Semi-local simple connectedness of non-collapsing Ricci limit spaces

Jiayin Pan (Joint work with Guofang Wei)

UC Santa Barbara
Gromov-Hausdorff distance

The Gromov-Hausdorff distance between two metric spaces measures how they look alike.

Gromov's precompactness theorem

Let \( \{ (M_i, p_i) \}_{i} \) be a sequence of complete Riemannian \( n \)-manifolds of \( \text{Ric}_{M_i} \geq -(n-1) \), then \( \{ (M_i, p_i) \}_{i} \) has a convergent subsequence with respect to the pointed Gromov-Hausdorff distance. \( (M_i, p_i) \xrightarrow{GH} (X, p) \)

...a useful tool to study a class of Riemannian manifolds with geometric conditions (curvature, volume, diameter, etc.) to study infinitesimal or asymptotic geometry important to understand the structure of the limit space \( X \) relations between \( X \) and \( M_i \) for \( i \) large.

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\[(M^n_i, p_i) \overset{GH}{\to} (X, p), \text{ where } M_i \text{ are complete.}\]
Topology of limit spaces

\((M_i^n, p_i) \overset{GH}{\to} (X, p)\), where \(M_i\) are complete.

\[ \text{sec} \geq -1, \quad X \text{ Alexandrov space (Burago-Gromov-Perelman)} \]

\(X\) is locally contractible (Perelman)
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$$X \text{ is locally contractible (Perelman)}$$

$$\Ric \geq -(n - 1), \quad X \text{ Ricci limit space (Cheeger-Colding)}$$

Even when non-collapsing, $X$ may have infinite local topological type (Menguy)

“Non-collapsing” means there is $\nu > 0$ such that $\text{vol}(B_1(p_i)) \geq \nu$ for all $i$. 

Jiayin Pan (Joint work with Guofang Wei) | Semi-local simple connectedness
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**Theorem** *(Sormani-Wei, 2001)*
Any Ricci limit space has a universal cover.

Remark: “Universal” in the sense of universal property of covering maps; unknown whether it is simply connected.
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**Theorem** (Mondino-Wei, 2016)

Let \((X, d, m)\) be an \(\text{RCD}^*(K, N)\)-space for some \(K \in \mathbb{R}, N \in (1, +\infty)\). Then \((X, d, m)\) admits a universal cover.
Examples

A Hawaii ring does not have a universal cover.

The Griffiths twin cone is one-point-join of two “cutted” cones over a Hawaii ring. It is not semi-locally simply connected, but it has a universal cover (as itself); also, in this example, a non-contractible loop may have infinite length.
Main result

Main Theorem (Pan-Wei, 2019)

Any non-collapsing Ricci limit space is semi-locally simply connected.
Methods for sectional curvature case

Method 1:
Tangent cones are metric cones (BGP); True for non-collapsing Ricci limits (CC).
Construct a homeomorphism from tangent cone to a local nbhd (Perelman)/Gradient flow of distance function.
Impossible for Ricci case, due to Menguy's example.

Method 2:
With the non-collapsing condition, control the local contractibility radius (Grove-Petersen);
Not true for Ricci case on $\pi_1$-level; examples by Otsu.
then pass this property to the limit (Borsuk, Petersen).
True (even without curvature conditions), but requires uniform control at all points.

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**Theorem (Grove-Petersen)**

Given $n, \kappa, \nu > 0$, there exist positive constants $\epsilon(n, \kappa, \nu)$ and $C(n, \kappa, \nu)$ such that for any complete $n$-manifold $(M, p)$ of

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\text{sec}_M \geq -\kappa, \quad \text{vol}(B_1(p)) \geq \nu,
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$B_r(p)$ is contractible in $B_{Cr}(p)$, where $r \in [0, \epsilon)$.
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**Otsu’s example**

$S^3 \times \mathbb{R}P^2$ admits a sequence of Riemannian metrics $g_i$ of positive Ricci curvature with a non-collapsing limit space as a metric suspension over $S^2 \times \mathbb{R}P^2$. At the “tip” point, there are shorter and shorter non-contractible loops.
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Also, the Eguchi-Hanson metric on the tangent bundle of \( \mathbb{R}P^2 \) gives a Ricci flat example.
**Definition (1-contractibility radius):**

\[ \rho(t, x) = \inf \{ \infty, \rho \geq t \mid \text{any loop in } B_t(x) \text{ is contractible in } B_\rho(x) \} . \]

Remark: \( X \) is semi-locally simply connected if for any \( x \in X \), there is \( T > 0 \) such that \( \rho(T, x) < \infty \); \( X \) is locally simply connected if for any \( x \in X \), there is \( t_i \to 0 \) such that \( \rho(t_i, x) = t_i \).
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**Local version of the main theorem with an estimate on \( \rho(t, x) \)**

Let \((M_i, p_i)\) be a sequence of Riemannian \( n \)-manifolds (not necessarily complete) converging to \((X, p)\) such that for all \( i \),

1. \( B_2(p_i) \cap \partial M_i = \emptyset \) and the closure of \( B_2(p_i) \) is compact;
2. \( \text{Ric} \geq -(n - 1) \) on \( B_2(p_i) \), \( \text{vol}(B_1(p_i)) \geq v > 0 \).

Then

\[ \lim_{t \to 0} \frac{\rho(t, x)}{t} = 1 \]

holds for any \( x \in B_1(p) \).
Key: show $\lim_{t \to 0} \rho(t, x) = 0$.
After this, we can improve the result to $\lim_{t \to 0} \rho(t, x)/t = 1$ by using the structure of tangent cones and Sormani’s uniform cut techniques.
Approach

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Classification: We classify the points in $X$ according to the local 1-contractibility radius on manifolds: $x \in X$, let $x_i$ in $M_i$ to $x$;

- $x$ is of type I, if there is $r > 0$ such that the family of functions $\{\rho(q, t)|q \in B_r(x_i), i \in \mathbb{N}\}$ is equi-continuous at $t = 0$;
- $x$ is of type II, if $\{\rho(x_i, t)\}_{i \in \mathbb{N}}$ is not equi-continuous at $t = 0$;
- $x$ is of type III, if it is not of type I nor type II. (In other words, $\{\rho(x_i, t)\}_{i \in \mathbb{N}}$ is equi-continuous at $t = 0$, but the family $\{\rho(q, t)|q \in B_r(x_i), i \in \mathbb{N}\}$ is not equi-continuous for any $r > 0$.)

Remark: Type II point may exist due to Otsu’s example. Spoiler: Type III points are the most difficult ones to deal with.

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**Key:** show \( \lim_{t \to 0} \rho(t, x) = 0. \)

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- \( x \) is of type II, if \( \{ \rho(x_i, t) \}_{i \in \mathbb{N}} \) is not equi-continuous at \( t = 0 \);
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Spoiler: Type III points are the most difficult ones to deal with.
For a family of functions $\{\rho_\alpha(t)\}_{\alpha \in A}$ with $\rho_\alpha(0) = 0$, the family is equi-continuous at $t = 0$ if and only if there is a continuous function $\lambda(t)$ defined on $[0, T)$ with $\lambda(0) = 0$ such that $\rho_\alpha(t) \leq \lambda(t)$ for all $t \in [0, T)$ and all $\alpha \in A$. 
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For **type I** points, we can pass the local 1-contractibility control on local balls around \( x_i \) to that around \( x \) in the limit space:
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For **type I** points, we can pass the local 1-contractibility control on local balls around $x_i$ to that around $x$ in the limit space:

**Proposition (without curvature conditions)**

Let $(X_i, x_i)$ be a sequence of length metric spaces with the conditions below:

1. the closure of $B_1(x_i)$ is compact;
2. there is a nice function $\lambda$ on $[0, T)$ such that for all $i$ and all $q \in B_2(x_i)$, $\rho(t, q) \leq \lambda(t) < 1/2$ holds on $[0, T)$;
3. $(X_i, x_i) \xrightarrow{GH} (Y, y)$.

Then $\rho(t, q) \leq \lambda(t)$ for all $t \in [0, T)$ and all $q \in B_1(y)$. 

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For a loop $c$ in the limit space, we can find a sequence of loops $c_i$ in $M_i$ that converges uniformly to $c$. 

In this construction, we need to control the distance between nullhomotopies to assure uniform convergence.
Sketch: (Borsuk’s method)

For a loop $c$ in the limit space, we can find a sequence of loops $c_i$ in $M_i$ that converges uniformly to $c$.

For $c$ in a sufficiently small ball, we can contract $c_i$ for a large $i$. Take a skeleton of this homotopy, then we transfer this skeleton to the next manifold in the sequence and fill in the skeleton there. This allows us to transfer the nullhomotopy of $c_i$ along the sequence and pass it to the limit space by uniform convergence. In this construction, we need to control the distance between nullhomotopies to assure uniform convergence.
A point $x$ is of **Type II** means there are $\epsilon > 0$ and $t_i \to 0$ such that $\rho(t_i, x_i) \geq \epsilon$. We consider the universal covering space of $B_\epsilon(x_i)$. The local fundamental group $\Gamma_i$ has a subgroup generated by these non-contractible “small” loops $H_i$. In Otsu’s example, $\Gamma_i = \mathbb{Z}_2$.

We consider the equivariant Gromov-Hausdorff convergence:

$$(U_i, y_i, \Gamma_i, H_i) \xrightarrow{GH} (Y, y, G, H)$$

$$(B_\epsilon(x_i), x_i) \xrightarrow{GH} (B_\epsilon(x) = Y/G, x),$$
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$H_i$ has a uniform order upper bound, $\#H_i \leq N$ (Anderson). This also implies that $H$-action fixes $y$. 

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Roughly speaking, covering group increases the volume around $y_i$ when compared with the volume around $x_i$. Together with volume convergence (Colding, CC), local volume of $y$ is at least double of the local volume of $x$: $\mathcal{H}^n(B_{s}(y)) \geq 2 \cdot \mathcal{H}^n(B_{s}(x))$. 

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Therefore, if $x$ has local volume strictly larger than half of corresponding volume in the space form, then the above diagram cannot happen. In fact, this implies such a point $x$ must be type I.
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**Theorem (By-product for half-volume lower bounds)**

Given $n \geq 2$, $\kappa \geq 0$, and $\omega > 1/2$, there exist positive constants $\epsilon(n, \kappa, \omega)$ and $C(n, \kappa, \omega)$ such that the following holds. Let $(M, p)$ be a complete $n$-manifold satisfying

$$\text{Ric} \geq -(n-1)\kappa, \quad \text{vol}(B_1(p)) \geq \omega \cdot \text{vol}(B_1^n(-\kappa)).$$

Then every loop in $B_r(p)$ is contractible in $B_{Cr}(p)$, where $r \in [0, \epsilon)$. 
Define \( \omega(x) = \lim_{r \to 0} \frac{\mathcal{H}^n(B_r(x))}{\text{vol}(B_r^n(0))} \).

Remark: \( 0 \leq \omega(x) \leq 1 \), equality holds iff \( x \) is a regular point. \( \omega(x) \) has a uniform lower bound for all \( x \in B_1^p \).

We use an induction argument on \( \omega(x) \):

- If \( \omega(x) > \frac{1}{2} \) for all \( x \in B_1^p \), then every point \( x \in B_1^p \) is of type I...we are good.
- If \( \omega(x) > \frac{1}{4} \) for all \( x \in B_1^p \):
  - Case \( x \) is of type I, we are good;
  - Case \( x \) is of type II...go to local covers...the corresponding limit point \( y \) has \( \omega(y) > \frac{1}{2} \)...lift the loop in \( B_s^x \) to a loop in \( B_s^y \)...project the homotopy down...we are good;
  - Case \( x \) is of type III...main technical part...no local control around \( x \) to utilize...no local covers to build...

We need a technical lemma to deal with type III points.
Define $\omega(x) = \lim_{r \to 0} \frac{\mathcal{H}^n(B_r(x))}{\text{vol}(B^n_r(0))}$.

Remark: $0 < \omega(x) \leq 1$, $=\$ holds iff $x$ is a regular point. $\omega(x)$ has a uniform lower bound for all $x \in B_1(p)$. 
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Induction on local volume

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- Case \( x \) is of type I, we are good;
- Case \( x \) is of type II...go to local covers...the corresponding limit point \( y \) has \( \omega(y) > 1/2 \)...lift the loop in \( B_s(x) \) to a loop in \( B_s(y) \)...project the homotopy down...we are good;
- Case \( x \) is of type III...main technical part...no local control around \( x_i \) to utilize...no local covers to build...

...
Induction on local volume

Define $\omega(x) = \lim_{r \to 0} \frac{\mathcal{H}^n(B_r(x))}{\text{vol}(B^n_r(0))}$.

Remark: $0 < \omega(x) \leq 1$, $\omega(x) = 1$ holds iff $x$ is a regular point. $\omega(x)$ has a uniform lower bound for all $x \in B_1(p)$.

We use an induction argument on $\omega(x)$:
If $\omega(x) > 1/2$ for all $x \in B_1(p)$, then every point $x \in B_1(p)$ is of type I...we are good.

If $\omega(x) > 1/4$ for all $x \in B_1(p)$:

- Case $x$ is of type I, we are good;
- Case $x$ is of type II...go to local covers...the corresponding limit point $y$ has $\omega(y) > 1/2$...lift the loop in $B_s(x)$ to a loop in $B_s(y)$...project the homotopy down...we are good;
- Case $x$ is of type III...main technical part...no local control around $x_i$ to utilize...no local covers to build...

...We need a technical lemma to deal with type III points.
(Roughly speaking,) If all type II points are good, then all type III points are good as well.

Technical lemma (without curvature conditions)

By definition, if \( x \) is of type III, then the family \( \{ \rho(t, x_i) \}_{i \in \mathbb{N}} \) is equi-continuous at \( t = 0 \).

A trivial, but important, observation: for any \( x \), either \( \rho(t, x) \) is good (type II case, by assumption), or \( \{ \rho(t, x_i) \}_{i \in \mathbb{N}} \) is good (type III case); for type I points, both \( \rho(t, x) \) and \( \{ \rho(t, x_i) \}_{i \in \mathbb{N}} \) are good.
(Roughly speaking,) If all type II points are good, then all type III points are good as well.

Assuming this, we clear the quarter volume lower bound case and we can continue the induction argument.
Technical lemma (without curvature conditions)

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On Technical Lemma:
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On Technical Lemma:
By definition, if $x$ is of type III, then the family $\{\rho(t, x_i)\}_{i \in \mathbb{N}}$ is equi-continuous at $t = 0$.

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A very brief sketch of the proof of technical lemma

To better illustrate the proof, we first present an alternative proof for type I points.

Jiayin Pan (Joint work with Guofang Wei)
Semi-local simple connectedness
To better illustrate the proof, we first present an alternative proof for type I points.

We construct the nullhomotopy gradually in the limit space through a sequence of refining skeletons. By controlling the extensions on the new skeletons at each step, these maps on the skeletons converge uniformly to a continuous map defined on the disk.
A very brief sketch of the proof of technical lemma

**Sketch:** Roughly speaking, if a sub-triangle is away from a point of type III, we can directly contract this sub-triangle; if not, we will use the local 1-contractibility from the sequence to extend the map on a finer 1-skeleton.