Objective: The objective of this project will be to gain an understanding of a new type of mathematical object known as a differential form. These are the objects that we deal with when we do integrations over various subsets of $\mathbb{R}^n$. We will introduce the concept of a $k$-form, then talk about ways to add, “multiply”, and differentiate these objects. Finally we will study how the classical integration theorems we have learned are special cases of a single theorem.

Definition 1. A 0-form in $\mathbb{R}^3$ is a differentiable real-valued function $f : \mathbb{R}^3 \rightarrow \mathbb{R}$.

Recall that the set of 0-forms (i.e. the set of differentiable functions) form a vector space.

Definition 2. The objects $dx$, $dy$, $dz$ are examples of 1-forms. A 1-form in $\mathbb{R}^3$ is an object of the form $\alpha = f(x,y,z)dx + g(x,y,z)dy + h(x,y,z)dz$ where $f,g,$ and $h$ are differentiable real-valued functions.

You can think of the set of 1-forms as a vector space whose coefficients are functions. Just treat $f,g,$ and $h$ above as coefficients, and $\{dx,dy,dz\}$ as the basis for a vector space. Then if $\alpha_1 = f_1dx + g_1dy + h_1dz$ and $\alpha_2 = f_2dx + g_2dy + h_2dz$, then $\alpha_1 + \alpha_2 = (f_1 + f_2)dx + (g_1 + g_2)dy + (h_1 + h_2)dz$. Therefore the zero 1-form corresponds to the form $0dx + 0dy + 0dz$. If $f$ is a 0-form and $\alpha = f_1dx + g_1dy + h_1dz$ is a 1-form, then $f\alpha = ff_1dx + fg_1dy + fh_1dz$ is a 1-form. We will now like to come up with a way to “multiply” two 1-forms together. We will call this the wedge product.

Definition 3. The wedge product (for 1-forms) is an operation on 1-forms which satisfies the following properties:

\begin{itemize}
\item[a)] \(dx \wedge dx = dy \wedge dy = dz \wedge dz = 0\)
\item[b)] For any two 1-forms $\alpha$ and $\beta$, $\alpha \wedge \beta = -\beta \wedge \alpha$.
\item[c)] For any two 1-forms $\alpha$ and $\beta$ and any 0-form $f$, $(f\alpha) \wedge \beta = \alpha \wedge (f\beta) = f(\alpha \wedge \beta)$.
\item[d)] The wedge product is distributive (i.e. $\alpha \wedge (\beta + \gamma) = \alpha \wedge \beta + \alpha \wedge \gamma$ for any 1-forms $\alpha, \beta,$ and $\gamma$).
\end{itemize}
Example 1. Let $\alpha = ydx - xdy$ and $\beta = xdx$, then $\alpha \wedge \beta = (yx)dx \wedge dx - x^2dy \wedge dx = x^2dx \wedge dy$ since $dx \wedge dx = 0$.

Exercise 1. Let $\alpha_1 = f_1dx + g_1dy + h_1dz$ and $\alpha_2 = f_2dx + g_2dy + h_2dz$ be two 1-forms. Calculate $\alpha_1 \wedge \alpha_2$.

Definition 4. The objects $dx \wedge dy$, $dx \wedge dz$, and $dy \wedge dz$ are examples of 2-forms. A 2-form in $\mathbb{R}^3$ is an object of the form $\alpha = f(x, y, z)dx \wedge dy + g(x, y, z)dy \wedge dz + h(x, y, z)dz \wedge dx$ where $f, g,$ and $h$ are differentiable real-valued functions.

Exercise 2. Again, just as in the case of 0-forms and 1-forms, you can think of the set of 2-forms as a vector space with functions as coefficients.

a) What would be the “basis” for this vector space?

b) Let $\alpha_1$ and $\alpha_2$ be any two 2-forms. Define $\alpha_1 + \alpha_2$.

c) If $\alpha$ is a 2-form and $f$ is a 0-form, define $f\alpha$.

d) What is the zero 2-form?

Notice that if $\alpha$ and $\beta$ are 1-forms, then $\alpha \wedge \beta$ is a 2-form. In general, a $k$-form will consist of parts which look like $f(\vec{x})dx_1 \wedge \ldots \wedge dx_k$. Above we came up with the wedge product for 1-forms. Now we want to discuss the wedge product between any 2 forms.

Definition 5. The wedge product is an operation on forms that is defined by the following properties:

a) If $\alpha$ is a $k$-form and $\beta$ is an $l$-form, then $\alpha \wedge \beta$ is a $(k + l)$-form.

b) If $\alpha$ is a $k$-form and $\beta$ is an $l$-form, then $\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$.

c) If $\alpha, \beta,$ and $\gamma$ are forms, then $(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)$.

d) If $\alpha$ is a form and $f$ is a 0-form, define $f\alpha$.

e) If 0 is the zero $k$-form, then $\alpha \wedge 0 = 0$ for any $\alpha$.

f) If $\alpha$ and $\beta$ are both $k$-forms and $\gamma$ is any form, then $(\alpha + \beta) \wedge \gamma = \alpha \wedge \gamma + \beta \wedge \gamma$.

g) If $f$ is a real valued function and $\alpha$ and $\beta$ are any forms, then $f\alpha \wedge \beta = (f\alpha) \wedge \beta = \alpha \wedge (f\beta)$.

Exercise 3. Complete the following problems:

a) Let $\alpha = f_1dx + g_1dy + h_1dz$ be any 1-form and $\beta = f_2dx \wedge dy + g_2dy \wedge dz + h_2dz \wedge dx$ be any 2-form. Calculate $\alpha \wedge \beta$.

b) Write down the most general 3-form in $\mathbb{R}^3$. 

c) Let $\alpha = dx$ and $\beta = dx \wedge dy \wedge dz$ and calculate $\alpha \wedge \beta$. By considering your answer, complete the following sentence: All $k$-forms in $\mathbb{R}^3$ are the zero form if $k \geq \underline{\phantom{0}}$. Why?

We know how to differentiate functions (0-forms). Now we want to be able to differentiate other types of forms. You should already be familiar with the “differential” of a function $df$. Recall that if $f(x)$ is a differentiable function, then $df = f'(x)dx$. In other words if $f$ is a 0-form, then $df$ is a 1-form. For a differentiable function $f(x, y, z)$, the differential of $f$ is the 1-form $df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$. We are now ready to talk about the differential of a form.

Definition 6. The differential is an operation on forms that is defined by the following properties:

a) If $\alpha$ is a $k$-form, $d\alpha$ is a $(k + 1)$-form.

b) If $\alpha$ and $\beta$ are $k$-forms, then $d(\alpha + \beta) = d\alpha + d\beta$.

c) If $f$ is a 0-form, then $df$ is the ordinary differential (see above).

d) If $\alpha$ is a $k$-form and $\beta$ is an $l$-form, then $d(\alpha \wedge \beta) = (d\alpha) \wedge \beta - (1)^k \alpha \wedge d(\beta)$.

e) For any form $\alpha$, $d^2\alpha = d(d\alpha) = 0$.

Example 2. If $\alpha = (x^2 - y)dz$, then $d\alpha = d(x^2 - y) \wedge dz + (-1)^0(x^2 - y)d(dz) = (2xdx - dy) \wedge dz = 2xdx \wedge dz - dy \wedge dz$ since $(x^2 - y)$ is a 0-form and $d^2z = 0$.

Exercise 4. Use the definition above to calculate:

a) If $f(x, y, z) = e^{xyz}$, calculate $df$.

b) If $\alpha = f_1(x, y, z)dx + f_2(x, y, z)dy + f_3(x, y, z)dz$, calculate $d\alpha$.

c) If $\beta = f_1(x, y, z)dy \wedge dz + f_2(x, y, z)dz \wedge dx + f_3(x, y, z)dx \wedge dy$, calculate $d\beta$.

Exercise 5. Complete the following problems:

a) For $f(x, y, z) = e^{xyz}$, calculate $\nabla f$. Notice that $df$ and $\nabla f$ have the same components.

b) Let $F = (f_1(x, y, z), f_2(x, y, z), f_3(x, y, z))$, calculate $\nabla \times F$ and $\nabla \cdot F$. Do you notice any similarities between these objects and your answers to parts b) and c) above? Explain.

c) Let $f$ be a function and $\alpha$ a 1-form. By considering your answers to the previous problem, which well known vector identities do $d^2f = 0$ and $d^2\alpha = 0$ represent?
So what do we use these forms for? They should look familiar to some extent. We are used to integrating things like $f(x)dx$, $f(x,y)dxdy$, and $f(x,y,z)dxdydz$. Hence forms are the things that we integrate over various subsets of $\mathbb{R}^n$. When doing a line integral or a surface integral, we are actually integrating a 1-form or a 2-form respectively. Now we are ready to understand the ultimate integration theorem.

**Theorem 1.** Let $M$ be a $k$-dimensional subset of $\mathbb{R}^n$ (a curve, a surface, a three-dimensional solid, etc.). Let $\partial M$ denote the boundary of $M$. Let $\alpha$ be a $k-1$ form. Then

$$\int_{\partial M} \alpha = \int_M d\alpha.$$  

All the integration theorems that we have learned this quarter are special cases of the theorem above. For example, if $M$ is a curve in $\mathbb{R}^3$ with endpoints $A$ and $B$, then $\partial M$ consists of the endpoints $A$ and $B$. If $f$ is a 0-form, then $df$ corresponds to $\nabla f \cdot ds$. Hence the theorem above tells us that $\int_M \nabla f \cdot ds = f(B) - f(A)$ which is the fundamental theorem of calculus.

**Exercise 6.** Consider the theorem above in $\mathbb{R}^3$.

a) Let $\alpha$ be a 1-form and $M$ be a 2-dimensional subset of $\mathbb{R}^3$ with boundary $\partial M$. Write out what the theorem above means in this case. Which classical integration theorem does this resemble?

b) Let $\alpha$ be a 2-form and $M$ be a 3-dimensional subset of $\mathbb{R}^3$ with boundary $\partial M$. Write out what the theorem above means in this case. Which classical integration theorem does this resemble?

The underlying theme of this theorem is that the integral of the derivative of a function over an object $M$ does not depend on all of $M$, just the boundary. We can use this theorem to do some pretty cool things. We have already seen that we can calculate the area of a planar region just by knowing the shape of its boundary. As a physical example, we can use these ideas to calculate the temperature of the core of the earth without actually taking a thermometer down there.

**Exercise 7.** Discuss an application of one of the classical integration theorems (a short paragraph will suffice).

**Exercise 8.** Write out in full what the theorem above in $\mathbb{R}^4$ says for $k$-forms when:

a) $k = 1$

b) $k = 2$

c) $k = 3$

*Hint: You will need to figure out what 1-forms, 2-forms, and 3-forms mean in $\mathbb{R}^4$ and then figure out what their differentials are. Just follow what I have discussed above, starting with functions of 4 variables, $f(x, y, z, t)$, as your 0-forms.*