HESSIAN RECOVERY FOR FINITE ELEMENT METHODS

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ABSTRACT. In this article, we propose and analyze an effective Hessian recovery strategy for the Lagrangian finite element method of arbitrary order. We prove that the proposed Hessian recovery method preserves polynomials of degree $k + 1$ on general unstructured meshes and superconverges at a rate of $O(h^k)$ on mildly structured meshes. In addition, the method is proved to be ultraconvergent (two orders higher) for the translation invariant finite element space of any order. Numerical examples are presented to support our theoretical results.

1. INTRODUCTION

Post-processing is an important technique in scientific computing, where it is necessary to draw some useful information that have physical meanings such as velocity, flux, stress, etc., from the primary results of the computation. These quantities of interest usually involve derivatives of the primary data. Some popular post-processing techniques include the celebrated Zienkiewicz-Zhu superconvergent patch recovery (SPR) [28], polynomial preserving recovery (PPR) [16,27], and edge based recovery [21], which were proposed to obtain accurate gradients with reasonable cost. Similarly, post-processing for second order derivatives, which are related to physical quantities such as momentum and Hessian, are also desirable. The Hessian matrix is particularly significant in adaptive mesh design, since it can indicate the direction where the function changes the most and guide us to construct anisotropic meshes to cope with the anisotropic properties of the solution of the underlying partial differential equation [3, 5]. It also plays an important role in finite element approximation of second order nonvariational elliptic problems [13], numerical solution of some fully nonlinear equations such as the Monge-Ampère equation [14,18], and designing a nonlocal finite element technique [8].

There have been some works in literature on this subject. In 1998, Lakhany-Whiteman used a simple averaging twice at edge centers of the regular uniform triangular mesh to produce a superconvergent Hessian [12]. Later, some other researchers such as Agouzal et al. [1], Ovall [20] , and Aguilera et al. [2] also studied Hessian recovery. Comparison studies of existing Hessian recovery techniques are found in Vallet et al. [23] and Picasso et al. [22]. However, there is no systematic theory that guarantees convergence of recovered Hessian in general circumstances. Moreover, there are certain technical difficulties in obtaining rigorous convergence

proof for meshes other than the regular pattern triangular mesh. In a very recent work, Kamenski-Huang argued that it is not necessary to have very accurate or even convergent Hessian in order to obtain a good mesh [11].

Our current work is not targeted in the direction of adaptive mesh refinement; instead, our emphasis is to obtain accurate Hessian matrices via recovery techniques. We propose an effective Hessian recovery method and establish a solid theoretical analysis for such recovery methods. Our approach is to apply PPR twice to the primarily computed data. This idea is natural. However, the mathematical theory behind it is nontrivial and quite involved, especially in the ultraconvergence analysis of the recovered Hessian. A direct calculation of the gradient from the linear finite element space has linear convergent rate and the Hessian has no convergence at all. Our Hessian recovery method can achieve second order convergence under some uniform meshes, which is a very surprising result! In particular, the proposed method is the only one of all Hessian recovery methods that is ultraconvergent on the Chevron pattern uniform mesh.

The rest of the paper is organized as follows. We begin in Section 2 with an introduction of some notation and definition of polynomial preserving recovery. Then, in Section 3, we define the Hessian recovery operator and use two examples to show how it relates to the finite difference operator. Also, in this section, we analyze the consistence of the Hessian recovery operator by proving its polynomial preserving property. In Section 4, we prove superconvergence of our Hessian recovery operator on mildly unstructured mesh and ultraconvergence on translation invariant mesh. Section 5 is devoted to numerical comparison of the proposed Hessian recovery method with some popular Hessian recovery methods in the literature and illustration of our theoretical results. Finally, some conclusions are drawn in Section 6.

Throughout this article, the letter C or c, with or without subscript, denotes a generic constant which is independent of h and may not be the same at each occurrence. For convenience, we denote $x \leq Cy$ by $x \lesssim y$.

2. Preliminaries

In this section, we first introduce some notation and then briefly describe the polynomial preserving recovery (PPR) operator [16, 27], which is a basis of our Hessian recovery method.

2.1. **Notation.** Let Ω be a bounded polygonal domain with Lipschitz boundary $\partial\Omega$ in $\mathbb{R}^2$. Throughout this article, the standard notation for Sobolev spaces and their associate norms are adopted as in [4,6]. A 2-index α is a pair of nonnegative integers α_i, $i = 1, 2$. The length of α is given by

$$|\alpha| = \sum_{i=1}^{2} \alpha_i.$$

We adopt the same notation for derivatives as in the textbook by Evans [7]. We usually write u_x (or u_y) for weak derivative $\frac{\partial u}{\partial x}$ (or $\frac{\partial u}{\partial y}$). Similarly, $\frac{\partial^2 u}{\partial x \partial y} = u_{xy}$, $\frac{\partial^3 u}{\partial x^2 \partial y} = u_{xxy}$, etc. Given a 2-index α, define

$$D^\alpha u := \frac{\partial^{|\alpha|} u}{\partial x^{\alpha_1} \partial y^{\alpha_2}}.$$

Given any nonnegative integer k, $D^k u$ is the tensor of all partial derivatives of order k. The Hessian operator H is denoted by

$$(2.1) \qquad H = \begin{pmatrix} \frac{\partial^2}{\partial x^2} & \frac{\partial^2}{\partial x \partial y} \\ \frac{\partial^2}{\partial y \partial x} & \frac{\partial^2}{\partial y^2} \end{pmatrix}.$$

For a subdomain $\mathcal{A}$ of Ω, let $\mathbb{P}_m(\mathcal{A})$ be the space of polynomials of degree less than or equal to m over $\mathcal{A}$ and let n_m be the dimension of $\mathbb{P}_m(\mathcal{A})$ with $n_m = \frac{1}{2}(m+1)(m+2)$. $W_p^k(\mathcal{A})$ denotes the Sobolev space with norm $\|\cdot\|_{k,p,\mathcal{A}}$ and seminorm $|\cdot|_{k,p,\mathcal{A}}$. When $p = 2$, we denote simply $H^k(\mathcal{A}) = W_2^k(\mathcal{A})$ and the subscript p is omitted.

For any $0 < h < \frac{1}{2}$, let $\mathcal{T}_h$ be a shape regular triangulation of $\bar{\Omega}$ with mesh size at most h, i.e.,

$$\bar{\Omega} = \bigcup_{K \in \mathcal{T}_h} K,$$

where K is a triangle. For any $k \in \mathbb{N}$, define the continuous finite element space S_h of order k as

$$S_h = \{v \in C(\bar{\Omega}) : v|_K \in \mathbb{P}_k(K), \quad \forall K \in \mathcal{T}_h\} \subset H^1(\Omega).$$

Let $\mathcal{N}_h$ denote the set of mesh nodes. The standard Lagrange basis of S_h is denoted by $\{\phi_z : z \in \mathcal{N}_h\}$ with $\phi_z(z') = \delta_{zz'}$ for all $z, z' \in \mathcal{N}_h$. For any $v \in H^1(\Omega) \cap C(\Omega)$, let v_I be the interpolation of v in S_h, i.e., $v_I = \sum_{z \in \mathcal{N}_h} v(z)\phi_z$.

For any vertex z and $n \in \mathbb{Z}^+$, let $\mathcal{L}(z, n)$ denote the union of mesh elements in the first n layers around z, i.e.,

$$(2.2) \qquad \mathcal{L}(z, n) := \bigcup \{\tau : \tau \in \mathcal{T}_h, \tau \cap \mathcal{L}(z, n-1) \neq \emptyset\},$$

where $\mathcal{L}(z, 0) := \{z\}$.

For $\mathcal{A} \subset \Omega$, let $S_h(\mathcal{A})$ denote the restrictions of functions in S_h to $\mathcal{A}$ and let $S_h^{\mathrm{comp}}(\mathcal{A})$ denote the set of those functions in $S_h(\mathcal{A})$ with compact support in the interior of $\mathcal{A}$ [24]. Let $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega_2 \subset\subset \Omega$ be separated by $d \geq c_o h$ and ℓ be a direction, i.e., a unit vector in $\mathbb{R}^2$. Let τ be a parameter, which will typically be a multiple of h. Let T_τ^ℓ denote translation by τ in the direction ℓ, i.e.,

$$(2.3) \qquad T_\tau^\ell v(x) = v(x + \tau \ell),$$

and for an integer ν,

$$(2.4) \qquad T_{\nu\tau}^\ell v(x) = v(x + \nu\tau\ell).$$

Following the definition of [24], the finite element space S_h is called translation invariant by τ in the direction ℓ if

$$(2.5) \qquad T_{\nu\tau}^\ell v \in S_h^{\mathrm{comp}}(\Omega), \quad \forall v \in S_h^{\mathrm{comp}}(\Omega_1),$$

for some integer ν with $|\nu| < M$. Equivalently, $\mathcal{T}_h$ is called a translation invariant mesh. To clarify the matter, we consider five popular triangular mesh patterns: Regular, Chevron, Criss-cross, Union-Jack, and Equilateral patterns, as shown in Figure 1.

We see that:

1) Regular pattern is translation invariant by h in directions $(1, 0)$ and $(0, 1)$, by $2\sqrt{2}h$ in directions $(\pm\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, and by $\sqrt{5}h$ in directions $(\frac{2\sqrt{5}}{5}, \pm\frac{\sqrt{5}}{5})$ and $(\pm\frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5})$, etc.
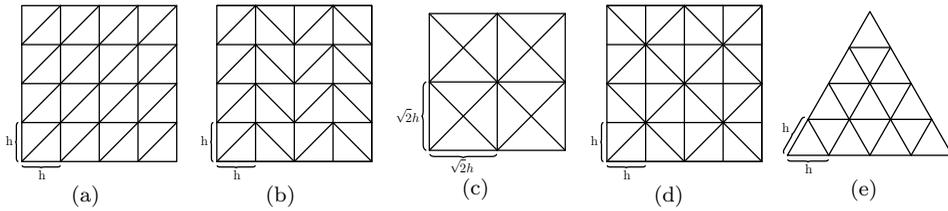
FIGURE 1. Five types of uniform meshes: (a) Regular pattern; (b) Chevron pattern; (c) Criss-cross pattern; (d) Union-Jack pattern; (e) Equilateral pattern

2) Chevron pattern is translation invariant by h in the direction $(0,1)$, by $2h$ in the direction $(1,0)$, and by $2\sqrt{2}h$ in directions $(\pm\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, and by $\sqrt{5}h$ in directions $(\pm\frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5})$, etc.

3) Criss-cross pattern is translation invariant by $\sqrt{2}h$ in directions $(1,0)$ and $(0,1)$, and by $2h$ in directions $(\pm\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, etc.

4) Union-Jack pattern is translation invariant by $2h$ in directions $(1,0)$ and $(0,1)$, and by $2\sqrt{2}h$ in directions $(\pm\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, etc.

5) Equilateral pattern is translation invariant by h in directions $(1,0)$ and $(\pm\frac{1}{2}, \frac{\sqrt{3}}{2})$, and by $\sqrt{3}h$ in directions $(0,1)$ and $(\frac{\sqrt{3}}{2}, \pm\frac{1}{2})$, etc.

2.2. **Polynomial preserving recovery.** Let us introduce $G_h : S_h \to S_h \times S_h$ the PPR gradient recovery operator. For any function $u_h \in S_h$, $G_h u_h$ is a function in $S_h \times S_h$ which is uniquely determined by its values at nodes. If the values $\{(G_h u_h)(z) : z \in \mathcal{N}_h\}$ are well defined, then define $G_h u_h$ on the whole domain by

$$G_h u_h := \sum_{z \in \mathcal{N}_h} (G_h u_h)(z)\phi_z.$$

When z is a vertex, let $\mathcal{K}_z$ be a patch of elements around z. Select all nodes in $\mathcal{N}_h \cap \mathcal{K}_z$ as sampling points and fit a polynomial $p_z \in \mathbb{P}_{k+1}(\mathcal{K}_z)$ in the least squares sense at those sampling points, i.e.,

$$(2.6) \qquad p_z = \arg \min_{p \in \mathbb{P}_{k+1}(\mathcal{K}_z)} \sum_{\tilde{z} \in \mathcal{N}_h \cap \mathcal{K}_z} (u_h - p)^2(\tilde{z}).$$

Then the recovered gradient at z is defined as

$$(G_h u_h)(z) = \nabla p_z(z).$$

For the linear element, all nodes in $\mathcal{N}_h$ are vertices and hence $G_h u_h$ is well defined. However, $\mathcal{N}_h$ may contain edge nodes or interior nodes for higher order elements. If z is an edge node which lies on an edge between two vertices z_1 and z_2, we define

$$(G_h u_h)(z) = \beta \nabla p_{z_1}(z) + (1 - \beta)\nabla p_{z_2}(z),$$

where β is determined by the ratio of distances of z to z_1 and z_2. If z is an interior node which lies in a triangle formed by three vertices z_1, z_2, and z_3, we define

$$(G_h u_h)(z) = \sum_{j=1}^{3} \beta_j \nabla p_{z_j}(z),$$

where β_j is the barycentric coordinate of z.

To complete the definition of PPR, we need to define $\mathcal{K}_z$. If z is an interior vertex, $\mathcal{K}_z$ is defined as the smallest $\mathcal{L}(z, n)$ that guarantees the uniqueness of p_z in (2.6) [16, 27]. In the case that z is a boundary vertex, let n_0 be the smallest positive integer such that $\mathcal{L}(z, n_0)$ has at least one interior mesh vertex. Then, we define

$$\mathcal{K}_z = \mathcal{L}(z, n_0) \cup \{\mathcal{K}_{\tilde{z}} : \tilde{z} \in \mathcal{L}(z, n_0) \text{ and } \tilde{z} \text{ an interior vertex}\}.$$

Remark 2.1. In order to avoid numerical instability, a discrete least squares fitting process is carried out on a reference patch ω_z.

3. Hessian recovery method

Given $u \in S_h$, let $G_h u \in S_h \times S_h$ be the recovered gradient using PPR as defined in the previous section. We rewrite $G_h u$ as

$$(3.1) \qquad G_h u = \begin{pmatrix} G_h^x u \\ G_h^y u \end{pmatrix}.$$

In order to recover the Hessian matrix of u, we apply gradient recovery operator G_h to $G_h^x u$ and $G_h^y u$ one more time, respectively, and define the Hessian recovery operator H_h as follows:

$$(3.2) \qquad H_h u = \begin{pmatrix} G_h(G_h^x u), & G_h(G_h^y u) \end{pmatrix} = \begin{pmatrix} G_h^x(G_h^x u) & G_h^x(G_h^y u) \\ G_h^y(G_h^x u) & G_h^y(G_h^y u) \end{pmatrix}.$$

Just as PPR, we obtain $H_h : S_h \to S_h^2 \times S_h^2$ on the whole domain Ω by interpolation after determining values of $H_h u$ at all nodes in $\mathcal{N}_h$.

Remark 3.1. For the Hessian recovery operator H_h defined in (3.2), we shall prove that H_h is symmetric when the mesh is translation invariant and all sampling points are symmetric with respect to the recovered point. In the general case, H_h may not be symmetric. But we can overcome the asymmetry by symmetrizing the recovered Hessian matrix, i.e.,

$$H_h \leftarrow \frac{(H_h + H_h^T)}{2}.$$

This symmetrization process is easily implemented in practical and it certainly does not compromise the quality of approximation. Since the recovered Hessian converges to the actual Hessian which is symmetric, the skew-symmetric part of H_h should be relatively small. For simplicity in theoretical analysis, we still keep the definition (3.2).

Remark 3.2. The two gradient recovery operators in definition (3.2) of H_h can be different. Actually, we can define the Hessian recovery operator H_h as

$$H_h u = \begin{pmatrix} \tilde{G}_h(G_h^x u), & \tilde{G}_h(G_h^y u) \end{pmatrix}.$$

By choosing G_h and $\tilde{G}_h$ as PPR or SPR operators, we obtain four different Hessian recovery operators, i.e., PPR-PPR, PPR-SPR, SPR-PPR, and SPR-SPR. However, numerical tests have shown that PPR-PPR is the best one. Indeed, PPR-PPR is the only one that ultraconverges at all of the five different uniform meshes.

3.1. **Illustration.** To give the readers some intuition, we shall discuss two examples in detail. For the sake of simplicity, only linear elements on uniform meshes will be considered. In practice, the method can be applied to arbitrary meshes and higher order elements.

Example 1. Consider the regular pattern uniform mesh as in Figure 2. We want to recover the Hessian matrix at z_0. As deduced in [27], the recovered gradient at z_0 is given by

$$(G_h u)(z_0) = \frac{1}{6h}\left(\binom{2}{1}u_1 + \binom{1}{2}u_2 + \binom{-1}{1}u_3 + \binom{-2}{-1}u_4 + \binom{-1}{-2}u_5 + \binom{1}{-1}u_6\right).$$

Here $u_i = u(z_i)$ $(i = 0, 1, \ldots, 18)$ represents the function value of u at node z_i. Thus, according to the definition (3.2) of the Hessian recovery operator H_h, we have

$$(3.3) \qquad \binom{H_h^{xx}u}{H_h^{xy}u}(z_0) = \frac{1}{6h}\left(2(G_h u)(z_1) + (G_h u)(z_2) - (G_h u)(z_3)\right.$$
$$\left. -2(G_h u)(z_4) - (G_h u)(z_5) + (G_h u)(z_6)\right)$$

and

$$(3.4) \qquad \binom{H_h^{yx}u}{H_h^{yy}u}(z_0) = \frac{1}{6h}\left((G_h u)(z_1) + 2(G_h u)(z_2) + (G_h u)(z_3)\right.$$
$$\left. -(G_h u)(z_4) - 2(G_h u)(z_5) - (G_h u)(z_6)\right),$$

where

$$(G_h u)(z_1) = \frac{1}{6h}\left(\binom{2}{1}u_7 + \binom{1}{2}u_8 + \binom{-1}{1}u_2 + \binom{-2}{-1}u_0 + \binom{-1}{-2}u_{18} + \binom{1}{-1}u_6\right)$$

and $(G_h u)(z_2), \ldots, (G_h u)(z_6)$ follow a similar pattern. Direct calculation reveals that

$$(H_h^{xx}u)(z_0) = \frac{1}{36h^2}(-12u_0 + 2u_1 - 4u_2 - 4u_3 + 2u_4 - 4u_5 - 4u_6 + 4u_7 + 4u_8 + u_9$$
$$- 2u_{10} + u_{11} + 4u_{12} + 4u_{13} + 4u_{14} + u_{15} - 2u_{16} + u_{17} + 4u_{18}),$$
$$(H_h^{xy}u)(z_0) = \frac{1}{36h^2}(6u_0 - u_1 + 5u_2 - u_3 - u_4 + 5u_5 - u_6 - 2u_7 + u_8 + u_9$$
$$+ u_{10} - 2u_{11} - 5u_{12} - 2u_{13} + u_{14} + u_{15} + u_{16} - 2u_{17} - 5u_{18}),$$
$$(H_h^{yx}u)(z_0) = \frac{1}{36h^2}(6u_0 - u_1 + 5u_2 - u_3 - u_4 + 5u_5 - u_6 - 2u_7 + u_8 + u_9$$
$$+ u_{10} - 2u_{11} - 5u_{12} - 2u_{13} + u_{14} + u_{15} + u_{16} - 2u_{17} - 5u_{18}),$$
$$(H_h^{yy}u)(z_0) = \frac{1}{36h^2}(-12u_0 - 4u_1 - 4u_2 + 2u_3 - 4u_4 - 4u_5 + 2u_6 + u_7 - 2u_8 + u_9$$
$$+ 4u_{10} + 4u_{11} + 4u_{12} + u_{13} - 2u_{14} + u_{15} + 4u_{16} + 4u_{17} + 4u_{18}).$$

It is observed that $(H_h^{xy}u)(z_0) = (H_h^{yx}u)(z_0)$, which means the recovered Hessian matrix is symmetric, a property of the exact Hessian we would like to maintain.
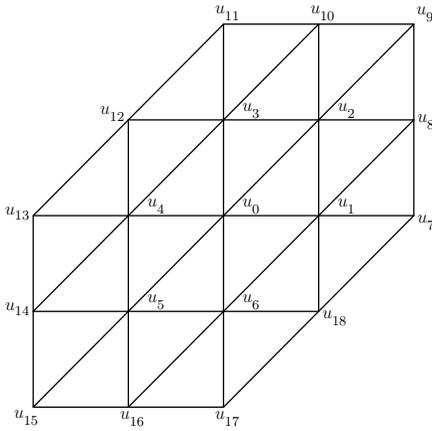
FIGURE 2. Regular pattern
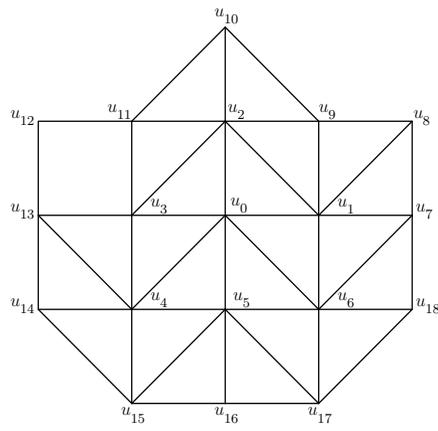
FIGURE 3. Chevron pattern

Using Taylor expansion, we can show that

$$(H_h^{xx}u)(z_0) = u_{xx}(z_0) + \frac{h^2}{3}(u_{xxxx}(z_0) + u_{xxxy}(z_0) + u_{xxyy}(z_0)) + O(h^4),$$

$$(H_h^{xy}u)(z_0) = u_{xy}(z_0) + \frac{h^2}{3}(u_{xxxy}(z_0) + u_{xxyy}(z_0) + u_{xyyy}(z_0)) + O(h^4),$$

$$(H_h^{yx}u)(z_0) = u_{yx}(z_0) + \frac{h^2}{3}(u_{xxxy}(z_0) + u_{xxyy}(z_0) + u_{xyyy}(z_0)) + O(h^4),$$

$$(H_h^{yy}u)(z_0) = u_{yy}(z_0) + \frac{h^2}{3}(u_{xxyy}(z_0) + u_{xyyy}(z_0) + u_{yyyy}(z_0)) + O(h^4),$$

which imply that $H_h u$ provides a second order approximation of Hu at z_0.

Example 2. Consider the Chevron pattern uniform mesh as shown in Figure 3. Repeating the procedure as in Example 1, we derive the recovered Hessian matrix at z_0 as

$$(H_h^{xx}u)(z_0) = \frac{1}{144h^2}(-72u_0 + 36u_{13} + 36u_7),$$

$$\begin{aligned}(H_h^{xy}u)(z_0) = \frac{1}{144h^2}(&-12u_1 + 12u_3 + 24u_4 - 24u_6 + 6u_7\\ &+ 36u_9 - 36u_{11} - 6u_{13} + 6u_{14} - 6u_{18}),\end{aligned}$$

$$\begin{aligned}(H_h^{yx}u)(z_0) = \frac{1}{144h^2}(&12u_1 - 12u_3 + 36u_4 - 36u_6 - 6u_7\\ &+ 6u_8 + 24u_9 - 24u_{11} - 6u_{12} + 6u_{13}),\end{aligned}$$

$$\begin{aligned}(H_h^{yy}u)(z_0) = \frac{1}{144h^2}(&-48u_0 - 10u_1 - 22u_2 - 10u_3 - 10u_4 + 18u_5\\ &- 10u_6 - 2u_7 + u_8 + 10u_9 + 36u_{10} + 10u_{11} + u_{12}\\ &- 2u_{13} + u_{14} + 10u_{15} + 16u_{16} + 10u_{17} + u_{18}).\end{aligned}$$

In addition, we have the following Taylor expansion:

$$(H_h^{xx}u)(z_0) = u_{xx}(z_0) + \frac{h^2}{3}u_{xxxx}(z_0) + \frac{2h^4}{45}u_{xxxxxx}(z_0) + O(h^5),$$

$$(H_h^{xy}u)(z_0) = u_{xy}(z_0) + \frac{h^2}{12}(3u_{xxxy}(z_0) + 2u_{xyyy}(z_0)) - \frac{h^3}{24}u_{xxxyy}(z_0) + O(h^4),$$

$$(H_h^{yx}u)(z_0) = u_{yx}(z_0) + \frac{h^2}{12}(3u_{xxxy}(z_0) + 2u_{xyyy}(z_0)) + \frac{h^3}{24}u_{xxxyy}(z_0) + O(h^4),$$

$$(H_h^{yy}u)(z_0) = u_{yy}(z_0) + \frac{h^2}{6}(u_{xxyy}(z_0) + 2u_{yyyy}(z_0)) - \frac{5h^3}{72}u_{xxyyy}(z_0) + O(h^4).$$

We conclude that $H_h u$ is a second order approximation to the Hessian matrix. It is worth mentioning that, though $H_h^{xy} \neq H_h^{yx}$ for the Chevron pattern uniform mesh, they are both second order finite difference schemes at z_0.

Remark 3.3. PPR-PPR is the only one among the four Hessian recovery methods mentioned in Remark 3.2 that provides second order approximation for the Chevron pattern uniform mesh.

Both Examples 1 and 2 indicate that for the linear element the PPR-PPR approach is equivalent to a finite difference scheme of second order accuracy at vertex z_0.

Remark 3.4. In a general sense, the recovery operator can be viewed as a finite difference operator on unstructured meshes. The practical usage of recovery operator is not only to obtain a better approximation and provide an asymptotically exact a posteriori error estimator, but also to design some new numerical solvers for PDEs. It provides a systematic way to construct finite difference schemes on general unstructured meshes. Actually, the Hessian recovery operator defined in (3.2) can be used to construct finite difference schemes for second order differential operators on unstructured meshes. We have made some progress in this direction and will report the results in a separate paper.

3.2. Polynomial preserving property.

As we observed in previous subsection, H_h can be viewed as a finite difference scheme on unstructured meshes. For finite difference schemes, one of the most important properties is consistency. In this subsection, we shall prove the polynomial preserving property of the Hessian recovery operator H_h which leads to consistency.

For an arbitrary unstructured mesh, we can prove the following polynomial preserving property.

Theorem 3.5. *The Hessian recovery operator H_h preserves polynomials of degree $k + 1$ for an arbitrary mesh.*

Proof. Suppose u is a polynomial of $k + 1$ on $\mathcal{K}_z$, i.e., $u \in \mathbb{P}(\mathcal{K}_z)$. According to Theorem 2.1 in [27], G_h preserves polynomials of degree $k + 1$. Then it follows that $G_h u = \nabla u$ which is a polynomial of degree k. Therefore, we have

$$(3.5) \qquad H_h u = (G_h(G_h^x u), G_h(G_h^y u)) = (G_h \frac{\partial u}{\partial x}, G_h \frac{\partial u}{\partial x}) = (\nabla \frac{\partial u}{\partial x}, \nabla \frac{\partial u}{\partial x}) = H u.$$

It means that H_h preserves polynomials of degree $k + 1$ which completes our proof. $\square$

If the mesh $\mathcal{T}_h$ is translation invariant, we have the following improved results.

Theorem 3.6. *If z is a node of a translation invariant mesh and a mesh symmetric center of the involved nodes, then H_h preserves polynomials of degree $k + 2$ for odd k, and of degree $k + 3$ for even k. In addition, H_h is symmetric.*

Proof. Since G_h is exact for polynomial of degree $k + 1$, it follows that

$$(3.6) \quad G_h^x u = D_x u + h^{k+1} \boldsymbol{a^x} \cdot D^{k+2} u + h^{k+2} \boldsymbol{b^x} \cdot D^{k+3} u + h^{k+3} \boldsymbol{c^x} \cdot D^{k+4} u + \cdots,$$

$$(3.7) \quad G_h^y u = D_y u + h^{k+1} \boldsymbol{a^y} \cdot D^{k+2} u + h^{k+2} \boldsymbol{b^y} \cdot D^{k+3} u + h^{k+3} \boldsymbol{c^y} \cdot D^{k+4} u + \cdots.$$

Notice that coefficients $\boldsymbol{a^x}$, $\boldsymbol{a^y}$, $\boldsymbol{b^x}$, $\boldsymbol{b^y}, \ldots$ depend only on the coordinates of nodes, since we recover gradient at nodes only. Thus for translation invariant meshes, $\boldsymbol{a^x}$, $\boldsymbol{a^y}$, $\boldsymbol{b^x}$, $\boldsymbol{b^y}, \ldots$ are constants. In addition, due to symmetry, it makes no difference if we perform G_h^x or G_h^y first. Hence,

$$(3.8)$$
$$\begin{aligned}
&(H_h^{xy} u)(z) = (G_h^y(G_h^x u))(z) \\
&= G_h^y [D_x u(z) + h^{k+1} \boldsymbol{a^x} \cdot D^{k+2} u(z) + h^{k+2} \boldsymbol{b^x} \cdot D^{k+3} u(z) + \cdots] \\
&= (G_h^y(D_x u))(z) + h^{k+1}(\boldsymbol{a^x} \cdot G_h^y(D^{k+2} u))(z) + h^{k+2}(\boldsymbol{b^x} \cdot G_h^y(D^{k+3} u))(z) + \cdots \\
&= (D_y D_x u)(z) + h^{k+1}(\boldsymbol{a^y} \cdot D^{k+2} D_x u)(z) + h^{k+2}(\boldsymbol{b^y} \cdot D^{k+3} D_x u)(z) \\
&\quad + h^{k+1}(\boldsymbol{a^x} \cdot D_y(D^{k+2} u))(z) + h^{k+2}(\boldsymbol{b^x} \cdot D_y(D^{k+3} u))(z) + O(h^{k+3}) \\
&= (D_y D_x u)(z) + h^{k+1}[\boldsymbol{a^y} \cdot D^{k+2} D_x u + \boldsymbol{a^x} \cdot D_y(D^{k+2} u)](z) \\
&\quad + h^{k+2}[\boldsymbol{b^y} \cdot D^{k+3} D_x u + \boldsymbol{b^x} \cdot D_y(D^{k+3} u)](z) + O(h^{k+3}).
\end{aligned}$$

Notice that (3.8) is valid only at nodal points. Similarly,

$$(3.9) \quad \begin{aligned} (H_h^{yx} u)(z) &= (D_x D_y u)(z) + h^{k+1}[\boldsymbol{a^x} \cdot D^{k+2} D_y u + \boldsymbol{a^y} \cdot D_x(D^{k+2} u)](z) \\ &\quad + h^{k+2}[\boldsymbol{b^x} \cdot D^{k+3} D_y u + \boldsymbol{b^y} \cdot D_x(D^{k+3} u)](z) + O(h^{k+3}), \end{aligned}$$

$$(3.10) \quad \begin{aligned} (H_h^{xx} u)(z) &= (D_x D_x u)(z) + h^{k+1}[\boldsymbol{a^x} \cdot D^{k+2} D_x u + \boldsymbol{a^x} \cdot D_x(D^{k+2} u)](z) \\ &\quad + h^{k+2}[\boldsymbol{b^x} \cdot D^{k+3} D_x u + \boldsymbol{b^x} \cdot D_x(D^{k+3} u)](z) + O(h^{k+3}), \end{aligned}$$

$$(3.11) \quad \begin{aligned} (H_h^{yy} u)(z) &= (D_y D_y u)(z) + h^{k+1}[\boldsymbol{a^y} \cdot D^{k+2} D_y u + \boldsymbol{a^y} \cdot D_y(D^{k+2} u)](z) \\ &\quad + h^{k+2}[\boldsymbol{b^y} \cdot D^{k+3} D_y u + \boldsymbol{b^y} \cdot D_y(D^{k+3} u)](z) + O(h^{k+3}). \end{aligned}$$

(3.8)–(3.11) imply that the Hessian recovery operator H_h is exact for polynomials of degree $k + 2$ for translation invariant meshes. Also, we observe $H_h^{xy} = H_h^{yx}$ from (3.8) and (3.9).

Next we consider even order $(k = 2r)$ elements on translation invariant meshes, in which case

$$(3.12) \qquad \boldsymbol{a^x}(z) = \boldsymbol{0}, \quad \boldsymbol{c^x}(z) = \boldsymbol{0}, \quad \boldsymbol{a^y}(z) = \boldsymbol{0}, \quad \boldsymbol{c^y}(z) = \boldsymbol{0},$$

$$(3.13) \qquad D\boldsymbol{a^x}(z) = \boldsymbol{0}, \quad D\boldsymbol{c^x}(z) = \boldsymbol{0}, \quad D\boldsymbol{a^y}(z) = \boldsymbol{0}, \quad D\boldsymbol{c^y}(z) = \boldsymbol{0},$$

and $\boldsymbol{b^x}, \boldsymbol{b^y}, \ldots$ are constants in (3.7). Here the symbol D is understood as taking all partial derivatives to each entry of the vector. Consequently,

$$(3.14) \qquad (G_h^y u)(z) = (D_y u)(z) + h^{k+2}(\boldsymbol{b^y} \cdot D^{k+3} u)(z) + O(h^{k+4}).$$

Also, (3.14) is valid only at nodal points. Plugging (3.6) into (3.14) yields

$$(H_h^{xy}u)(z) = (G_h^y G_h^x u)(z)$$
$$= (D_y G_h^x u)(z) + h^{k+2}(\boldsymbol{b}^y \cdot D^{k+3} G_h^x u)(z) + O(h^{k+4})$$
$$= D_y(D_x u + h^{k+1}\boldsymbol{a}^x \cdot D^{k+2}u + h^{k+2}\boldsymbol{b}^x \cdot D^{k+3}u + h^{k+3}\boldsymbol{c}^x \cdot D^{k+4}u$$
$$\quad + \cdots)(z) + h^{k+2}(\boldsymbol{b}^y \cdot D^{k+3} D_x u)(z) + O(h^{k+4})$$
$$= (D_y D_x u)(z) + h^{k+2}(\boldsymbol{b}^x \cdot D_y D^{k+3}u + \boldsymbol{b}^y \cdot D^{k+3} D_x u)(z) + O(h^{k+4}).$$

In the last identity we have used (3.12) and (3.13).

The argument for the other three entries of the recovered Hessian matrix are similar. We conclude that the Hessian recovery operator H_h is exact for polynomials of degree up to $k + 3$ when k is even and the mesh is translation invariant and symmetric with respect to x and y. □

Remark 3.7. It is worth mentioning that, except for the Chevron pattern, (3.8)–(3.11) are valid for the other four patterns of uniform meshes, since the recovered gradient $G_h u$ produces the same stencil at each node.

Remark 3.8. According to [23], the best Hessian recovery method in the literature preserves polynomials of degree 2 for the linear element. Our method preserves polynomials of degree 2 on general unstructured meshes and preserves polynomials of degree 3 on translation invariant meshes for the linear element.

Theorem 3.9. *Let $u \in W_\infty^{k+2}(\mathcal{K}_z)$; then*

$$\|Hu - H_h u\|_{0,\infty,\mathcal{K}_z} \lesssim h^k |u|_{k+2,\infty,\mathcal{K}_z}.$$

If z is a node of translation invariant mesh and a mesh symmetric center of the involved nodes and $u \in W_\infty^{k+3}(\mathcal{K}_z)$, then

$$|(Hu - H_h u)(z)| \lesssim h^{k+1} |u|_{k+3,\infty,\mathcal{K}_z}.$$

Moreover, if $u \in W_\infty^{k+4}(\mathcal{K}_z)$ and k is an even number, then

$$|(Hu - H_h u)(z)| \lesssim h^{k+2} |u|_{k+4,\infty,\mathcal{K}_z}.$$

Proof. It follows directly from Theorems 3.5, 3.6, and the Bramble-Hilbert lemma.
□

4. Superconvergence analysis

In this section, we first use the supercloseness between the gradient of the finite element solution u_h and the gradient of the interpolation u_I [3, 5, 9, 10, 25, 26], and properties of the PPR operator [15, 27] to establish the superconvergence property of our Hessian recovery operator on mildly structured mesh. Then we utilize the tool of superconvergence by difference quotients from [24] to prove the proposed Hessian recovery method is ultraconvergent for the translation invariant finite element space of any order.

In this section, we consider the following variational problem: find $u \in H^1(\Omega)$ such that

$$(4.1) \qquad B(u,v) = \int_\Omega (\mathcal{D}\nabla u + \mathbf{b}u) \cdot \nabla v + cuv dx = f(v), \quad \forall v \in H^1(\Omega).$$

Here $\mathcal{D}$ is a 2×2 symmetric positive definite matrix, $\mathbf{b}$ is a vector, c is a number and $f(\cdot)$ is a linear functional on $H^1(\Omega)$. All coefficient functions are assumed to be smooth.

In order to insure (4.1) has a unique solution, we assume the bilinear form $B(\cdot, \cdot)$ satisfies the continuity condition

$$(4.2) \qquad |B(u,v)| \leq \nu \|u\|_{1,\Omega} \|v\|_{1,\Omega},$$

for all $u, v \in H^1(\Omega)$. We also assume the inf-sup conditions [3, 4, 6]

$$(4.3) \qquad \inf_{u \in H^1(\Omega)} \sup_{v \in H^1(\Omega)} \frac{B(u,v)}{\|u\|_{1,\Omega}\|v\|_{1,\Omega}} = \sup_{u \in H^1(\Omega)} \inf_{v \in H^1(\Omega)} \frac{B(u,v)}{\|u\|_{1,\Omega}\|v\|_{1,\Omega}} \geq \mu > 0.$$

The finite element approximation of (4.1) is to find $u_h \in S_h$ satisfying

$$(4.4) \qquad B(u_h, v_h) = f(v_h), \quad \forall v_h \in S_h.$$

To insure a unique solution for (4.4), we assume the inf-sup conditions

$$(4.5) \qquad \inf_{u \in S_h} \sup_{v \in S_h} \frac{B(u,v)}{\|u\|_{1,\Omega}\|v\|_{1,\Omega}} = \sup_{u \in S_h} \inf_{v \in S_h} \frac{B(u,v)}{\|u\|_{1,\Omega}\|v\|_{1,\Omega}} \geq \mu > 0.$$

From (4.1) and (4.4), it is easy to see that

$$(4.6) \qquad B(u - u_h, v) = 0$$

for any $v \in S_h$. In particular, (4.6) holds for any $v \in S_h^{\mathrm{comp}}(\Omega)$.

4.1. **Linear element.** The linear finite element space S_h on quasi-uniform mesh $\mathcal{T}_h$ is considered in this subsection.

Definition 4.1. The triangulation $\mathcal{T}_h$ is said to satisfy condition (σ, α) if there exist a partition $\mathcal{T}_{h,1} \cup \mathcal{T}_{h,2}$ of $\mathcal{T}_h$ and positive constants α and σ such that every two adjacent triangles in $\mathcal{T}_{h,1}$ form an $O(h^{1+\alpha})$ parallelogram and

$$\sum_{T \in \mathcal{T}_{h,2}} |T| = O(h^\sigma).$$

An $O(h^{1+\alpha})$ parallelogram is a quadrilateral shifted from a parallelogram by $O(h^{1+\alpha})$.

For general α and σ, Xu and Zhang [26] proved the following theorem.

Theorem 4.2. *Let u be the solution of (4.1), let $u_h \in S_h$ be the finite element solution of (4.4), and let $u_I \in S_h$ be the linear interpolation of u. If the triangulation $\mathcal{T}_h$ satisfies condition (σ, α) and $u \in H^3(\Omega) \cap W_\infty^2(\Omega)$, then*

$$|u_h - u_I|_{1,\Omega} \lesssim h^{1+\rho}(|u|_{3,\Omega} + |u|_{2,\infty,\Omega}),$$

where $\rho = \min(\alpha, \sigma/2, 1/2)$.

Using the above result, we are able to obtain a convergence rate for our Hessian recovery operator.

Theorem 4.3. *Suppose that the solution of (4.1) belongs to $H^3(\Omega) \cap W_\infty^2(\Omega)$ and $\mathcal{T}_h$ satisfies condition (σ, α), then we have*

$$\|Hu - H_h u_h\|_{0,\Omega} \leq h^\rho \|u\|_{3,\infty,\Omega}.$$

Proof. We decompose $Hu - H_h u_h$ as $(Hu - H_h u) + H_h(u_I - u_h)$, since $H_h u = H_h u_I$. Using the triangle inequality and the definition of H_h, we obtain

$$\|Hu - H_h u_h\|_{0,\Omega} \le \|Hu - H_h u\|_{0,\Omega} + \|H_h(u_I - u_h)\|_{0,\Omega}$$
$$= \|Hu - H_h u\|_{0,\Omega} + \|G_h(G_h(u_I - u_h))\|_{0,\Omega}.$$

The first term in the above expression is bounded by $h|u|_{3,\infty,\Omega}$ according to Theorem 3.9. Since G_h is a bounded linear operator [16], it follows that

$$\|H_h(u_I - u_h)\|_{0,\Omega} \lesssim \|\nabla(G_h(u_I - u_h))\|_{0,\Omega}$$

Notice that $G_h(u_I - u_h)$ is a function in S_h and hence the inverse estimate [4, 6] can be applied. Thus,

$$\|H_h(u_I - u_h)\|_{0,\Omega} \lesssim h^{-1}\|G_h(u_I - u_h)\|_{0,\Omega} \lesssim h^{-1}\|u_I - u_h\|_{1,\Omega}$$

and hence Theorem 4.2 implies that

$$\|H_h(u_I - u_h)\|_{0,\Omega} \lesssim h^\rho \|u\|_{3,\infty,\Omega}.$$

Combining the above two estimates completes our proof. $\qquad\square$

4.2. Quadratic element.

We proceed to quadratic finite element space S_h. According to [10], a triangulation $\mathcal{T}_h$ is strongly regular if any two adjacent triangles in $\mathcal{T}_h$ form an $O(h^2)$ approximate parallelogram. Huang and Xu proved the following superconvergence results in [10].

Theorem 4.4. *If the triangulation $\mathcal{T}_h$ is uniform or strongly regular, then*

$$|u_h - u_I|_{1,\Omega} \lesssim h^3 |u|_{4,\Omega}.$$

Based on the above theorem, we obtain the following superconvergent result.

Theorem 4.5. *Suppose that the solution of (4.1) belongs to $H^4(\Omega)$ and $\mathcal{T}_h$ is uniform or strongly regular. Then we have*

$$\|Hu - H_h u_h\|_{0,\Omega} \le h^2 \|u\|_{4,\Omega}.$$

Proof. The proof is similar to the proof of Theorem 4.3 by using Theorem 4.4 and the inverse estimate. $\qquad\square$

Remark 4.6. Theorem 4.5 can be generalized to mildly structured meshes as in [10].

4.3. Translation invariant element of any order.

In this subsection, we establish the ultraconvergence theory of Hessian recovery operator H_h for the translation invariant finite element space.

First, we observe that the Hessian recovery operator results in a difference quotient. It is due to the fact that G_h is a difference quotient [27] and the composition of two difference quotients is still a difference quotient. Let us take the linear element on uniform triangular mesh of the regular pattern as an example; see Figure 2. The recovered second order derivative at a nodal point z is

$$(H_h^{xx} u_h)(z) = \frac{1}{36h^2}(-12u_0 + 2u_1 - 4u_2 - 4u_3 + 2u_4 - 4u_5 - 4u_6 + 4u_7 + 4u_8 + u_9$$
$$- 2u_{10} + u_{11} + 4u_{12} + 4u_{13} + 4u_{14} + u_{15} - 2u_{16} + u_{17} + 4u_{18}).$$

Let ϕ_j be the nodal shape functions. Since $\phi_z(z') = \delta_{zz'}$, it follows that

$$
\begin{aligned}
(H_h^{xx} & u_h)\phi_0(x,y) \\
= & \frac{1}{36h^2}[-12u_0\phi_0(x,y) + 2u_1\phi_1(x+h,y) - 4u_2\phi_2(x+h,y+h) \\
& - 4u_3\phi_3(x,y+h) + 2u_4\phi_4(x-h,y) - 4u_5\phi_5(x-h,y-h) \\
& - 4u_6\phi_6(x,y-h) + 4u_7\phi_7(x+2h,y) + 4u_8\phi_8(x+2h,y+h) \\
& + u_9\phi_9(x+2h,y+2h) - 2u_{10}\phi_{10}(x+h,y+2h) + u_{11}\phi_{11}(x,y+2h) \\
& + 4u_{12}\phi_{12}(x-h,y+h) + 4u_{13}\phi_{13}(x-2h,y) + 4u_{14}\phi_{14}(x-2h,y-h) \\
& + u_{15}\phi_{15}(x-2h,y-2h) - 2u_{16}\phi_{16}(x-h,y-2h) + u_{17}\phi_{17}(x,y-2h) \\
& + 4u_{18}\phi_{18}(x+h,y-h)].
\end{aligned}
$$

The translations are in the directions of $\ell_1 = (1,0)$, $\ell_2 = (0,1)$, $\ell_3 = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, $\ell_4 = (\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$, $\ell_5 = (\frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5})$, and $\ell_6 = (\frac{2\sqrt{5}}{5}, \frac{\sqrt{5}}{5})$. Therefore, we can express the recovered second order derivative as

$$
(4.7) \qquad (H_h^{xx}u_h)(z) = \sum_{|\nu| \leq M} \sum_{i=1}^{6} C_{\nu,h}^i u_h(z + \nu h \ell_i),
$$

for some integer M.

Based on such an observation, we can prove the following lemma.

Lemma 4.7. *Let all the coefficients in the bilinear operator $B(\cdot,\cdot)$ be constant; let $\Omega_1 \subset\subset \Omega$ be separated by $d = O(1)$; let the finite element space S_h, which includes piecewise polynomials of degree k, be translation invariant in the directions required by the Hessian recovery operator H_h on Ω_1; and let $u \in W_\infty^{k+3}(\Omega)$. Assume that Theorem 5.2.2 from [24] is applicable. Then on any interior region $\Omega_0 \subset\subset \Omega_1$, we get*

$$
(4.8) \qquad \|H_h(u - u_h)\|_{0,\infty,\Omega_0} \lesssim \left(\ln\frac{1}{h}\right)^{\bar{r}} h^{k+1}\|u\|_{k+3,\infty,\Omega} + \|u - u_h\|_{-s,q,\Omega}
$$

for some $s \geq 0$ and $q \geq 1$. Here $\bar{r} = 1$ for the linear element and $\bar{r} = 0$ for the higher order element.

Proof. Since all coefficients in the bilinear form $B(\cdot,\cdot)$ are constant, it follows that

$$
B(T_{\nu\tau}^\ell(u - u_h), v) = B(u - u_h, T_{-\nu\tau}^\ell v) = B(u - u_h, (T_{\nu\tau}^\ell)^* v) = 0.
$$

Notice that H_h^{xx} is a difference operator constructed from translation of type (4.7). Then we have

$$
(4.9) \qquad B(H_h^{xx}(u - u_h), v) = B(u - u_h, (H_h^{xx})^* v) = 0, \quad v \in S_h^{\mathrm{comp}}(\Omega_1).
$$

Therefore, Theorem 5.5.2 of [24] (with $F \equiv 0$) implies that

$$
(4.10) \qquad
\begin{aligned}
\|H_h^{xx}(u - u_h)\|_{0,\infty,\Omega_0} \lesssim & \left(\ln\frac{d}{h}\right)^{\bar{r}} \min_{v \in S_h} \|H_h^{xx}u - v\|_{0,\infty,\Omega_1} \\
& + d^{-s-\frac{2}{q}}\|H_h^{xx}(u - u_h)\|_{-s,q,\Omega_1}.
\end{aligned}
$$

Note that $H_h^{xx}u \in S_h$ and hence the first term on the right-hand side of (4.10) can be estimated by standard approximation theory under the assumption that the finite element space includes piecewise polynomial of degree k:

$$
(4.11) \qquad \min_{v \in S_h} \|H_h^{xx}u - v\|_{0,\infty,\Omega_1} \lesssim h^{k+1}|u|_{k+3,\infty,\Omega_1},
$$

provided $u \in W_\infty^{k+3}(\Omega)$; see [4, 6]. It remains to attack the second term on the right-hand side of (4.10). Note that

$$(4.12) \qquad \|H_h^{xx}(u - u_h)\|_{-s,q,\Omega_1} = \sup_{\phi \in C_0^\infty(\Omega_1), \|\phi\|_{s,q',\Omega_1} = 1} (H_h^{xx}(u - u_h), \phi).$$

Here $\frac{1}{q} + \frac{1}{q'} = 1$ and

$$
\begin{aligned}
(H_h^{xx}(u - u_h), \phi) &= (u - u_h, (H_h^{xx})^* \phi) \\
(4.13) &\lesssim \|u - u_h\|_{0,\infty,\Omega_1} \|(H_h^{xx})^* \phi\|_{0,1,\Omega_1} \\
&\lesssim \|u - u_h\|_{0,\infty,\Omega_1},
\end{aligned}
$$

where we use the fact that $\|(H_h^{xx})^* \phi\|_{0,1,\Omega_1}$ is bounded uniformly with respect to h when $s \geq 1$. Again, we apply Theorem 5.5.1 from [24] to $\|u - u_h\|_{0,\infty,\Omega_1}$ with $\Omega_1 \subset\subset \Omega$ separated by d; then

$$
\begin{aligned}
(4.14) \qquad \|u - u_h\|_{0,\infty,\Omega_1} &\lesssim \left(\ln \frac{d}{h} \right)^{\bar{r}} \min_{v \in S_h} \|u - v\|_{0,\infty,\Omega} \\
&\quad + d^{-s-\frac{2}{q}} \|u - u_h\|_{-s,q,\Omega}.
\end{aligned}
$$

If the separation parameter $d = O(1)$, then we combine (4.10), (4.11) and (4.14) to obtain

$$(4.15) \qquad \|H_h^{xx}(u - u_h)\|_{0,\infty,\Omega_0} \lesssim \left(\ln \frac{1}{h} \right)^{\bar{r}} h^{k+1} \|u\|_{k+3,\infty,\Omega} + \|u - u_h\|_{-s,q,\Omega}.$$

Following the same argument, we can establish the same result for H_h^{xy}, H_h^{yx}, and H_h^{yy}. Therefore, our proof is completed by replacing H_h^{xx} with H_h in (4.15). $\qquad\square$

Now we are in a perfect position to prove our main result for the translation invariant finite element space of any order.

Theorem 4.8. *Let all the coefficients in the bilinear operator $B(\cdot, \cdot)$ be constant; let $\Omega_1 \subset\subset \Omega$ be separated by $d = O(1)$; let the finite element space S_h, which includes piecewise polynomials of degree k, be translation invariant in the directions required by the Hessian recovery operator H_h on Ω_1; and let $u \in W_\infty^{k+3}(\Omega)$. Assume that Theorem 5.2.2 from [24] is applicable. Then on any interior region $\Omega_0 \subset\subset \Omega_1$, we get*

$$(4.16) \qquad \|Hu - H_h u_h\|_{0,\infty,\Omega_0} \lesssim \left(\ln \frac{1}{h} \right)^{\bar{r}} h^{k+1} \|u\|_{k+3,\infty,\Omega} + \|u - u_h\|_{-s,q,\Omega}$$

for some $s \geq 0$ and $q \geq 1$.

Proof. We decompose

$$(4.17) \qquad Hu - H_h u_h = (Hu - (Hu)_I) + ((Hu)_I - H_h u) + H_h(u - u_h),$$

where $(Hu)_I \in S_h^2 \times S_h^2$ is the standard Lagrange interpolation of Hu in the finite element space S_h. By the standard approximation theory, we obtain

$$(4.18) \qquad \|Hu - (Hu)_I\|_{0,\infty,\Omega} \lesssim h^{k+1} |Hu|_{k+1,\infty,\Omega} \lesssim h^{k+1} |u|_{k+3,\infty,\Omega}.$$

For the second term, using Theorem 3.9, we have

$$
\begin{aligned}
\|(Hu)_I - H_h u\|_{0,\infty,\Omega_0} &= \Big\| \sum_{z \in \mathcal{N}_h} ((Hu)(z) - (H_h u)(z)) \phi_z \Big\|_{0,\infty,\Omega_0} \\
(4.19) &\lesssim \max_{z \in \mathcal{N}_h \cap \Omega_0} |(Hu)(z) - (H_h u)(z)| \\
&\lesssim h^{k+1} |u|_{k+3,\infty,\Omega}.
\end{aligned}
$$

The last term in (4.17) is bounded by (4.8). The conclusion follows by combining (4.8), (4.18) and (4.19). $\qquad\square$

Remark 4.9. Theorem 4.8 is an ultraconvergence result under the condition

$$\|u - u_h\|_{-s,q,\Omega} \lesssim h^{k+\sigma}, \quad \sigma > 0.$$

The reader is referred to [19] for negative norm estimates.

5. Numerical tests

In this section, two numerical examples are provided to illustrate our Hessian recovery method. The first one is designed to demonstrate the polynomial preserving property of the proposed Hessian recovery method. The second one is devoted to a comparison of our method and some existing Hessian recovery methods in the literature on both uniform and unstructured meshes.

In order to evaluate the performance of Hessian recovery methods, we split mesh nodes $\mathcal{N}_h$ into $\mathcal{N}_{h,1}$ and $\mathcal{N}_{h,2}$, where $\mathcal{N}_{h,2} = \{z \in \mathcal{N}_h : \text{dist}(z, \partial\Omega) \leq L\}$ denotes the set of nodes near the boundary and $\mathcal{N}_{h,1} = \mathcal{N}_h \setminus \mathcal{N}_{h,2}$ denotes the remaining interior nodes. Now, we can define

$$\Omega_{h,1} = \bigcup\{\tau \in \mathcal{T}_h : \tau \text{ has all of its vertices in } \mathcal{N}_{h,1}\},$$

and $\Omega_{h,2} = \Omega \setminus \Omega_{h,1}$. In the following examples we choose $L = 0.1$.

Let $\tilde{G}_h$ be the weighted average recovery operator. Then we define

$$H_h^{ZZ} u_h = \left(\tilde{G}_h(\tilde{G}_h^x u_h), \quad \tilde{G}_h(\tilde{G}_h^y u_h)\right)$$

and

$$H_h^{LS} u_h = \left(\tilde{G}_h(G_h^x u_h), \quad \tilde{G}_h(G_h^y u_h)\right).$$

For any nodal point z, fit a quadratic polynomial p_z at z as PPR. Then H_h^{QF} is defined as

$$H_h^{QF} u_h(z) = \begin{pmatrix} \frac{\partial^2 p_z}{\partial x^2}(0,0) & \frac{\partial^2 p_z}{\partial x \partial y}(0,0) \\ \frac{\partial^2 p_z}{\partial y \partial x}(0,0) & \frac{\partial^2 p_z}{\partial y^2}(0,0) \end{pmatrix}.$$

H_h^{ZZ}, H_h^{LS}, and H_h^{QF} are the first three Hessian recovery methods in [22]. To compare them, define

$$De = \|H_h u_h - Hu\|_{0,\Omega_{1,h}}, \quad De^{ZZ} = \|H_h^{ZZ} u_h - Hu\|_{0,\Omega_{1,h}},$$

$$De^{LS} = \|H_h^{LS} u_h - Hu\|_{0,\Omega_{1,h}}, \quad De^{QF} = \|H_h^{QF} u_h - Hu\|_{0,\Omega_{1,h}},$$

where u_h is the finite element solution.

Example 1. Consider the following function:

$$(5.1) \qquad u(x,y) = \sin(\pi x)\sin(\pi y), \quad (x,y) \in \Omega = (0,1) \times (0,1).$$

Let u_I be the standard Lagrangian interpolation of u in the finite element space. To validate Theorem 3.9, we apply the Hessian recovery operator H_h to u_I and consider the discrete maximum error of $H_h u_I - Hu$ at all vertices in $N_{1,h}$. First, linear elements on uniform meshes are taken into account. Figures 4–7 display the numerical results. The numerical errors decrease at a rate of $O(h^2)$ for four different pattern uniform meshes. It means the proposed Hessian recovery method preserves polynomials of degree 3 for linear elements on uniform meshes.
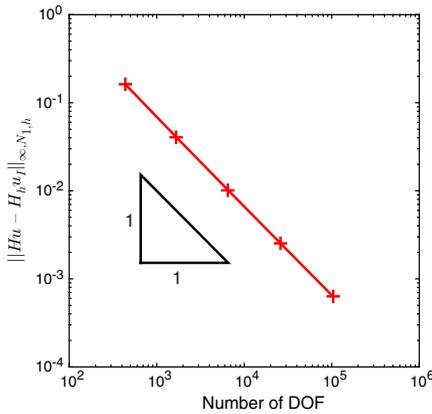
FIGURE 4. Numerical result of Example 1 for the linear element on Regular pattern uniform mesh
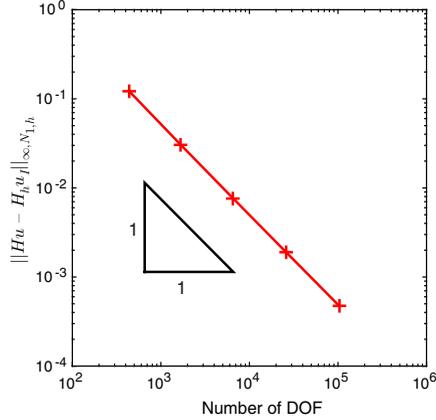
FIGURE 5. Numerical result of Example 1 for the linear element on Chevron pattern uniform mesh
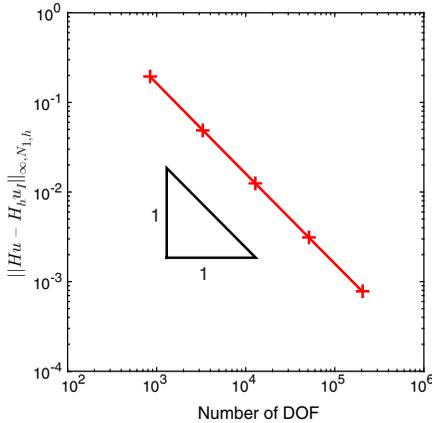
FIGURE 6. Numerical result of Example 1 for the linear element on Criss-cross pattern uniform mesh
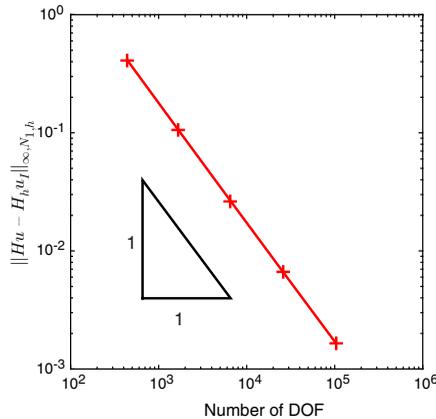
FIGURE 7. Numerical result of Example 1 for the linear element on Union-Jack pattern uniform mesh

Next, we consider unstructured meshes. We start from an initial mesh generated by EasyMesh [17] as shown in Figure 8 followed by four levels of refinement using bisection. Figure 9 shows that the recovered Hessian $H_h u_I$ converges to the exact Hessian at rate $O(h)$. This coincides with the result in Theorem 3.6 that H_h only preserves polynomials of degree 2 on general unstructured meshes.

Then we turn to the quadratic element. We test the discrete error of recovered Hessian $H_h u_I$ and the exact Hessian Hu using uniform meshes of regular pattern and the same Delaunay meshes. Similarly, we define $\|\cdot\|_{\infty,h}$ as a discrete maximum
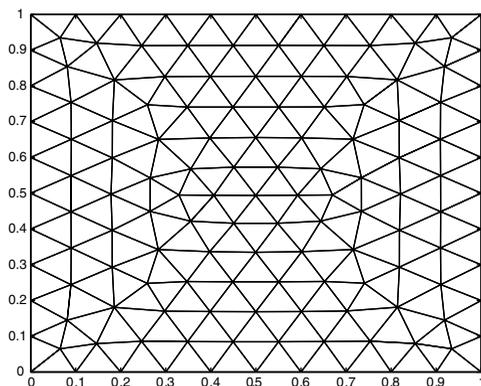
FIGURE 8. An example of unstructured mesh generated by EasyMesh based on Delaunay triangulation
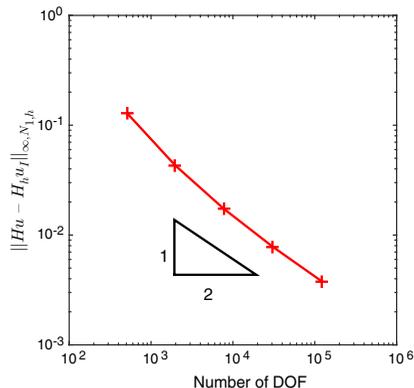
FIGURE 9. Numerical result of Example 1 for the linear element on Delaunay mesh with regular refinement

norm at all vertices and edge centers in an interior region $\Omega_{1,h}$. The result of uniform mesh of regular pattern is reported in Figure 10. As predicted by Theorem 3.9, $H_h u_I$ converges to Hu at a rate of $O(h^4)$ which implies H_h preserves polynomials of degree 5 for the quadratic element on uniform triangulation. For unstructured mesh, we observe that $H_h u_I$ approximates Hu at a rate of $O(h^2)$ from Figure 11.
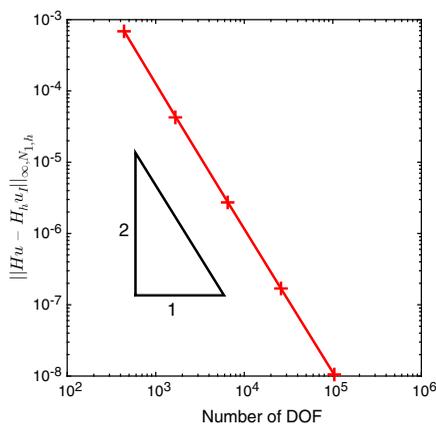
FIGURE 10. Numerical result of Example 1 for the quadratic element on regular pattern uniform mesh
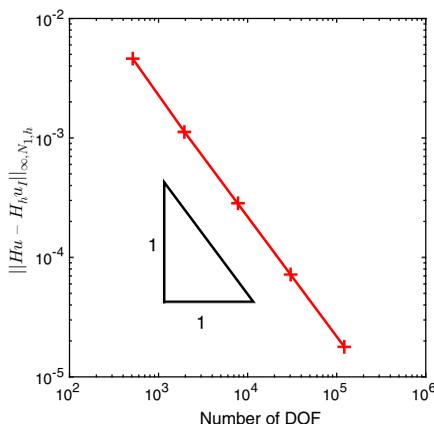
FIGURE 11. Numerical result of Example 1 for the quadratic element on Delaunay mesh with regular refinement

Example 2. We consider the following elliptic equation:

$$(5.2) \qquad \begin{cases} -\Delta u = 2\pi^2 \sin \pi x \sin \pi y, & \text{in } \Omega = [0,1] \times [0,1], \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

The exact solution is $u(x,y) = \sin(\pi x)\sin(\pi y)$. First, the linear element is considered. In Table 1, we report the numerical results for regular pattern meshes. All four methods ultraconverge at a rate of $O(h^2)$ in the interior subdomain. The fact that H_h^{LS} and H_h^{ZZ} perform as good as H_h is not a surprise since it is well known that the polynomial preserving recovery is the same as the weighted average for uniform triangular mesh of the regular pattern.

The results of the Chevron pattern is shown in Table 2. $H_h u_h$ approximates Hu at a rate of $O(h^2)$ while $H_h^{LS} u_h$, $H_h^{ZZ} u_h$ and $H_h^{QF} u_h$ approximate Hu at a rate of $O(h)$. It is observed that our method out-performs the other three Hessian recovery methods on the Chevron pattern uniform meshes. To the best of our knowledge, the proposed PPR-PPR Hessian recovery is the only method to achieve $O(h^2)$ superconvergence for the linear element under the Chevron pattern triangular mesh.

TABLE 1. Numerical comparison of several Hessian recovery methods for the linear element on regular pattern uniform mesh

Dof	De	order	$De^{ZZ}e$	order	De^{LS}	order	De^{QF}	order
121	7.93e-001	–	9.73e-001	–	7.93e-001	–	4.01e-001	–
441	2.02e-001	1.06	2.02e-001	1.22	2.02e-001	1.06	1.03e-001	1.05
1681	5.10e-002	1.03	5.10e-002	1.03	5.10e-002	1.03	2.61e-002	1.03
6561	1.28e-002	1.02	1.28e-002	1.02	1.28e-002	1.02	6.53e-003	1.02
25921	3.20e-003	1.01	3.20e-003	1.01	3.20e-003	1.01	1.63e-003	1.01
103041	8.00e-004	1.00	8.00e-004	1.00	8.00e-004	1.00	4.08e-004	1.00

TABLE 2. Numerical comparison of several Hessian recovery methods for the linear element on Chevron pattern uniform mesh

Dof	De	order	$De^{ZZ}e$	order	De^{LS}	order	De^{QF}	order
121	6.51e-001	–	7.98e-001	–	7.82e-001	–	9.03e-001	–
441	1.34e-001	1.22	2.12e-001	1.03	2.34e-001	0.93	4.30e-001	0.57
1681	3.38e-002	1.03	7.96e-002	0.73	9.87e-002	0.64	2.11e-001	0.53
6561	8.46e-003	1.02	3.57e-002	0.59	4.68e-002	0.55	1.05e-001	0.51
25921	2.11e-003	1.01	1.73e-002	0.53	2.30e-002	0.52	5.23e-002	0.51
103041	5.29e-004	1.00	8.57e-003	0.51	1.15e-002	0.50	2.62e-002	0.50

Then the Criss-cross pattern mesh is considered and results are displayed in Table 3. An $O(h^2)$ convergence rate is observed for our recovery method, H_h^{LS} and H_h^{ZZ} while no convergence rate is observed for H_h^{QF}. The results for the Union-Jack pattern mesh is very similar to the Criss-cross pattern mesh except that our recovery method superconverges at a rate of $O(h^2)$ as shown in Table 4.

TABLE 3. Numerical comparison of several Hessian recovery methods for the linear element on Criss-cross pattern uniform mesh

Dof	De	order	$De^{ZZ}e$	order	De^{LS}	order	De^{QF}	order
221	5.49e-001	–	3.57e-001	–	4.40e-001	–	7.14e-001	–
841	1.28e-001	1.09	8.03e-002	1.12	1.04e-001	1.08	6.17e-001	0.11
3281	3.22e-002	1.01	2.01e-002	1.02	2.62e-002	1.01	5.95e-001	0.03
12961	8.06e-003	1.01	5.04e-003	1.01	6.55e-003	1.01	5.90e-001	0.01
51521	2.02e-003	1.00	1.26e-003	1.00	1.64e-003	1.00	5.89e-001	0.00
205441	5.04e-004	1.00	3.15e-004	1.00	4.09e-004	1.00	5.88e-001	0.00

TABLE 4. Numerical comparison of several Hessian recovery methods for the linear element on Union-Jack pattern uniform mesh

Dof	De	order	$De^{ZZ}e$	order	De^{LS}	order	De^{QF}	order
121	1.25e+000	–	8.40e-001	–	9.87e-001	–	1.05e+000	–
441	3.16e-001	1.06	1.77e-001	1.20	2.48e-001	1.07	6.95e-001	0.32
1681	7.96e-002	1.03	4.46e-002	1.03	6.24e-002	1.03	6.14e-001	0.09
6561	2.00e-002	1.02	1.12e-002	1.02	1.56e-002	1.02	5.95e-001	0.02
25921	5.00e-003	1.01	2.80e-003	1.01	3.91e-003	1.01	5.90e-001	0.01
103041	1.25e-003	1.00	6.99e-004	1.00	9.78e-004	1.00	5.89e-001	0.00

TABLE 5. Numerical comparison of several Hessian recovery methods for the linear element on Delaunay mesh with regular refinement

Dof	De	order	$De^{ZZ}e$	order	De^{LS}	order	De^{QF}	order
139	4.31e-001	–	4.38e-001	–	4.40e-001	–	3.26e-001	–
513	1.38e-001	0.87	2.20e-001	0.53	1.49e-001	0.83	1.79e-001	0.46
1969	5.39e-002	0.70	2.36e-001	-0.05	5.85e-002	0.69	8.88e-002	0.52
7713	2.38e-002	0.60	1.62e-001	0.28	2.55e-002	0.61	4.35e-002	0.52
30529	1.14e-002	0.54	1.13e-001	0.26	1.19e-002	0.56	2.15e-002	0.51
121473	5.59e-003	0.51	7.97e-002	0.25	5.73e-003	0.53	1.07e-002	0.51

Now, we turn to unstructured mesh generated by EasyMesh [17] as in the previous examples. Numerical data are listed in Table 5. H_h, H_h^{LS} and H_h^{QF} converge at a rate of $O(h)$ while H_h^{ZZ} only converges at a rate of $O(h^{0.5})$.

The results above indicate clearly that our Hessian recovery method converges at rate $O(h)$ on general Delaunay meshes, which is predicted by Theorem 4.3. On uniform meshes, we can obtain $O(h^2)$ ultraconvergence on an interior sub-domain as predicted by Theorem 4.8.

In the end, we consider the quadratic element. Note that our Hessian recovery method is well defined for arbitrary order elements. However, the extension of the other three methods to the quadratic element is not straightforward or even impossible and hence only our method is implemented here. We report the numerical results in Figure 12 for regular pattern uniform mesh. A rate of about $O(h^{3.2})$
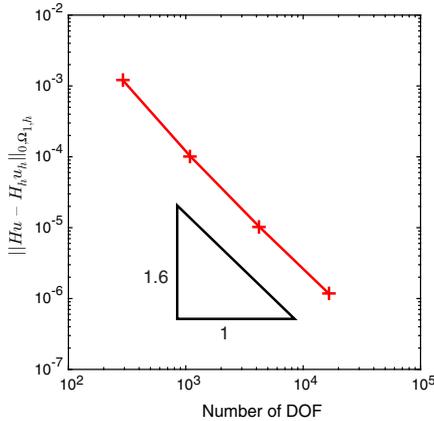
FIGURE 12. Numerical result of Example 2 for the quadratic element on regular pattern uniform mesh
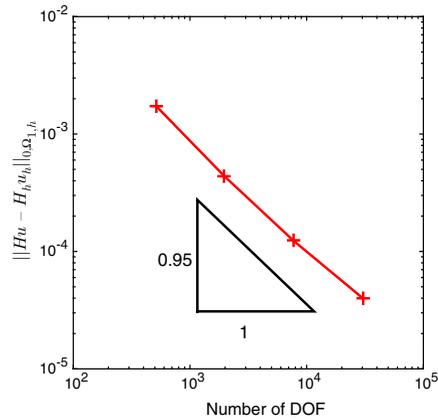
FIGURE 13. Numerical result of Example 2 for the quadratic element on Delaunay mesh with regular refinement

order convergence is observed, which is a bit better than the theoretical result predicted by Theorem 4.8. Figure 13 shows the result for Delaunay mesh generated by EasyMesh [17]. A rate of about $O(h^{1.9})$ superconvergence is observed.

6. Concluding remarks

In this work, we introduced a Hessian recovery method for arbitrary order Lagrange finite elements. Theoretically, we proved that the PPR-PPR Hessian recovery operator H_h preserves polynomials of degree $k+1$ on general unstructured meshes and preserves polynomials of degree $k+2$ on translation invariant meshes. This polynomial preserving property, combined with the supercloseness property of the finite element method, enabled us to prove convergence and superconvergence results for our Hessian recovery method on mildly structured meshes. Moreover, we proved the ultraconvergence result for the translation invariant finite element space of any order by using the argument of superconvergence by difference quotient from [24].

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