Zeta Zeros and Quantum Chaos

Yaron Hadad and Jordan Schettler

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Outline

Classical Mechanics

Newtonian Mechanics

Newton's 2nd Law: The trajectory $\mathbf{x}(t)$ of a particle with mass m is determined by

$$m\ddot{\mathbf{x}} = F\left(\mathbf{x}, \dot{\mathbf{x}}\right)$$

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Usually, the force is conservative and independent of $\dot{\mathbf{x}}$, and thus can be written as (minus) the gradient of a function:

$$m\ddot{\mathbf{x}} = -\frac{\partial V}{\partial \mathbf{x}}$$

where $V(\mathbf{x})$ is called the potential function.

Lagrangian Mechanics

In Calculus of Variations, we study the 'action' functional

$$S[\mathbf{x}] = \int_{t_1}^{t_2} L(\mathbf{x}(t), \dot{\mathbf{x}}(t), t)dt$$

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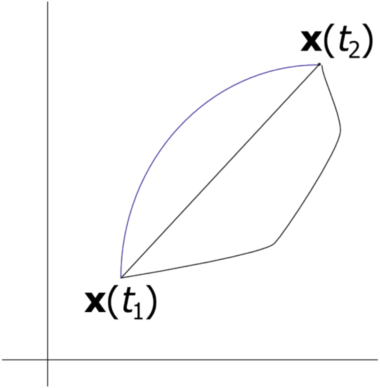
where L is called the Lagrangian. This is a functional over the space of all possible continuous paths.



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The path extremizing $S[\mathbf{x}]$ satisfies Euler-Lagrange's equation:

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Taking $L = \frac{1}{2}m\dot{\mathbf{x}}^2 - V(\mathbf{x})$ gives back Newton's 2nd Law.

Instead, Lagrange preferred to think of Nature as optimal: we can think of the particle as traveling in such a way that it always minimizes (extremizes) the action S.

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These are the celebrated *Hamilton's equations*. Consequence:

$$\frac{dH}{dt} = 0 \text{ (conservation of energy)}$$

Riemann’s 1859 Paper

Euler Product Formula

Theorem (Dirichlet, 1838)

For $s > 1$

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}$$

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Note: The divergence at $s = 1$ implies the infinitude of primes.

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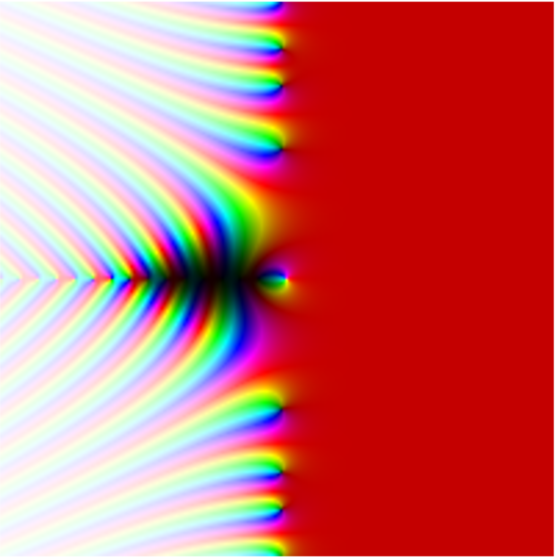
Riemann used complex analysis to find a formula for the number of primes $\pi(x)$ less than a given magnitude x.

Color Map of $f(s) = s : \Re(s), \Im(s) \in [-30, 30]$

Color Map of $\zeta(s)$: $\Re(s), \Im(s) \in [-30, 30]$



The Functional Equation and Special Values

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We have $\zeta(0) = -1/2$ and for $n = 1, 2, \cdots$

$$\zeta(-2n) = 0 \text{ (trivial zeros)} \qquad \zeta(1-2n) = -\frac{B_{2n}}{2n}$$

$$\zeta(2n) = \frac{(2\pi)^{2n}|B_{2n}|}{2(2n)!} \qquad \zeta(1+2n) = \text{ ???}$$

where $\dfrac{x}{e^x - 1} = \displaystyle\sum_{m=0}^{\infty} B_m \frac{x^m}{m!}$.

The “Completed” Zeta Function $Z(s)$

The function

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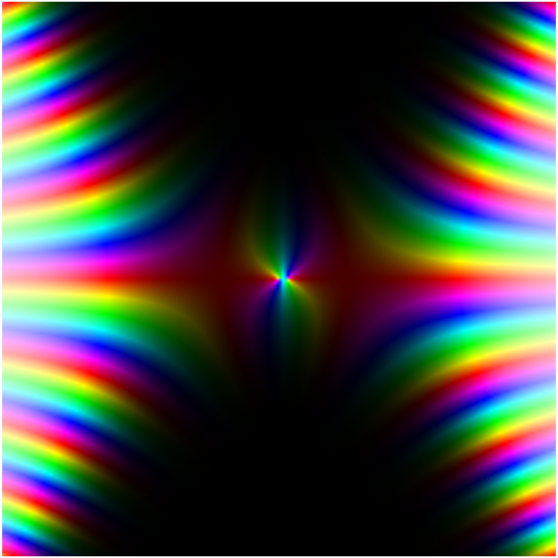
The functional equation becomes

$$Z(s) = Z(1 - s)$$

Color Map of $Z(s)$: $\Re(s), \Im(s) \in [-30, 30]$



Hadamard Product Formula

Theorem (Hadamard, 1893)

$$s(s-1)Z(s) = \prod_{\alpha}\left(1 - \frac{s}{\alpha}\right)$$

where a nontrivial zero α of $\zeta(s)$ is paired with its "twin" $1-\alpha$.

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We can combine the Hadamard and Euler product formulas to count primes with nontrivial zeros or vice versa.

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$$\frac{\log(\zeta(s))}{s} = \int_0^\infty \tilde{\pi}(x) x^{-s-1}\, dx$$

where $\tilde{\pi}(x) = \pi(x) + \frac{1}{2}\pi(x^{1/2}) + \frac{1}{3}\pi(x^{1/3}) + \cdots$.

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Fourier inversion gives

$$\tilde{\pi}(x) = -\frac{1}{2\pi i} \cdot \frac{1}{\log(x)} \int_{a-i\infty}^{a+i\infty} \frac{d}{ds}\left(\frac{\log(\zeta(s))}{s}\right) x^s\, ds \quad (a > 1)$$

Counting Primes with Zeros

We get

$$\tilde{\pi}(x) = \mathrm{Li}(x) - \sum_{\alpha} \mathrm{Li}(x^{\alpha}) - \log(2) + \int_{x}^{\infty} \frac{dt}{(t^3 - t)\log(t)}$$

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Möbius inversion gives

$$\begin{aligned}\pi(x) &= \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \tilde{\pi}(x^{1/n}) \\ &= \langle \pi(x) \rangle + \pi_{\text{osc}}(x)\end{aligned}$$

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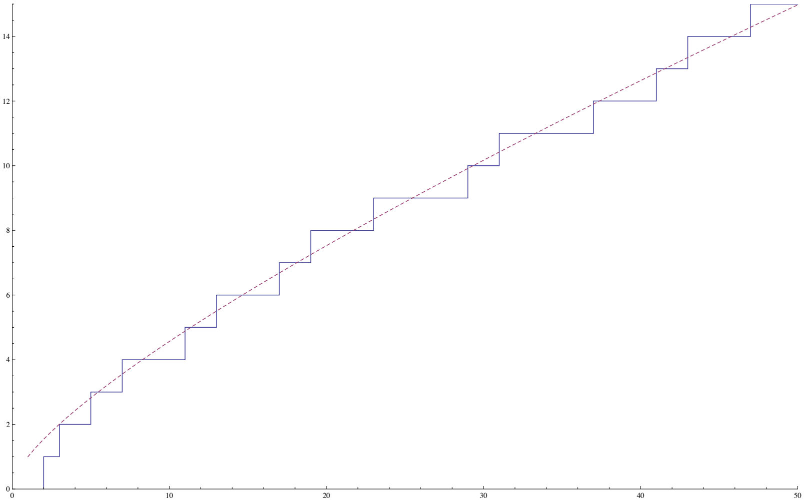
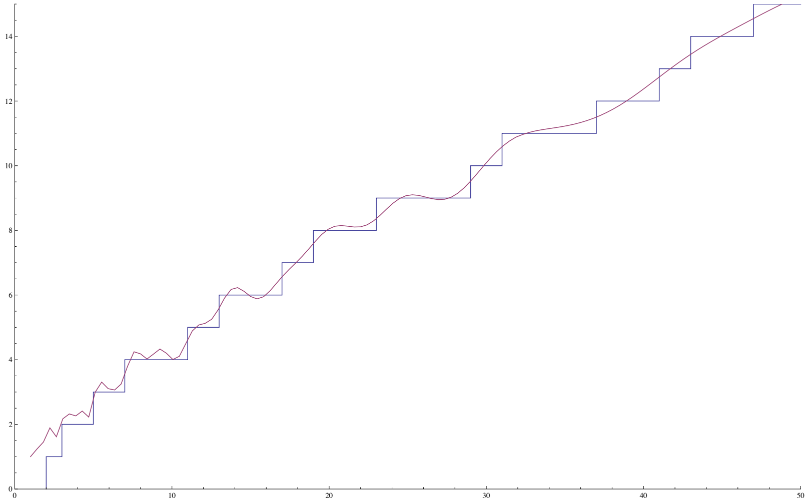
PNT = Prime Number Theorem (Hadamard and de la Vallée-Poussin, 1896): $\nexists$nontrivial zero α on the line $\Re(s) = 1$.

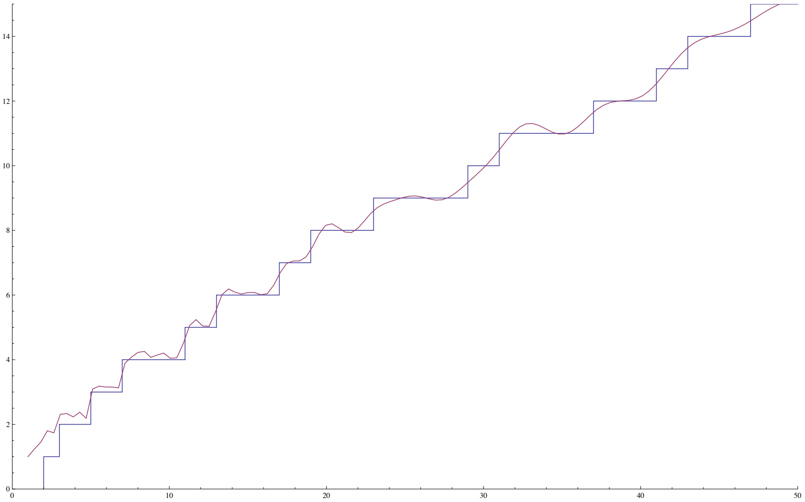
$\pi(x)$ Approximated with $\langle\pi(x)\rangle$



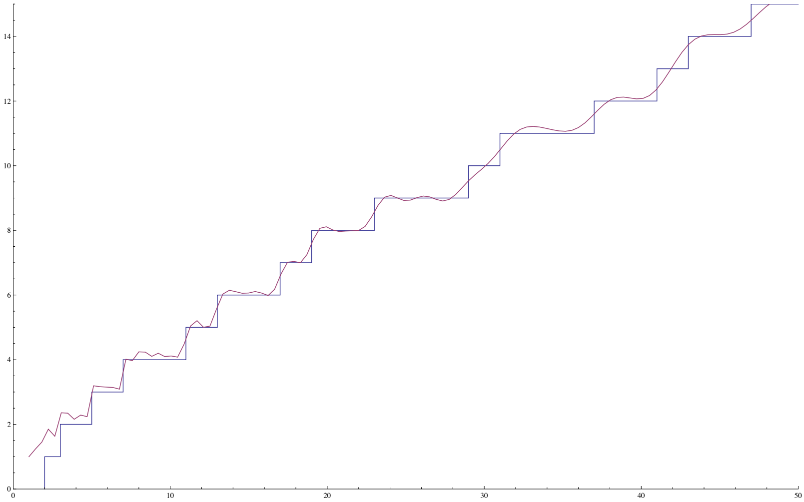
$\pi(x)$ Approximated with $\langle\pi(x)\rangle$ and 5 Pairs of Zeros



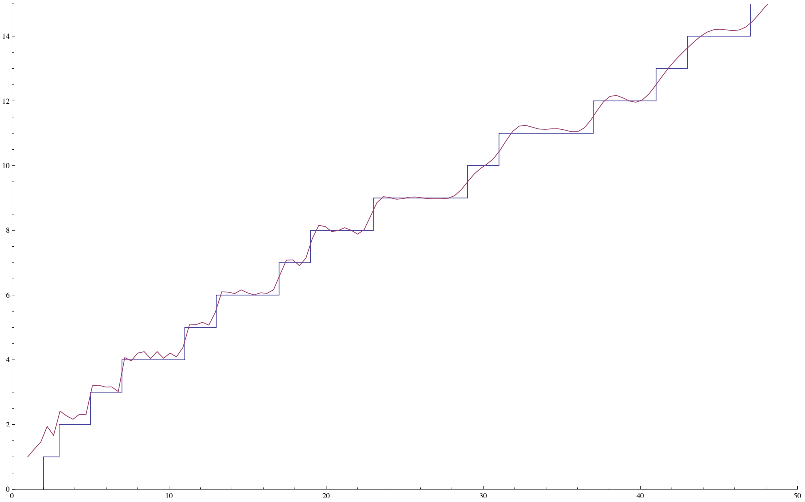
$\pi(x)$ Approximated with $\langle\pi(x)\rangle$ and 10 Pairs of Zeros



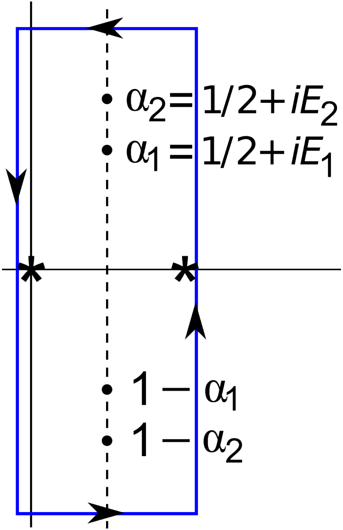
$\pi(x)$ Approximated with $\langle\pi(x)\rangle$ and 15 Pairs of Zeros



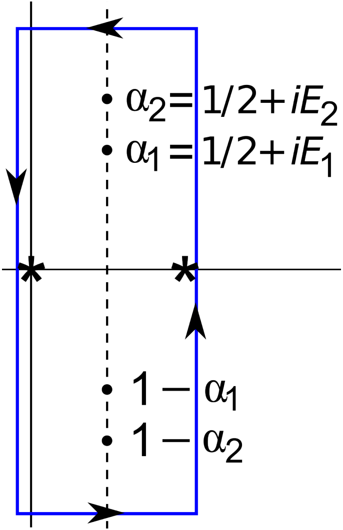
$\pi(x)$ Approximated with $\langle\pi(x)\rangle$ and 20 Pairs of Zeros



$$R : \Re(s) \in [-\epsilon, 1+\epsilon], \Im(s) \in [-E, E]$$



$$\frac{1}{2\pi i} \int_{\partial R} \frac{Z'(s)}{Z(s)}\, ds$$



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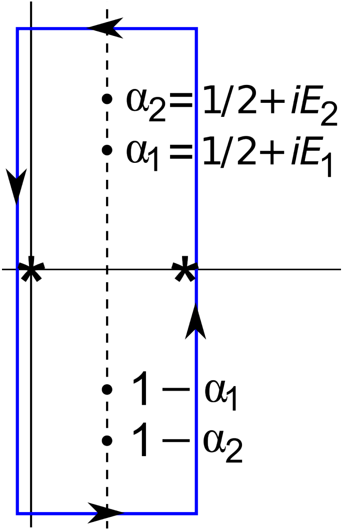

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$$\frac{1}{2\pi i}\int_{\partial R} \frac{Z'(s)}{Z(s)}\, ds$$

$$= \frac{4}{2\pi}\Im \int_{1+\varepsilon}^{\frac{1}{2}+iE} \frac{Z'(s)}{Z(s)}\, ds$$

$$= \frac{2}{\pi}\Im \log(Z(1/2+iE))$$

Counting Zeros with Primes

$$N(E) = 1 + \frac{1}{\pi}\Im \log \pi^{-\frac{1}{4} - i\frac{E}{2}} + \frac{1}{\pi}\Im \log \left(-\frac{3}{4} + i\frac{E}{2}\right)! + \frac{1}{\pi}\Im \log \zeta\left(\frac{1}{2} + iE\right)$$

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$$\text{“}=\text{”} \underbrace{\frac{E}{2\pi}\left(\log\frac{E}{2\pi} - 1\right) + \frac{7}{8} + \cdots}_{\langle N(E)\rangle} + \underbrace{\frac{-1}{\pi}\sum_{p}\sum_{m=1}^{\infty}\frac{1}{m}\frac{\sin(mET_p)}{e^{(m/2)\lambda_p}}}_{N_{\text{osc}}(E)}$$

Here $T_p = \lambda_p = \log(p)$

Quantum Mechanics (QM)

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   For example the momentum $\mathbf{p}$ has the corresponding operator $\hat{\mathbf{p}} = -i\hbar\frac{d}{d\mathbf{x}}$

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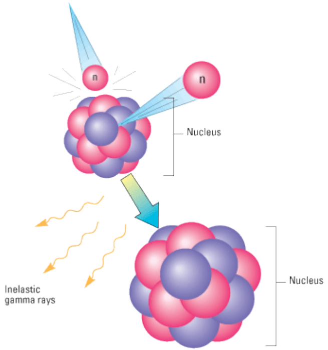

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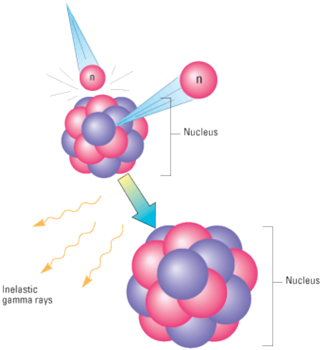

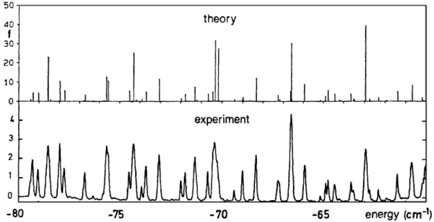

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Its Fourier transform ($t \to E$) is the Green function $G^{+}(\mathbf{x},\mathbf{x}';E)$.

Göttingen (1912-1914)

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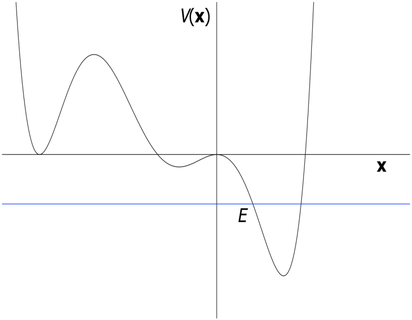
This would imply the RH! Hilbert also had this idea.

The Riemann Operator

Unstable Equilibriums



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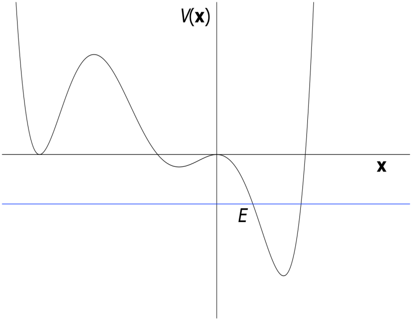

Nearby an equilibrium point $x_0 = 0$, the potential is

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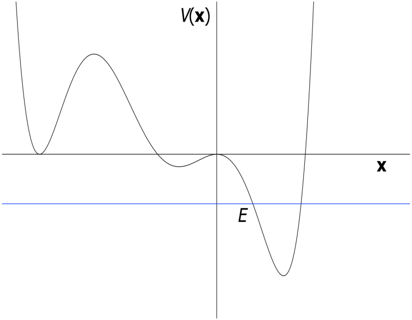

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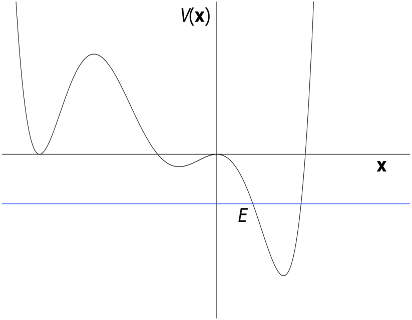
If the equilibrium point is unstable, the Hamiltonian is

$$H = \frac{\mathbf{p}^2}{2m} - \frac{1}{2}K\mathbf{x}^2 \quad (K > 0)$$

Unstable Equilibriums



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$$V(x) = V(0) + V'(0)\mathbf{x} + \frac{1}{2}V''(0)\mathbf{x}^2 + \dots$$

If the equilibrium point is unstable, the Hamiltonian is

$$H = \frac{\mathbf{p}^2}{2m} - \frac{1}{2}K\mathbf{x}^2 \quad (K > 0)$$

By rescaling the coordinates and rotating the $\mathbf{x} - \mathbf{p}$ plane, one gets

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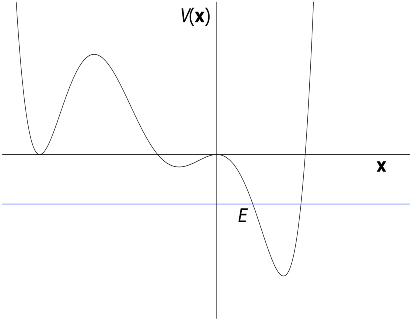
Unstable Equilibriums



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Namely, this Hamiltonian describes the general motion of a particle about an unstable equilibrium point.

Energy Levels of the Riemann Operator

For any (classically bound) Hamiltonian $H(\mathbf{x}, \mathbf{p})$,

$$N(E) = \text{“\# of energy levels } E_n \leq E\text{”} = A\left(E\right)/2\pi\hbar + O\left(1\right)$$

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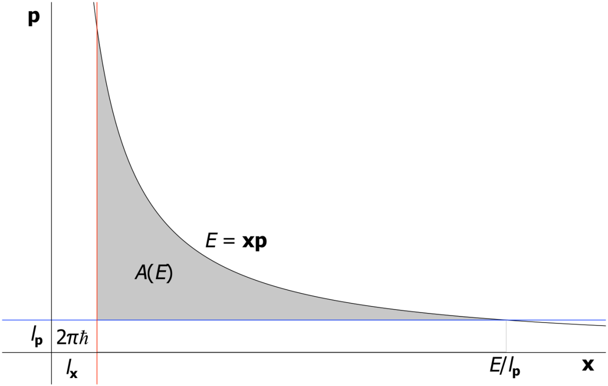

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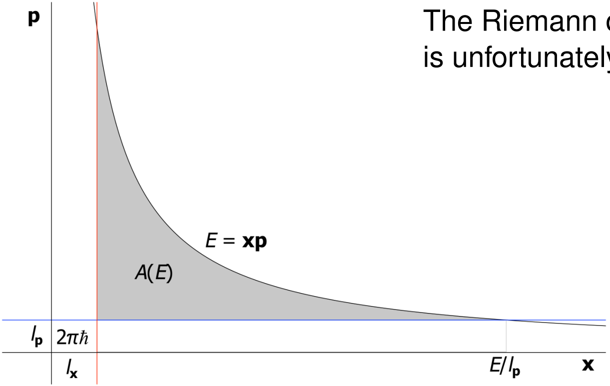

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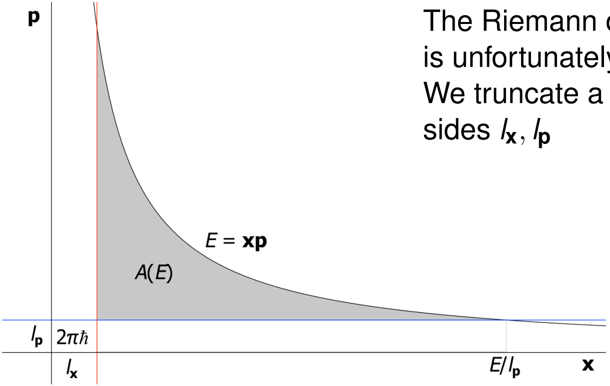

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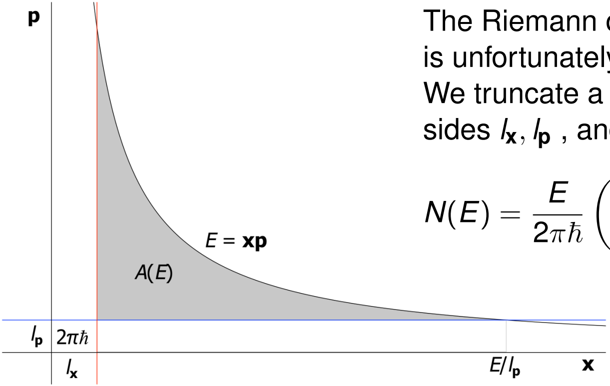

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$$N(E) = \frac{E}{2\pi\hbar}\left(\log\frac{E}{2\pi\hbar} - 1\right) + \cdots$$

The Oscillatory Part of the Energy Levels

Note:

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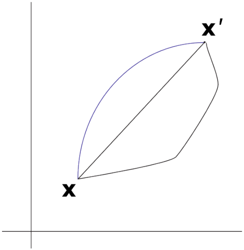
This means that we can approximate the energy levels density function by integrating the Green function!

Feynman Path Integrals

In the Lagrangian formulation
a particle takes a *unique* path
that minimizes the action S.



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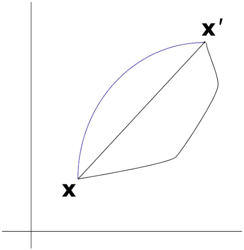

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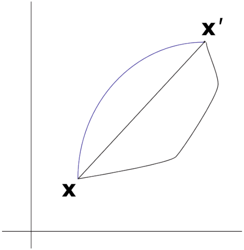

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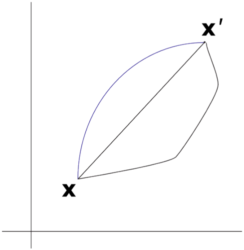

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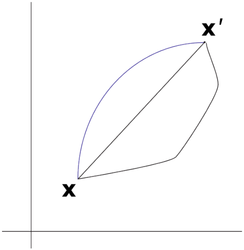

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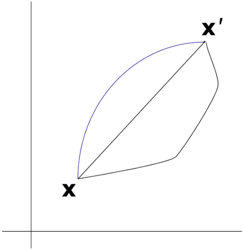

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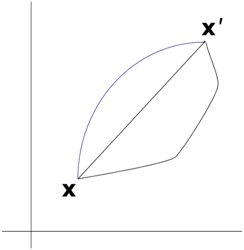

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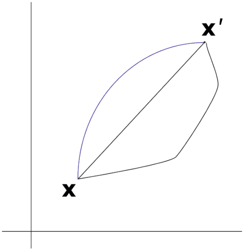

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More Evidence?

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Theorem (Montgomery, 1972)

If the Fourier transform $\hat{f}$ is C^∞ and supported in $(-1, 1)$, then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{m,n\leq N} f(\phi_m - \phi_n) = f(0) + \int_{-\infty}^{\infty} f(x)K(x)\, dx$$

where $K(x) = 1 - \left(\frac{\sin(\pi x)}{\pi x}\right)^2$. Conj: we can take $f = 1_{[\alpha,\beta]}$.

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Dyson noticed that these patterns for Riemann zeros were the same as those predicted by quantum physicists for energy levels in the nucleus of heavy atoms via random matrix theory.

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Let $A \in U(N)$ ($N \times N$ matrices : $A\overline{A}^T = 1$) w/ eigenvalues $e^{i\theta_n}$.

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Theorem (Dyson, 1963)

If $f(x) \to 0$ as $x \to \pm\infty$, then

$$\lim_{N\to\infty} \mathbb{E}\left[\frac{1}{N}\sum_{m,n\leq N} f(\phi_m - \phi_n)\right] = f(0) + \int_{-\infty}^{\infty} f(x)K(x)\,dx$$

where $K(x) = 1 - \left(\frac{\sin(\pi x)}{\pi x}\right)^2$. In fact, we can take $f = 1_{[\alpha,\beta]}$.

Thank You!