element method for boundary value problems, the effectivity index κ is used to measure the quality of an error estimator [2,4]. For eigenvalue problem, it is better to consider eigenvalue effectivity index instead of traditional effectivity index in [2,4]. In the article, we consider a similar eigenvalue effective index as in [19]
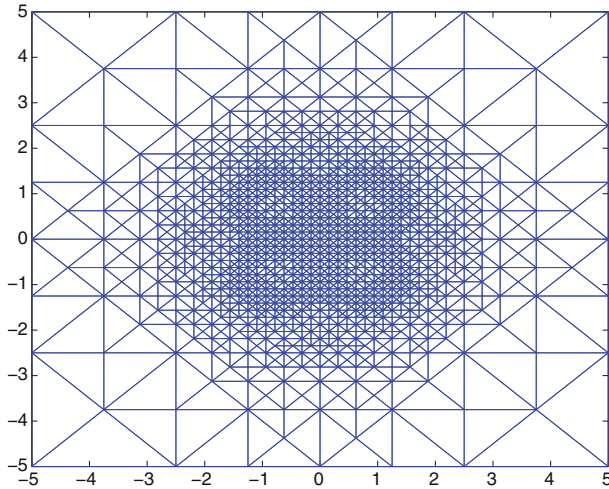
Fig. 9 Adaptive mesh for Example 3
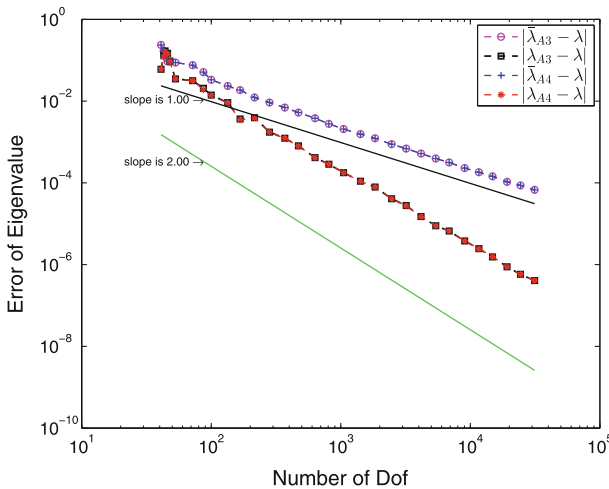
**Fig. 10** Eigenvalue approximatio error for Example 3

$$\kappa = \frac{\left\| \mathcal{D}^{\frac{1}{2}} G_h u_h - \mathcal{D}^{\frac{1}{2}} \nabla u_h \right\|_{0,\Omega}^2}{|\lambda - \lambda_h|}, \tag{5.2}$$

where u_h is either u_{A3} or u_{A4} and λ_h is either λ_{A3} or λ_{A4}. The effectivity index for the two proposed multilevel adaptive algorithms are reported in Figs. 6 and 7. We see that κ converges to 1 quickly after the first few iterations, which indicates that the posteriori error estimator (4.3) or (4.4) is asymptotically exact.

Example 3 Consider the following harmonic oscillator equation [18], which is a simple model in quantum mechanics,
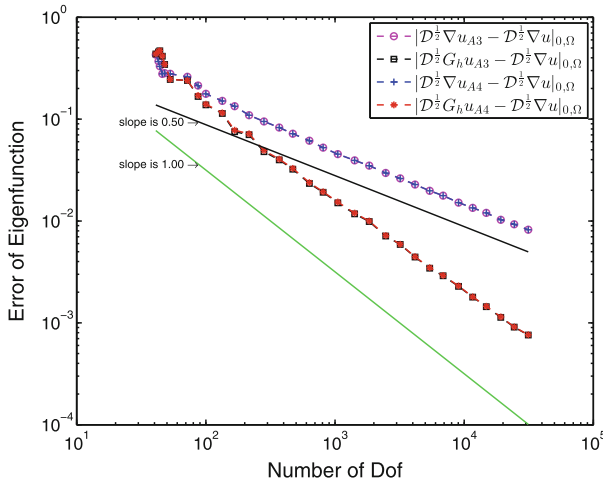
**Fig. 11** Eigenfunction approximation error for Example 3
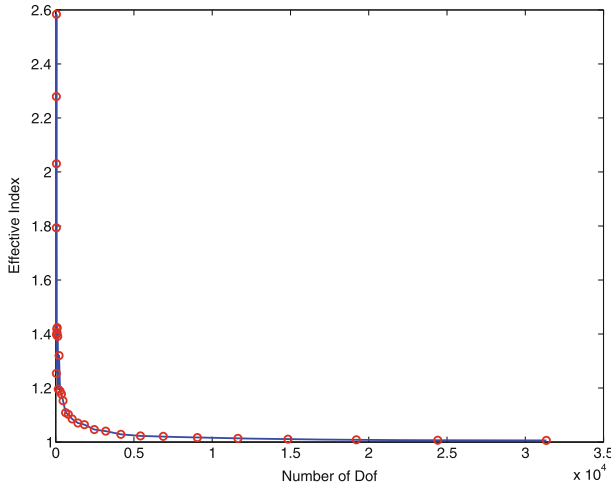
Fig. 12 Effective index of Algorithm 3 for Example 3

$$-\frac{1}{2}\Delta u + \frac{1}{2}|x|^2 u = \lambda u, \quad \text{in } \mathbb{R}^2, \tag{5.3}$$

where $|x| = \sqrt{|x_1|^2 + |x_2|^2}$. The first eigenvalue of (5.3) is $\lambda = 1$ and the corresponding eigenfunction is $u = \gamma e^{-|x|^2/2}$ with any nonzero constant γ.

We solve this eigenvalue problem with $\Omega = (-5, 5) \times (-5, 5)$ and zero boundary condition as in [41]. The initial mesh is shown in Fig. 8 and the adaptive mesh after 20 iterations is displayed in Fig. 9. The parameter θ is chosen as 0.4. Numerical results are presented in Figs. 10 and 11. For eigenvalue approximation, $\mathcal{O}(N^{-1})$ convergence rate is observed for $|\bar{\lambda}_{A3} - \lambda|$ while $\mathcal{O}(N^{-2})$ ultraconvergence rate is observed for $|\lambda_{A3} - \lambda|$. For eigenfunction approximation, $\|\mathcal{D}^{\frac{1}{2}}\nabla u_{A3} - \mathcal{D}^{\frac{1}{2}}\nabla u\|_{0,\Omega} \approx \mathcal{O}(N^{-0.5})$ and $\|\mathcal{D}^{\frac{1}{2}}G_h u_{A3} - \mathcal{D}^{\frac{1}{2}}\nabla u\|_{0,\Omega} \approx \mathcal{O}(N^{-1})$. The numerical result of Algorithm 4 is similar.
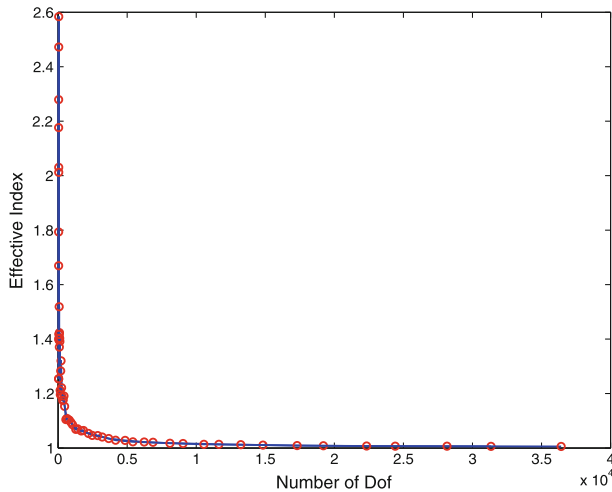
Fig. 13 Effective index of Algorithm 4 for Example 3

Figures 12 and 13 graph the eigenvalue effectivity index for the two proposed multilevel adaptive algorithms. It also indicates that the posteriori error estimator (4.3) or (4.4) is asymptotically exact for problem (5.3).

6 Conclusion

When eigenfunctions are relatively smooth, two-space methods (using higher-order elements in the second stage) is superior to two-grid methods (using the same element at finer grids in the second stage). They have the comparable accuracy. However, at the last stage, the degrees of freedom of the two-space method is much smaller than that of the two-grid method.

For linear element on structured meshes, using gradient recovery at the last stage achieves similar accuracy as the quadratic element on the same mesh. Therefore, with much reduced cost, the gradient recovery is comparable with the two-stage method on the same mesh.

Algorithms 3 and 4 use recovery type error estimators to adapt the mesh, and have two advantages comparing with the residual based adaptive algorithms. (1) Cost effective. In fact, the recovery based error estimator plays two roles: one is to measure the error, and another is to enhance the eigenvalue approximation. (2) Higher accuracy. Indeed, after recovery enhancement, the approximation error is further reduced.

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