NONNEGATIVE RICCI CURVATURE AND VIRTUALLY ABELIAN STRUCTURE

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Abstract. This short note includes Buser’s proof on classical Bieberbach’s theorem and Cheeger-Gromoll’s proof on virtually abelian structure. It also offers a viewpoint of virtually abelian structure from virtually nilpotent structure.

1. Bieberbach’s theorem

Let $M$ be a compact flat $n$-manifold. The universal cover of $M$ is the standard Euclidean space $\mathbb{R}^n$. Hence to study the structure of $\pi_1(M)$, it is equivalent to study the isometry group of $\mathbb{R}^n$.

Classical Bieberbach’s theorem:

**Theorem 1.1.** Let $M$ be a compact flat $n$-manifold. Then $\pi_1(M)$-action on $\tilde{M} = \mathbb{R}^n$ contains $n$ linearly independent translations. Consequently, $\pi_1(M)$ has a normal subgroup $\mathbb{Z}^n$ of finite index.

In fact, there is a second part of Bieberbach’s theorem, which states that there are only finitely many isomorphism classes of $\pi_1(M)$ in each dimension. This gives a universal bound $C(n)$ on the index.

The following algebraic lemma is helpful to get normal subgroups of finite index.

**Lemma 1.2.** Let $G$ be a group and $H$ be a subgroup of finite index $N$, then there is a subgroup $H'$ of $H$ such that $H'$ is normal in $G$ and $[G : H'] \leq NN$.

**Proof.** Let $H' = \bigcap_{g \in G} g^{-1}Hg$. □

We follow [1] to prove Theorem 1.1. The idea has origins in Gromov’s work on almost flat manifolds [5].

1.1. Rotations and translations. Any isometry $\alpha$ of $\mathbb{R}^n$ can be expressed as

$$\alpha = (A, a) : v \mapsto Av + a,$$

where $A \in O(n)$ is the rotational part and $a \in \mathbb{R}^n$ is the translation part.

**Definition 1.3.** For $A \in O(n)$, we define

$$m(A) = ||A - I_n|| = \max_{v \in \mathbb{R}^n - \{0\}} |Av - v|/|v|.$$

This measures the difference between $A$ and the identity matrix $I_n$.

**Lemma 1.4.** Let $A, B \in O(n)$. Then

$$m([A, B]) \leq 2m(A)m(B).$$
Lemma 1.5. For all $v$

Proof. It is direct to check that

$$[A, B] - I = ABA^{-1}B^{-1} - I = [A - I, B - I]A^{-1}B^{-1}.$$  

Thus

$$|([A, B] - I)v| = ||A - I, B - I]A^{-1}B^{-1}v|$$

$$= ||A - I, B - I)v|$$

$$\leq m(A)m(B)|v| + m(B)m(A)|v|.$$  

for all $v \in \mathbb{R}^n$. \hfill $\square$

Lemma 1.4 implies that if $m(A) \leq 1/2$, then $m([A, B]) \leq m(B)$.

Lemma 1.6. Put

$$E_A = \{ v \in \mathbb{R}^n | |Av - v| = m(A)|v| \}.$$  

$E_A$ is an $A$-invariant subspace.

Proof. It is direct to check that $E_A$ is invariant under $A$ and scalar multiplication. Let $v, w \in E_A$. Note that by parallelogram law,

$$2m(A)^2(|v|^2 + |w|^2) = 2(|Av - v|^2 + |Aw - w|^2)$$

$$= |A(v + w) - (v + w)|^2 + |A(v - w) - (v - w)|^2$$

$$\leq m(A)^2(|v + w|^2 + |v - w|^2)$$

$$= 2m(A)^2(|v|^2 + |w|^2).$$

It follows that $v \pm w \in E_A$. \hfill $\square$

We can decompose $\mathbb{R}^n$ as $E_A + E_A^\perp$, where $E_A^\perp$ is the orthogonal complement of $E_A$, which is $A$-invariant as well. For $A|_{E_A^\perp}$, we similarly define

$$m^+(A) = ||A - I||_{E_A^\perp}.$$  

It is clear that if $A \neq I$, then $m^+(A) < m(A)$.

1.2. Almost/pure translations. We write $\Gamma = \pi_1(M)$. Recall that $\Gamma$ acts compactly on $\mathbb{R}^n$. Thus there is $D > 0$, the diameter of $M$, such that for any $v \in \mathbb{R}^n$, there is $\alpha = (A, a) \in \Gamma$ such that

$$D \geq |\alpha \cdot 0 - v| = |a - v|.$$  

To obtain a pure translations in $\Gamma$, we will first look for almost translations. These elements have rotational parts very close to identity.

Lemma 1.6. Let $u \in \mathbb{R}^n$ be a unit vector. For any $\epsilon > 0$, there is $\alpha = (A, a) \in \Gamma$ such that

$$a \neq 0, \quad \angle(u, a) \leq \epsilon, \quad m(A) \leq \epsilon.$$  

Proof. For each $k \in \mathbb{N}$, we can find $b_k = (B_k, b_k) \in \Gamma$ such that $|b_k - ku| \leq D$. It is clear that

$$|b_k| \to \infty, \quad \angle(u, b_k) \to 0.$$  

Next we make use of the compactness of $O(n)$. After passing to a subsequence, $B_k$ converges in $O(n)$. In this subsequence, we can pick a large $i$ such that for all $j \geq i$, we have

$$m(B_j B_i^{-1}) \leq \epsilon, \quad \angle(u, b_j) \leq \epsilon/2.$$  

We consider the element $\beta_\beta^{-1} = (B_jB_j^{-1}, B_jb_j + b_j) \in \Gamma$. As $j \to \infty$, the translation part $B_jb_j + b_j$ is dominated by $b_j$. Thus for a sufficiently large $j$,

$$\angle(u, B_jb_j + b_j) \leq \epsilon$$

holds. This completes the proof. \hfill \Box

The key observation in this proof is the lemma below, which states that any almost translation is indeed a pure translation.

**Lemma 1.7.** Let $\alpha = (A, a) \in \Gamma$. If $m(A) \leq 1/2$, then $A = I$.

**Proof.** Let

$$T = \{ \alpha = (A, a) \in \Gamma \mid 0 < m(A) \leq 1/2 \}.$$ 

Suppose that $T$ is non-empty. We choose $\alpha \in T$ so that it has the smallest translation part $|a|$ among elements in $T$. Apply Lemma 1.6 to any unit vector of $E_A$, we can find $\beta = (B, b) \in \Gamma$ such that

$$b \neq 0, \quad |b^\perp| \leq |b^E|, \quad m(B) \leq \frac{1}{8}(m(A) - m^+(A)). \quad (*)$$

We choose $\beta$ with the minimal $|b| \neq 0$.

Let $\gamma = [\alpha, \beta] = (C, c)$. We claim that $\gamma$ also satisfies $(*)$ but has a smaller $|c|$, which is a contradiction to the choice of $\beta$.

*Case 1:* $\beta$ is a pure translation. Then by direct calculation $\gamma = (I, (A - I)b)$. We decompose $c = (A - I)b$ as

$$c = c^E + c^\perp = (A - I)b^E + (A - I)b^\perp.$$ 

Then

$$|c^\perp| = |(A - I)b^\perp| \leq m^+(A)|b^\perp| \leq m(A)|b^E| = |(A - I)b^E| = |c^E|.$$ 

This shows that $\gamma$ satisfies $(*)$. Also,

$$|c|^2 = |(A - I)b^E|^2 + |(A - I)b^\perp|^2$$

$$\leq m(A)^2|b^E|^2 + m^+(A)^2|b^\perp|^2$$

$$= (|b^E|^2 + |b^\perp|^2)/4 = |b|^2/4.$$ 

Thus $|c| < |b|$, a contradiction.

*Case 2:* $\beta$ is not a pure translation. By Lemma 1.4,

$$m(C) \leq 2m(A)m(B) \leq m(B).$$

By the choice of $\alpha = (A, a)$, we have $|a| \leq |b|$. It is direct to calculate that

$$(C, c) = ([A, B], -[A, B]b - ABA^{-1}a + Ab + a).$$

We write $c = (A - I)b + r$, where

$$r = -[A, B]b - ABA^{-1}a + a + b = -([A, B] - I)b - (A(B - I)A^{-1}a).$$

We estimate

$$|r| \leq m(C)|b| + m(B)|a| \leq 2m(B)|b| \leq 4m(B)|b^E| \leq \frac{1}{2} (m(A) - m^+(A))|b^E|.$$ 

Together with

$$|c^\perp| \leq |(A - I)b^\perp| + |r| \leq m^+(A)|b^E| + |r|,$$

$$|c^E| \geq |(A - I)b^E| - |r| = m(A)|b^E| - |r|,$$
we conclude that $|c^+| < |c^E|$. Lastly,

$$|c| \leq |(A - I)b| + |r| \leq m(A)|b| + 2m(B)|b| < \frac{1}{2}|b| + \frac{1}{4}|b| < |b|.$$ 

We result in the desired contradiction and complete the proof. \(\square\)

Theorem 1.1 follows directly from Lemmas 1.6 and 1.7.

Remark 1.8. In the proof above, we never used the fact that $\pi_1(M)$-action is free. Therefore, as long as a discrete subgroup acts cocompactly on $\mathbb{R}^n$ by isometries, the conclusion in Theorem 1.1 holds.

We mention the generalized Bieberbach theorem by Fukaya-Yamaguchi [4]:

**Theorem 1.9.** Let $G$ be a closed subgroup of $\text{Isom}(\mathbb{R}^n)$. Then $G/G_0$ is virtually abelian, where $G_0$ is the identity component of $G$.

### 2. Virtually Abelian Structure

The main references for this section are [2, 3].

#### 2.1. Splitting theorem and its consequences

We need Cheeger-Gromoll’s splitting theorem:

**Theorem 2.1.** Let $M$ be a complete $n$-manifolds of $\text{Ric} \geq 0$. If $M$ contains a line, then $M$ splits isometrically as $N \times \mathbb{R}$.

Due to the splitting theorem, any open manifold $M$ of $\text{Ric} \geq 0$ splits isometrically as $N \times \mathbb{R}^k$, where $N$ has no lines.

**Proposition 2.2.** Let $M = N \times \mathbb{R}^k$ be a metric product, where $N$ has no lines. Then isometry group $\text{Isom}(M)$ splits as $\text{Isom}(N) \times \text{Isom}(\mathbb{R}^k)$.

**Proof.** For any $g \in \text{Isom}(N \times \mathbb{R}^k)$, we write

$$g(x,v) = (g_1(x,v), g_2(x,v)).$$

We need to show that $g_1$ and $g_2$ are independent of $v$ and $x$, respectively.

Note that if $c(t) = (c_1(t), c_2(t))$ is a line in $N \times \mathbb{R}^k$, then $c_1$ must be a constant because $N$ contains no lines. Let $v$ and $w$ be any two nonzero vectors in $\mathbb{R}^k$. Because $g_1(x, tv)$ is the same for all $t$, it follows that $g_1(x, tv) = g_1(x,0)$. This implies

$$g_1(x, tv) = g_1(x,0) = g_1(x, sv).$$

Thus $g_1$ only depends on $x$. In particular, $g$ maps the slice $\{x\} \times \mathbb{R}^k$ to another slice $\{g_1(x)\} \times \mathbb{R}^k$.

Observe that $N \times \{v\}$ is orthogonal to $\{x\} \times \mathbb{R}^k$. Thus under the isometry $g$, $g(N \times \{v\})$ should be orthogonal to $g(\{x\} \times \mathbb{R}^k) = \{g_1(x)\} \times \mathbb{R}^k$. It follows that $g(N \times \{v\}) = N \times \{v'\}$. This shows that $g_2(x,v) = v'$ for all $x \in N$. \(\square\)

**Theorem 2.3.** Let $M$ be a compact $n$-manifold of $\text{Ric} \geq 0$. Then $\overline{M}$, the Riemannian universal cover of $M$, splits isometrically as $N \times \mathbb{R}^k$, where $N$ is compact.

**Proof.** By Theorem 2.1, $\overline{M}$ splits isometrically as $N \times \mathbb{R}^k$, where $N$ has no lines. We show that $N$ is compact.

Suppose that $N$ is not compact. We fix a reference point $y = (x,0) \in \overline{M}$. Let $c(t)$ be a unit speed ray in $N$. For $i \in \mathbb{N}$, we consider a segment of length $2i$:

$$c_i(t) = (c(t),0), \quad t \in [0,2i].$$
Because $M$ is compact, for each $i$, there is $g_i \in \pi_1(M,x)$ such that
\[ d(g_i(c_i(i)),y) \leq \text{diam}(M). \]
By Proposition 2.2, the projection of $g_i \circ c_i$ to $\mathbb{R}^k$-factor is a single point. Passing to a subsequence if necessary, $g_i \circ c_i$ converges to a line in $\tilde{M}$, which results in a line in $N$, a contradiction. \qed

2.2. A finite cover.

**Theorem 2.4.** Let $M$ be a compact $n$-manifold of Ric $\geq 0$. Let $\tilde{M} = N \times \mathbb{R}^k$ be the universal cover of $M$, where $N$ is compact. Then

1. $M$ has a finite cover diffeomorphic to $N \times \mathbb{T}^k$;
2. $\pi_1(M)$ contains a normal subgroup $\mathbb{Z}^k$ of finite index.

**Proof.** (1). We write $\Gamma = \pi_1(M,x)$. We consider the natural projection maps from Proposition 2.2:
\[ p_1 : \Gamma \to \text{Isom}(N), \quad p_2 : \Gamma \to \text{Isom}(\mathbb{R}^k). \]
Let $H$ be the kernel of $p_2$. Since $H$ is a discrete group of $\text{Isom}(N)$, it is clear that $H$ is finite. $\Gamma/H$ acts isometrically on the quotient space $(N \times \mathbb{R}^k)/H = (N/H) \times \mathbb{R}^k$. We shall show that $M$ has a finite cover diffeomorphic to $(N/H) \times \mathbb{T}^k$.

Let $\bar{p}_1$ and $\bar{p}_2$ be the corresponding projections of the isometry group of this intermediate cover:
\[ \bar{p}_1 : \Gamma/H \to \text{Isom}(N/H), \quad \bar{p}_2 : \Gamma/H \to \text{Isom}(\mathbb{R}^k). \]
Note that $\bar{p}_2(\Gamma/H)$ acts isometrically and co-compactly on $\mathbb{R}^k$. By Theorem 1.1, $\bar{p}_2(\Gamma/H)$ contains a normal subgroup $A$ isomorphic to $\mathbb{Z}^k$. Let $\tilde{A} = \bar{p}_2^{-1}(A)$. Because $\bar{p}_2$ is injective, $\tilde{A}$ is isomorphic to $\mathbb{Z}^k$ as well. Ideally, we wish to show that $\tilde{A}$-action on $(N/H) \times \mathbb{R}^k$ is equivariant to $A$-action on $(N/H) \times \mathbb{R}^k$. If this is true, then $((N/H) \times \mathbb{R}^k)/\tilde{A}$ is diffeomorphic to $(N/H) \times (\mathbb{R}^k)/A = (N/H) \times \mathbb{T}^k$. We claim that this can be done if $\bar{p}_1 : \Gamma/H \to \text{Isom}(N/H)$ admits an extension to $\mathbb{R}^k$.

**Claim:** If $\bar{p}_1 : \tilde{A}(= \mathbb{Z}^k) \to \text{Isom}(N/H)$ can be extended to a group homomorphism $\psi : \mathbb{R}^k \to \text{Isom}(N/H)$, then
\[ f : (N/H) \times \mathbb{R}^k \to (N/H) \times \mathbb{R}^k \]
\[ (z,v) \mapsto (\psi(v)^{-1}z, v) \]
is an equivariant diffeomorphism.

We can verify this claim directly. In fact, for any $\hat{a} \in \tilde{A}$ with $\bar{p}_2(\hat{a}) = a \in A$,
\[ a \cdot f(z,v) = a \cdot (\psi(v)^{-1}z, v) = (\psi(v)^{-1}z, a \cdot v); \]
\[ f(\hat{a} \cdot (z,v)) = f(\bar{p}_1(a) \cdot z, a \cdot v) = (\psi(a \cdot v)^{-1}\bar{p}_1(a) \cdot z, a \cdot v) \]
\[ = (\psi(v)^{-1}\psi(a)^{-1}\bar{p}_1(a) \cdot z, a \cdot v) = (\psi(v)^{-1}z, a \cdot v). \]

However, such an extension is not always guaranteed. To overcome this, we shall replace $\tilde{A}$ by a normal subgroup of finite index. Note that the closure $\bar{p}_1(\tilde{A}) =: G$ is a compact abelian subgroup of $\text{Isom}(N/H)$. Thus its identity component $G_0$ is a normal toral subgroup with finite index in $G$. Let $\hat{B}$ be the pre-image of $G_0$ in $\tilde{A}$. $\hat{B}$ has finite index in $\tilde{A}$, thus $\hat{B}$ is isomorphic to $\mathbb{Z}^k$ as well. Then
\[ \bar{p}_1 : \hat{B}(= \mathbb{Z}^k) \to \bar{p}_1(\hat{B}) \subset G_0 \]
can be extended to a group homomorphism
\[ \psi : \mathbb{R}^k \to G_0 \subset \text{Isom}(N/H) \]
by defining it through one-parameter subgroups.
Let \( B = \bar{p}_2(B) \). Then by the above claim, \( B \)-action on \( (N/H) \times \mathbb{R}^k \) is equivariant to \( B \)-action on \( (N/H) \times \mathbb{R}^k \). It follows that \( ((N/H) \times \mathbb{R}^k)/\hat{B} \) is diffeomorphic to \( (N/H) \times T^k \). This proves (1).

(2) follows directly from (1). \( \square \)

It is open whether there is a universal bound on the index of the abelian subgroup.

Conjecture 2.5 (Fukaya-Yamaguchi). Given \( n \), there is a constant \( C(n) \) such that for any compact \( n \)-manifold of \( \text{Ric} \geq 0 \), \( \pi_1(M) \) has a normal abelian subgroup of index \( \leq C(n) \).

3. A viewpoint from nilpotency

Milnor [7]:

Theorem 3.1. Let \( M \) be a complete \( n \)-manifold of \( \text{Ric} \geq 0 \). Then any finitely generated subgroup of \( \pi_1(M) \) has polynomial growth of degree \( \leq n \).

Gromov [6]:

Theorem 3.2. Any finitely generated group of polynomial growth is virtually nilpotent.

If we apply Theorems 3.1 and 3.2 above, then to prove the virtually abelian structure, we can assume that \( \pi_1(M) \) is nilpotent without lose of generality.

The following theorem is important in extracting an abelian group from a nilpotent one.

Theorem 3.3. Any compact connected nilpotent Lie group is a torus.

Lemma 3.4. Let \( G \) be a nilpotent subgroup of \( \text{Isom}(\mathbb{R}^n) \). Let \( (A, x) \) and \( (B, y) \) be two elements of \( G \). Then \( (A, x) \) and \( (B, y) \) commutes if and only if \( A \) and \( B \) commutes.

Proof. The proof is linear algebra. By direct calculation, we have
\[ [(A, x), (B, y)] = ([A, B], -[A, B]y - ABA^{-1}x + Ay + x). \]
Clearly if \( (A, x) \) and \( (B, y) \) commutes, so does \( A \) and \( B \).

Conversely, if \( A \) and \( B \) commutes, then
\[ [(A, x), (B, y)] = (I, -y - Bx + Ay + x), \]
which is a translation. We denote this vector as \( w = -y - Bx + Ay + x \).

Since \( G \) is nilpotent, after \( l \) times of commutator calculation, we result in
\[ [(A, x), \ldots, [(A, x), (I, w)]] = (I, 0). \]
It is easy to verify that the left hand side equals to \( (I, (A - I)^l w) \). Thus
\[ (A - I)^l w = 0 \]
for some \( l \). With the fact that \( A \in O(n) \), we have
\[ (A - I)^l w = 0 \quad \text{if and only if} \quad (A - I)w = 0. \]
Therefore, $Aw = w$. Similarly, we have $Bw = w$. Since $A$ and $B$ commutes, they share the same eigen-space decomposition. We define a subspace $E$ as

$$E = \{ v \in \mathbb{R}^n \mid Av = v = Bv \}$$

and decompose $\mathbb{R}^n$ as $E + E^\perp$, where $E^\perp$ is the orthogonal complement of $E$. We write $x = x^E + x^\perp$ and $y = y^E + y^\perp$ according to this decomposition. Then

$$w = Ay^\perp - y^\perp + x^\perp - Bx^\perp,$$

which is in $E^\perp$. Since $w \in E$, we conclude that $w = 0$ and complete the proof. □

**Proposition 3.5.** Let $\Gamma$ be a discrete nilpotent subgroup of $\text{Isom}(\mathbb{R}^n)$. Then $\Gamma$ is virtually abelian.

**Proof.** We consider the group homomorphism

$$\pi: \text{Isom}(\mathbb{R}^n) \to O(n) \quad (A, x) \mapsto A$$

Let $H$ be the closure of $\pi(\Gamma)$ in $O(n)$. $H$ is a compact nilpotent Lie group (could be a finite group). Let $H_0$ be the identity component of $H$, which has finite index in $H$. By Theorem 3.3, $H_0$ is a torus. Therefore, by Lemma 3.4, $\pi^{-1}(H_0)$ is an abelian subgroup of finite index in $\Gamma$. □

**Proposition 3.6.** Let $\Gamma$ be a discrete nilpotent subgroup of $K \times \text{Isom}(\mathbb{R}^k)$, where $K$ is a compact Lie group. Then $\Gamma$ is virtually abelian.

**Proof.** Let $p_1 : \Gamma \to K$, $p_2 : \Gamma \to \text{Isom}(\mathbb{R}^k)$ be the natural projection. By Proposition 3.5, $p_2(\Gamma)$ has an abelian subgroup $A_2$ of finite index. Also, by the same argument in Proposition 3.5, $p_1(\Gamma)$ has an abelian subgroup $A_1$ of finite index. Clearly, $\Gamma \cap (A_1 \times A_2)$ is the desired abelian subgroup of finite index in $\Gamma$. □

**Remark 3.7.** In the proof of Proposition 3.5, $H_0$ is indeed central in $H$. Using this, one can improve the result in Propositions 3.5 and 3.6: the center of $\Gamma$ has finite index in $\Gamma$.

**References**