Objective: The objective of this lesson will be to learn about inner product spaces, projections, and how they relate to Fourier series.

Definition 1. An inner product on a vector space \( V \) is a function \( \langle \cdot, \cdot \rangle : V \times V \to \mathbb{R} \) that satisfies the following properties for all \( u, v, w \in V \) and \( \alpha \in \mathbb{R} \):

a) \( \langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle \)

b) \( \langle \alpha u, v \rangle = \alpha \langle u, v \rangle \)

c) \( \langle u, v \rangle = \langle v, u \rangle \)

d) \( \langle v, v \rangle \geq 0 \) and equal to 0 if and only if \( v = 0 \)

An inner product space is a vector space together with an inner product.

Exercise 1. Complete the following problems.

a) Show that \( \langle v, w \rangle = v \cdot w \) defines an inner product on \( \mathbb{R}^3 \).

b) Show that \( \langle v, w \rangle = v^T M w \) defines an inner product on \( \mathbb{R}^3 \) where

\[
M = \begin{pmatrix} 2 & -1 & 0 \\
-1 & 2 & -1 \\
0 & -1 & 2 \end{pmatrix}
\]

This will still be an inner product for any other symmetric positive definite matrix \( M \).

c) Show that \( \langle f, g \rangle = \int_a^b f(t) g(t) dt \) defines an inner product on the vector space of continuous functions defined on \( [a, b] \).

Inner products add additional structure to a vector space. It gives a way to tell whether two vectors are “orthogonal” and also allows one to project a vector onto another. With an inner product we can also define the length of a vector and the angle between two vectors. For example, the length of a vector \( v \) is defined as \( \|v\| = \sqrt{\langle v, v \rangle} \) and \( \cos \theta = \frac{\langle v, w \rangle}{\|v\| \|w\|} \) gives the angle between two vectors.
It can be shown that $|v| = \sqrt{\langle v, v \rangle}$ satisfies the following properties for all $v, w \in V$ and $\alpha \in \mathbb{R}$:

a) $||v + w|| \leq ||v|| + ||w||$

b) $||\alpha v|| = |\alpha| \cdot ||v||$

c) $||v|| \geq 0$ with equality if and only if $v = 0$

A function which satisfies these properties is called a norm.

**Exercise 2.** For each inner product in Exercise 1, show that the associated norm satisfies the 3 properties above.

With an inner product we can define the projection of one vector onto another. Suppose we have two vectors $v$ and $w$ and we want to define the projection of $v$ onto $w$, call it $P(v, w)$. The magnitude of this vector should be $||v|| \cos \theta$ and it should point in the $w$ direction ($\theta$ is the angle between $v$ and $w$). Recall that $\cos \theta = \frac{\langle v, w \rangle}{||v|| \cdot ||w||}$. So the magnitude of the projection should be $||v|| \cos \theta = \frac{\langle v, w \rangle}{||w||} \cdot ||w||$.

Now $\frac{w}{||w||}$ is a vector of unit length that points in the same direction as $w$. So we should define $P(v, w) = \frac{\langle v, w \rangle}{||w||} \cdot \frac{w}{||w||}$. Now since $\langle w, w \rangle = ||w||^2$, we can write the projection as $P(v, w) = \frac{\langle v, w \rangle}{\langle w, w \rangle} w$.

**Definition 2.** Given any two vectors $v$ and $w$, define the projection $P(v, w)$ of $v$ onto $w$ as $P(v, w) = \frac{\langle v, w \rangle}{\langle w, w \rangle} w$.

**Exercise 3.** Use the definition of the projection given above to complete the following problems:

a) Draw a picture that illustrates what it means to project a vector $v$ onto a vector $w$. Make sure you clearly label $v$, $w$, $P(v, w)$, $\theta$, and the magnitude of $P(v, w) = ||v|| \cos \theta$.

b) Find the projection of the vector $(1, -3, 4)$ on the vector $(-5, 2, 1)$ using the inner product defined in Exercise 1a).

c) Find the projection of the vector $(1, -3, 4)$ on the vector $(-5, 2, 1)$ using the inner product defined in Exercise 1b).

d) Find the projection of the vector $x$ on the vector $\cos (\pi x)$ defined on $[0, 1]$ using the inner product defined in Exercise 1c).

**Definition 3.** A set of vectors $\{v_1, v_2, \ldots, v_n, \ldots\}$ is said to be an orthogonal set if $\langle v_i, v_j \rangle = 0$ if $i \neq j$.

**Exercise 4.** Complete the following problems:

a) Show that $\{(1, 1, -1), (-2, 3, 1), (4, 1, 5)\}$ is an orthogonal set using the inner product defined in Exercise 1a). Can you add any additional non-zero vectors to this set and still have it be orthogonal?
b) Using the inner product defined in Exercise 1b), find an orthogonal set containing 3 non-zero vectors.

c) Show that the set \( \{1, \sin(x), \cos(x), \sin(2x), \cos(2x), \ldots, \sin(nx), \cos(nx), \ldots\} \) is an orthogonal set using the inner product defined in Exercise 1c) on \([−\pi, \pi]\).

We can also talk about the projection of a vector onto the subspace of a vector space spanned by an orthogonal set. Let \( w_1 \) and \( w_2 \) be orthogonal vectors and consider the subspace \( S = \{c_1 w_1 + c_2 w_2 | c_1, c_2 \in \mathbb{R}\} \). If \( v \) is any other vector then the projection of \( v \) onto \( S \) is \( P(v, S) = P(v, w_1) + P(v, w_2) \). So in general, if \( \{w_1, w_2, \ldots, w_n\} \) is an orthogonal set and \( S = \{c_1 w_1 + c_2 w_2 + \cdots + c_n w_n | c_i \in \mathbb{R}\} \), then the projection of \( v \) onto \( S \) is \( P(v, S) = \sum_{k=1}^{n} P(v, w_k) \).

**Exercise 5.** Complete the following exercises:

a) Draw a picture representing the projection of a vector \( v \) onto the plane spanned by \( u \) and \( w \) (assume \( v \) is not on this plane), completely label every quantity (\( u, w, v, P(v, u), P(v, w), P(v, S) \)).

b) Find the projection of the vector \((1, 2, 2)\) on the plane spanned by \((1, 1, -1)\) and \((4, 1, 5)\) using the inner product defined in Exercise 1a).

c) Find the projection of the vector \((1, 2, 2)\) on the plane spanned by \((1, \sqrt{2}, 1)\) and \((1, -\sqrt{2}, 1)\) using the inner product defined in Exercise 1b).

d) Find the projection of \( x \) on the subspace of \( C[0, 1] \) spanned by \( \cos(\pi x) \) and \( \sin(\pi x) \) using the inner product defined in Exercise 1c).

So how does all this connect to Fourier series? We will see in the following exercise:

**Exercise 6.** Complete the following exercises:

a) Show that the set of periodic functions with period \( 2\pi \) is a vector space, let’s call it \( \Pi \). We will only be considering those functions which are integrable.

b) Show that the set \( T = \{1, \sin(x), \cos(x), \sin(2x), \cos(2x), \ldots, \sin(nx), \cos(nx), \ldots\} \) belongs in \( \Pi \). This set is orthogonal by a previous exercise using the inner product defined in Exercise 1c) on \([−\pi, \pi]\) (and also on any other interval of length \( 2\pi \)). Let \( S \) be the subspace of \( \Pi \) spanned by the vectors in set \( T \).

c) Using the inner product defined in Exercise 1c) on \([−\pi, \pi]\), calculate \( P(f, S) \) for a function \( f \in \Pi \). What have we called this object in class? (For full credit on this problem, you should evaluate all possible integrals and write out each term of the resulting series as a single integral times a single function).

d) Using part c), write out the Fourier series coefficient formulas we learned in class in terms of inner products.