Summary of Convergence Tests

**Divergence Test:** If \( \lim_{k \to \infty} u_k \neq 0 \), then the series \( \sum_{i=0}^{\infty} u_k \) diverges.

-This is a test to see if a series diverges, it doesn’t give any information about whether the series converges.

**Integral Test:** If \( f(x) \) is a continuous function which is positive and decreasing such that \( f(k) = u_k \) for all integers \( k \geq a \), then \( \int_{a}^{\infty} f(x) dx \) and \( \sum_{i=0}^{\infty} u_k \) either both converge or both diverge.

-Use this test when the function you get by replacing \( k \) with \( x \) in \( u_k \) is decreasing, positive, and easy to integrate.

**Comparison Test:** If \( 0 \leq u_k \leq v_k \) then

a) If \( \sum_{i=0}^{\infty} v_k \) converges then \( \sum_{i=0}^{\infty} u_k \) converges.

b) If \( \sum_{i=0}^{\infty} u_k \) diverges then \( \sum_{i=0}^{\infty} v_k \) diverges.

-Use this test if you can directly compare your series to a series that you know the convergence properties of (like a p-series).

**Limit Comparison Test:** If \( u_k \) and \( v_k \) are positive and \( \rho = \lim_{k \to \infty} \frac{u_k}{v_k} \) then if \( 0 < \rho < \infty \), \( \sum_{i=0}^{\infty} v_k \) and \( \sum_{i=0}^{\infty} u_k \) either both converge or both diverge.

-Use this test if you suspect that your series may behave similarly to a series that you know the convergence properties of (like a p-series). Since no direct comparison is necessary, this will likely be easier to apply than the comparison test.

**Ratio Test:** Suppose that \( u_k > 0 \) and \( \rho = \lim_{k \to \infty} \frac{u_{k+1}}{u_k} \), then:

a) If \( \rho < 1 \), \( \sum_{i=0}^{\infty} u_k \) converges.

b) If \( \rho > 1 \), \( \sum_{i=0}^{\infty} u_k \) diverges.

c) If \( \rho = 1 \), then the test fails. The series may either converge or diverge. Use another test.

-Use this test when \( u_k \) contains powers of \( k \) or factorials.
**Root Test:** Suppose that $u_k > 0$ and $\rho = \lim_{k \to \infty} \sqrt[k]{u_k}$, then:

a) If $\rho < 1$, $\sum_{i=0}^{\infty} u_k$ converges.

b) If $\rho > 1$, $\sum_{i=0}^{\infty} u_k$ diverges.

c) If $\rho = 1$, the test fails. The series may either converge or diverge. Use another test.

-Use this test when $u_k$ contains powers of $k$.

**Alternating Series Test:** Let $\sum_{k=0}^{\infty} u_k$ be an alternating series then if:

a) $|u_1| \geq |u_2| \geq |u_3| \geq \ldots$

b) $\lim_{k \to \infty} |u_k| = 0$

then the series $\sum_{k=0}^{\infty} u_k$ converges.

-This test only applies to alternating series. This test is only a test for convergence. If an alternating series fails properties a) or b), it does not necessarily mean that it will diverge.

**Geometric Series** A series of the form $\sum_{k=0}^{\infty} a \cdot r^k$ is called a geometric series ($a \neq 0$). If $|r| < 1$ the series will converge to $\frac{a}{1-r}$ and if $|r| \geq 1$ the series will diverge.

**P-series** A series of the form $\sum_{k=1}^{\infty} \frac{1}{k^p}$ with $p > 0$ is called a p-series. If $p \leq 1$ the series diverges. If $p > 1$ the series converges.

**Definition 1.** A series $\sum_{k=0}^{\infty} u_k$ is said to absolutely converge if $\sum_{k=0}^{\infty} |u_k|$ converges. A series $\sum_{k=0}^{\infty} u_k$ is said to absolutely diverge if $\sum_{k=0}^{\infty} |u_k|$ diverges. If a series $\sum_{k=0}^{\infty} u_k$ converges but $\sum_{k=0}^{\infty} |u_k|$ diverges, then $\sum_{k=0}^{\infty} u_k$ is said to conditionally converge.

**Theorem 1.** If a series $\sum_{k=0}^{\infty} u_k$ absolutely converges, then it converges.