Math 8: Homework #4 Solution

2.3 - 6b
b) Take any $b \in B \cup (\cap_{\alpha \in \Delta} A_{\alpha})$. Then $b \in B$ or $b \in \cap_{\alpha \in \Delta} A_{\alpha}$. If $b \in B$, then $b \in B \cup A_{\alpha}, \forall \alpha \in \Delta$. So $b \in \cap_{\alpha \in \Delta}(B \cup A_{\alpha})$. If $b \in \cap_{\alpha \in \Delta} A_{\alpha}$, then $b \in A_{\alpha}, \forall \alpha \in \Delta$. So $b \in \cap_{\alpha \in \Delta}(B \cup A_{\alpha})$. Therefore $B \cup (\cap_{\alpha \in \Delta} A_{\alpha}) \subseteq \cap_{\alpha \in \Delta}(B \cup A_{\alpha})$. Now take any $a \in \cap_{\alpha \in \Delta}(B \cup A_{\alpha})$ then $a \in B \cup A_{\alpha}, \forall \alpha \in \Delta$. So $a \in B$ or $a \in A_{\alpha}, \forall \alpha \in \Delta$. If $a \in B$, then $a \in B \cup (\cap_{\alpha \in \Delta} A_{\alpha})$. If $a \in A_{\alpha}, \forall \alpha \in \Delta$, then $a \in B \cup (\cap_{\alpha \in \Delta} A_{\alpha})$. Therefore, $\cap_{\alpha \in \Delta}(B \cup A_{\alpha}) \subseteq B \cup (\cap_{\alpha \in \Delta} A_{\alpha})$.

2.3 - 7b
b) $(\cap_{\alpha \in \Delta} A_{\alpha}) \cup (\cap_{\beta \in \Gamma} A_{\beta}) = (\cap_{\alpha \in \Delta} A_{\alpha}) \cup B_{\beta} = (\cap_{\alpha \in \Delta} A_{\alpha})$.

2.3 - 8a, d
a) false. Choose your sets such that each $A_{\alpha} \subseteq B$, and all the $A_{\alpha}$’s are disjoint. For instance $B = \{1, 2, 3, 4\}$ and $A_1 = \{1\}, A_2 = \{2\}$.
d) true. Writing out the definition of the left hand side, $\{x : x \in A_{\alpha}$ for some $\alpha \in \Delta$ and $x \notin B\}$. Definition of the right hand side, $\{x : x \in (A_{\alpha} - B)$ for some $\alpha \in \Delta\}$. So $x \in A_{\alpha}$ for some $\alpha \in \Delta$ and $x \notin B$. Both definitions are the same.

2.4 - 8a, c, e
a) $n = 1$ case: $2^1 = 2^2 - 2$. Assume that $\sum_{i=1}^{n} (3i - 2) = \frac{1}{2}(3n - 1)$ for some $n \geq 1$. And $\sum_{i=1}^{n-1} (3i - 2) = 3(n+1) - 2 + \sum_{i=1}^{n-1} (3i - 2) = 3(n+1) - 2 + \frac{1}{2}n(3n-1) = 3n+1 + \frac{1}{2}n(3n-1) = \frac{1}{2}(6n+2+3n^2-n) = \frac{1}{2}(3n^2+5n+2) = \frac{1}{2}(n+1)(3(n+1)-1).

b) $n = 1$ case: $2^1 = 2^2 - 2$. Assume that $\sum_{i=1}^{n} 2^i = 2^{n+1} - 2$ for some $n \geq 1$. Then $\sum_{i=1}^{n} 2^i = 2^{n+1} + \sum_{i=1}^{n} 2^i = 2^{n+1} + 2^{n+1} - 2 = 2(2^{n+1}) - 2 = 2^{n+2} - 2$.

c) $n = 1$ case: $\frac{3n^5 + 5n^3 + 2n}{15} = 1$ which is an integer. Now assume that the claim is true up to some $n \geq 1$. Then $\frac{3(n+1)^5 + 5(n+1)^3 + 2(n+1)}{15} = \frac{3n^5 + 5n^3 + 2n}{15} + \frac{15n^4 + 30n^2 + 45n + 30}{15}$. Now by breaking up the fraction into $\frac{2n^5 + 5n^3 + 2n}{15} + \frac{15n^4 + 30n^2 + 45n + 30}{15}$ we see that the first term is the integer we got by induction, and the second term is clearly an integer since each term is divisible by 15.
2.4 - 9b, d

b) \( n = 5 \) case: \((5 + 1)! = 720 > 256\). Assume that \((n+1)! > 2^{n+3}\) for some \(n \geq 5\). Then \((n+1+1)! = (n+2)(n+1)! > (n+2)2^{n+3} > 2\times 2^{n+3} = 2^{n+1+3}\)

d) \( n = 2 \) case: \( \sqrt{2} < 1 + \frac{1}{\sqrt{2}} \). Assume that \( \sqrt{n} < \sum_{i=1}^{n} \frac{1}{\sqrt{i}} \) for some \( n \geq 2 \).

Then \( \sqrt{n} + \frac{1}{\sqrt{n+1}} < \sum_{i=1}^{n} \frac{1}{\sqrt{i}} + \frac{1}{\sqrt{n+1}} \). And since \( n < \sqrt{n(n+1)} \), then \( n + 1 < \sqrt{n(n+1)} + 1 \), so \( \sqrt{n+1} < \sqrt{n} + \frac{1}{\sqrt{n+1}} < \sum_{i=1}^{n+1} \frac{1}{\sqrt{i}} \).

2.4 - 11 \( n = 1 \) case: with only one point, there should be no lines, and \( \frac{1^2-1}{2} = 0 \). Now assume for some \( n \geq 1 \) that there are \( n \) points, and \( \frac{2^2-2}{2} \) lines. Now add another point. So you will have to connect the new point to the \( n \) other points. So there must be \( \frac{2^2-2}{2} + n \) lines. And \( \frac{2^2-2}{2} + n = \frac{(n+1)^2-n}{2} \).

2.4 - 13 Moving 1 disk takes 1 move, and \( 1 = 2^1 - 1 \). Now assume that \( n \) disks can be moved in \( 2^n - 1 \) moves for \( n \geq 1 \). Now consider the case of moving \( n+1 \) disks. Since \( n \) disks can be moved in \( 2^n - 1 \) moves, the top \( n \) disks can be moved to the second peg in \( 2^n - 1 \) moves. The \( n+1 \)th disk can be moved to the third peg in 1 move. And the \( n \) disks on the second peg can be moved onto the third peg in \( 2^n - 1 \) moves. So to move \( n+1 \) disks, it takes \( 2^n - 1 + 1 + 2^n - 1 = 2(2^n) - 1 = 2^{n+1} - 1 \) moves.

2.4 - 15a

a) This proof is incorrect. The writer assumes that for any integer \( n \), the set of \( n \) horses all have the same color. This argument would require a different proof for going from \( n = 1 \) to \( n = 2 \). Given 2 horses, removing 1 horse will certainly give you a set of horses with all the same color, but when you remove the other horse, the 2 sets may not have horses of the same color.