Math 8: Midterm Solution

1) 
a) \(A \cup B = \{x : x \in A \lor x \in B\}\)
b) \(x\) is rational if \(x = \frac{p}{q}\) where \(p, q \in \mathbb{Z}\) and \(q \neq 0\)
c) An integer \(p\) is prime if \(p \geq 2\) and for every integer \(d \geq 0\), if \(d\) divides \(p\), then \(d = 1\) or \(d = p\)

2) 
a) All dogs are blue and some birds can fly
b) \(p\) is prime and \(p\) is even and \(p\) is not equal to 2, or \(p\) is equal to 2, and \(p\) is not prime or \(p\) is not even

c) You either have more than one soulmate or you have no soulmate

3) 
a) False. If \(a = 1\), \(b = 2\), and \(c = 4\), then \(a\) divides \(b\), \(b\) divides \(c\), but \(b\) does not divide \(a + c\) (2 does not divide 5)
b) True. \(x \in (A - C) \cap (B - C) \iff x \in (A - C) \land x \in (B - C) \iff (x \in A - x \not\in C) \land (x \in B - x \not\in C) \iff (x \in A \land x \in B) \land (x \in C \land x \in C) \iff x \in (A \cap B) - C\)
c) True. If \(a - b\) is odd, then \(a - b = 2k + 1\) for some integer \(k\). So \(a + b = a - b + 2b = 2k + 1 + 2b = 2(k + b) + 1\). Therefore \(a + b\) is odd.
d) False. \(A\) is an element of the power set of \(A\), not a subset. Let \(A = \{1\}\).

4) 
a) Assume \(a + x\) is rational. Then \(a + x = \frac{p}{q}\) where \(p, q \in \mathbb{Z}\) and \(q \neq 0\). Also \(a = \frac{r}{s}\) where \(r, s \in \mathbb{Z}\) and \(s \neq 0\). So \(x = \frac{p}{q} - \frac{r}{s} = \frac{ps - qr}{qs}\) and \(qs \neq 0\). So \(x\) is rational. So we have proved the contrapositive, so by contraposition the claim is true.
b) For \(n = 1\), \(5 \geq 5\). Assume that \(5^n \geq 1 + 4n\). Then \(5^{n+1} = 5 \cdot 5^n \geq 5 \cdot (1 + 4n) = 5 + 20n \geq 1 + 4 + 4n = 1 + 4(n + 1)\)
Extra Credit:
Prove by contraposition. If $T \subseteq \mathbb{N}$ and $T$ does not have a smallest element, then $T = \emptyset$. $1 \in T$ because $T$ does not have a smallest element. Assume $1, \cdots, n$ are not in $T$. If $n + 1 \in T$ then $n + 1$ would be the smallest element of $T$. But $T$ has no smallest element, so $n + 1 \notin T$. Therefore $T$ contains no natural numbers.

So $T = \emptyset$. 