1. Measure Theory

Exercise 1.1. If $A \subset \mathbb{R}$ and $\epsilon > 0$ show $\exists$ open sets $O \subset \mathbb{R}$ such that $m^*(O) \leq m^*(A) + \epsilon$.

Proof: Let $\{I_n\}$ be a countable cover for $A$, then $A \subset \bigcup_{n=1}^{\infty} I_n$. Since $m^*(O) \leq m^*(A) + \epsilon$. This implies that

$$m^*(O) - \epsilon \leq m^*(A)$$

where

$$m^*(A) = \inf_{A \subset \bigcup_{n=1}^{\infty} I_n} \left\{ \sum_{n=1}^{\infty} l(I_n) \right\}$$

If $l(I_k) = \infty$ for some $k$ then there is nothing to show, so suppose $(a_n, b_n) = I_n$ then $l(I_n) < \infty$, $\forall n$. Let $O_n = (a_n + 2^{-n} \epsilon, b_n)$ then we have

$$l(O_n) = b_n - a_n - 2^{-n} \epsilon \leq l(I_n)$$

$$\Rightarrow \sum l(O_n) = \sum b_n - a_n - \sum 2^{-n} \epsilon = \sum b_n - a_n - \epsilon$$

$$\Rightarrow m^*(\bigcup_{n} O_n) - \epsilon \leq m^*(A)$$

So let $O = \bigcup_{n} O_n$, then $m^*(O) - \epsilon \leq m^*(A)$ \therefore $\exists O \subset \mathbb{R}$ st $m^*(O) \leq m^*(A) + \epsilon$ □

Exercise 1.2. If $A, B \subset \mathbb{R}, m^*(A) > 0$, then $m^*(A \cup B) = m^*(B)$

Proof: $m^*(A \cup B) \leq m^*(A) + m^*(B)$, and $m^*(B) \leq m^*(A \cup B)$, hence we have

$$m^*(B) \leq m^*(A \cup B) \leq m^*(A) + m^*(B) = m^*(B)$$

\therefore $m^*(A \cup B) = m^*(B)$ □

Exercise 1.3. Prove $E \in \mathcal{M}$ iff $\forall \epsilon > 0, \exists O \subset \mathbb{R}$ open, such that $E \subset O$ and $m^*(O \setminus E) < \epsilon$

Proof: $(\Rightarrow)$ $O \setminus E = E^c \cap O$ implies that $m^*(O \setminus E) = m^*(E^c \cap O)$, but we have

$$m^*(O) = m^*(E^c \cap O) + m^*(E \cap O)$$

So suppose $m^*(E) < \infty \Rightarrow m^*(E^c \cap O) = m^*(O) - m^*(E \cap O)$. Let $I_n$ be a countable cover for $E$, so $I_n = (a_n, b_n)$. Let $O_n = (a_n, b_n + 2^{-n} \epsilon)$ and let $O = \bigcup O_n$. Then

$$m^*(O) = \sum l(O_n) = \sum 2^{-n} \epsilon + b_n - a_n = \epsilon + \sum b_n - a_n$$

and $m^*(E \cap O) = m^*(E)$ since $E \subset O$. So we have

$$m^*(E \cap O) = m^*(E) \leq \sum l(I_n) = \sum b_n - a_n$$

$$\Rightarrow m^*(E \cap O) \leq \sum l(O_n) - \sum l(I_n)$$

$$= \epsilon + \sum b_n - a_n - \sum b_n - a_n = \epsilon$$

\therefore $\exists O \subset \mathbb{R}$ open, st $E \subset O$ and $m^*(O \setminus E) \leq \epsilon$

$(\Leftarrow)$ Conversely, suppose $\forall \epsilon > 0, \exists O \subset \mathbb{R}$, such that $E \subset O$ and $m^*(O \setminus E) < \epsilon$ and that $O \in \mathcal{M}$. Then

$$m^*(O) = m^*(E^c \cap O) + m^*(E \cap O)$$

but $m^*(E^c \cap O) = m^*(O \setminus E) < \epsilon$

This implies that

$$m^*(O) = m^*(E \cap O) + \epsilon \Rightarrow m^*(O) = m^*(E) \quad \because E \in \mathcal{M}$$
Exercise 1.4. Prove $E \in \mathcal{M}$ iff $\forall \epsilon > 0 \exists F \subset \mathbb{R}$ closed, such that $F \subset E$ and $m^*(E \setminus F) < \epsilon$

Proof: ($\Rightarrow$) $E \setminus F = F^c \cap E$ this implies that $m^*(E \setminus F) = m^*(F^c \cap E)$, but we have

$$m^*(F) = m^*(F^c \cap E) + m^*(E \cap F)$$

So suppose $m^*(E) < \infty \Rightarrow m^*(F^c \cap E) = m^*(F) - m^*(E \cap F)$. Let $I_n$ be a countable cover for $E$, where $I_n = (a_n, b_n)$. Let $F_n = [a_n, b_n - 2^{-n}\epsilon]$ and let $F = \bigcup F_n$. Then we have

$$m^*(F) = \sum l(F_n) = \sum b_n - a_n - 2^{-n}\epsilon = \sum b_n - a_n - \epsilon,$$

and $m^*(E \cap F) = m^*(F)$, since $F \subset E$. So

$$m^*(E \cap F) = m^*(F) \leq \sum l(I_n) = \sum b_n - a_n$$

$$\Rightarrow m^*(E \cap F) \leq \sum l(I_n) - \sum l(F_n) = \sum b_n - a_n - \sum b_n - a_n + \epsilon = \epsilon$$

$\therefore \exists F \subset \mathbb{R}$ Closed, st $F \subset E$ and $m^*(E \setminus F) \leq \epsilon$

($\Leftarrow$) Conversely, suppose $\forall \epsilon > 0, \exists F \subset \mathbb{R}$, such that $F \subset E$ and $m^*(E \setminus F) < \epsilon$ and that $F \in \mathcal{M}$. Then

$$m^*(E) = m^*(F^c \cap E) + m^*(E \cap F),$$

but $m^*(F^c \cap E) = m^*(E \setminus F) < \epsilon$. This implies that

$$m^*(E) \leq m^*(F \cap E) + \epsilon \quad \Rightarrow \quad m^*(E) \leq m^*(F) + \epsilon$$

$\therefore E \in \mathcal{M}$

Vitali Let $E$ be a set of finite outer measure and $\mathcal{I}$ a collection of intervals that cover $E$ in the sense of Vitali. Then, given $\epsilon > 0$ there is a finite disjoint collection $\{I_N\}$ of intervals in $\mathcal{I}$ such that

$$\mu^* \left( \bigcup_{n=1}^{N} I_n \right) < \epsilon$$

Exercise 1.5. Does there exists a Lebesgue measurable subset $A$ of $\mathbb{R}$ such that for every interval $(a, b)$ we have $\mu(A \cap (a, b)) = (b - a)/2$?

Proof: First suppose that there is such a measurable set $A$ such that $0 \neq \mu(A \cap (a, b)) = \alpha \leq (b - a)/2$. Then there exists an open set $\mathcal{O}$ such that $A \subset \mathcal{O}$ and $\mu(\mathcal{O} \setminus A) < \epsilon$, so let $\epsilon = \alpha/2$. Now $\mathcal{O}$ is open, so there are disjoint intervals $(x_k, y_k)$ such that $\mathcal{O}$ is a countable union of these intervals. So

$$\mathcal{O} \cap (a, b) = \bigcup_{k=1}^{\infty} [(x_k, y_k) \cap (a, b)] = \bigcup_{l} [c_k, d_k].$$

Hence $\mu(\mathcal{O} \cap (a, b)) = \sum_{l} d_{k_l} - c_{k_l}$, and we have

$$A \cap \mathcal{O} \cap (a, b) = A \cap (a, b) = \bigcup_{l} [A \cap (c_{k_l}, d_{k_l})]$$

Now

$$\alpha = \mu(A \cap (a, b)) = \frac{1}{2} \sum_{l} (d_{k_l} - c_{k_l})$$

but

$$\sum_{l} (d_{k_l} - c_{k_l}) = \mu(\mathcal{O} \cap (a, b)) = \mu(\mathcal{O} \setminus A \cap (a, b)) + \mu(A \cap (a, b)) \leq \mu(\mathcal{O} \setminus A) + \frac{1}{2} \sum_{l} (d_{k_l} - c_{k_l})$$

$$< \epsilon + \frac{1}{2} \sum_{l} (d_{k_l} - c_{k_l})$$

But this implies that

$$\alpha/2 = \epsilon \leq \frac{1}{2} \sum_{l} (d_{k_l} - c_{k_l}) \geq \alpha$$
So \( \mu(A) = 0 \). which implies that \( \mu(A^c) = \infty \). Now if there were to exsits such a set \( A \) we have \( \mu(A^c) = 0 \), and so
\[
b - a = \mu((a, b)) = \mu(A \cap (a, b)) + \mu(A^c \cap (a, b)) = \mu(A^c \cap (a, b)) = \frac{1}{2}(b - a)
\]
So there cannot exist such a set \( \square \).

**Exercise 1.6.** Assume that \( E \subset [0, 1] \) is measurable and for any \( (a, b) \subset [0, 1] \) we have
\[
\mu(E \cap [a, b]) \geq \frac{1}{2}(b - a)
\]
Show that \( \mu(E) = 1 \).

**Proof:** By the previous problem, using the same proof, we know that \( \mu(E^c) = 0 \). So the result is shown.

**Exercise 1.7.** Let \( E_1, \ldots, E_n \) be measurable subsets of \([0, 1]\). Suppose almost every \( x \in [0, 1] \) belongs to at least \( k \) of these subsets. Prove that at least one of the \( E_1, \ldots, E_n \) has measure of at least \( k/n \).

**Proof:** Suppose not, then for each \( i \) we have \( \mu(E_i) < k/n \). Define a function \( f(x) \) as follows.
\[
f(x) = \sum_{i=1}^{n} \chi_{E_i}
\]
where \( \chi_{E_i} \) denotes the characteristic function of \( E_i \). Now since all most all \( x \in [0, 1] \) are in at least \( k \) of the \( E_i \) we have \( f(x) \geq k \) almost everywhere in \([0, 1]\]. Now
\[
k = \int_{[0,1]} k \, dx \leq \int_{[0,1]} f(x) \, dx = \sum_{i=1}^{n} \int_{[0,1]} \chi_{E_i} \, dx = \sum_{i=1}^{n} \mu E_i
\]
But this implies that
\[
\sum_{i=1}^{n} \mu E_i < \sum_{i=1}^{n} \frac{k}{n} = k
\]
Which is a contradiction, hence at least one \( E_i \) has \( \mu(E_i) \geq \frac{k}{n} \) \( \square \).

**Exercise 1.8.** Consider a measure space \((X, A, \mu)\) and a sequences of measurable sets \( E_n, n \in \mathbb{N} \), such that
\[
\sum_{n=1}^{\infty} \mu(E_n) < \infty
\]
Show that almost every \( x \in X \) is an element of at most finitely many \( E_n \)'s.

**Proof:** It suffices to show that \( \mu(x : x \in \cap E_{n_k}) = 0 \). So consider the following
\[
\lim_{m \to \infty} \mu \left( x : x \in \cap_{k=1}^{m} E_{n_k} \right)
\]
If we have shown the above limit is zero, then we’re done. To see this look at the following sum,
\[
\sum_{N=1}^{\infty} \mu \left( x : x \in \cap_{k=1}^{N} E_{n_k} \right) < \sum_{n=1}^{\infty} \mu(E_n) < \infty
\]
and hence
\[
\lim_{m \to \infty} \mu \left( x : x \in \cap_{k=1}^{m} E_{n_k} \right) = 0
\]
Therefore almost every \( x \in X \) is an element of at most finitely many \( E_n \)'s \( \square \).
Exercise 1.9. Consider a measure space \((\mathcal{X}, \mathcal{A}, \mu)\) with \(\mu(\mathcal{X}) < \infty\), and a sequences \(f_n : \mathcal{X} \to \mathbb{R}\) of measurable functions such that \(\lim_{n \to \infty} f_n(x) = f(x)\) for all \(x \in \mathcal{X}\). Show that for every \(\epsilon > 0\) there exists a set \(E\) of measure \(\mu(E) \leq \epsilon\) such that \(f_n\) converges uniformly to \(f\) outside the set \(E\).

Proof: This is Egoroff’s theorem. See below.

**Theorem (Egoroff’s)** If \(f_n\) is a sequence of measurable functions that converge to a real-valued function \(f\) a.e. on a measurable set \(E\) of finite measure, then given \(\eta > 0\), there is a subset \(A\) of \(E\) with \(\mu(A) < \eta\) such that \(f_n\) converges uniformly on \(E \setminus A\).

Proof: Let \(\eta > 0\), then for each \(n\), there exists a set \(A_n \subset E\) with \(\mu A_n < \eta 2^{-n}\), and there is an \(N_n\) such that for all \(x \notin A_n\) and \(k \geq N_n\) we have \(|f_k(x) - f(x)| < 1/n\). Let \(A = \bigcup A_n\), then by construction \(A \subset E\) and \(\mu A < \eta\). Choose \(n_0\) such that \(1/n_0 < \eta\). Now if \(x \notin A\) and \(k \geq N_{n_0}\) then \(|f_k(x) - f(x)| < 1/n_0 < \eta\). Therefore \(f_n\) converges uniformly on \(E \setminus A\).

Exercise 1.10. Let \(g\) be an absolutely continuous monotone function on \([0, 1]\). Prove that if \(E \subset [0, 1]\) is a set of Lebesgue measure zero, then the set \(g(E) = \{g(x) : x \in E\} \subset \mathbb{R}\) is also a set of Lebesgue measure zero.

Proof: Let \(E \subset [0, 1]\) with zero measure, then for any epsilon \(\epsilon > 0\), there exists an open cover \(\mathcal{O}\) for \(E\), such that \(\mu(\mathcal{O}\setminus E) < \epsilon\). Now \(\mathcal{O}\) being open in \([0, 1]\) implies that \(\mathcal{O} = \bigcup (a_n, b_n]\), where \((a_n, b_n]\) are disjoint. Now by absolutely continuity of \(g(x)\) we have

\[
\forall \eta > 0 \exists \delta \text{ s.t. } \sum_{n=1}^{\infty} \mu(I_n) < \delta \implies \sum_{n=1}^{\infty} |g(I_n \cap [0, 1])| < \eta
\]

Now \(g(E) \subset \bigcup |g(I_n \cap [0, 1])|\) which implies that \(\mu(g(E)) < \eta\), so given an \(\eta\) there exists a \(\delta > 0\) such that the above hold, then let \(\delta = \epsilon\). Since \(\eta\) is arbitrary we have \(\mu(g(E)) = 0\).

Remark: The above problem (1.10) is commonly referred to as Lusin’s N condition.

Exercise 1.11. Suppose \(f\) is Lipschitz continuous in \([0, 1]\). Show that

(a) \(\mu(f(E)) = 0\) if \(\mu(E) = 0\).

(b) If \(E\) is measurable, then \(f(E)\) is also measurable.

Proof: For part (a) if \(f\) is Lipschitz continuous then it is absolutely continuous, and so if \(\mu(E) = 0\), then \(\mu(f(E)) = 0\) (see above proof).

For part (b) Let \(E\) be a measurable set and let \(\epsilon > 0\). Now there exists an open set \(\mathcal{O}\) such that \(\mu(\mathcal{O}\setminus E) < \epsilon\), where \(\mathcal{O}\) is a disjoint union of intervals \(I_n = (a_n, b_n]\). Now since \(f\) is absolutely continuous, it can be approximated by simple functions, namely \(\chi_{I_n}\). Choose these functions such that

\[
|f - \sum_{n=1}^{\infty} c_n \chi_{I_n}| < \epsilon
\]

Now \(\mu(\chi_{I_n}) = b_n - a_n > 0\), so it is measurable. Let \(\alpha \in \mathbb{R}\), then the \(f(E)\) is measurable if \(\{x : f(x) \leq \alpha\}\) is a measurable set for any \(\alpha \in \mathbb{R}\). but we have now

\[
\{x : f(x) \leq \alpha\} \subset \{x : \chi_{I_n} + \epsilon \leq \alpha\}
\]

We know simple functions are measurable, and our choice of simple functions approximates \(f(x)\), therefore \(f\) is measurable.

Theorem (Lusin’s) Let \(f\) be a measurable real-valued function on an interval \([a, b]\). Then given \(\delta > 0\), there is a continuous function \(\phi\) on \([a, b]\) such that \(\mu\{x : f(x) \neq \phi(x)\} < \delta\).
Proof: Let \( f(x) \) be measurable on \([a, b]\) and let \( \delta > 0 \). For each \( n \), there is a continuous function \( h_n \) on \([a, b]\) such that
\[
\mu \{ x : |h_n(x) - f(x)| \geq \delta 2^{-n-2} \} < \delta 2^{-n-2}
\]
Denote these sets as \( E_n \). Then by construction we have
\[
|h_n(x) - f(x)| < \delta 2^{-n-2}, \quad \text{for } x \in [a, b] \setminus E_n
\]
Let \( E = \cup E_n \), then \( \mu E < \delta/4 \) and \( \{ h_n \} \) is a sequence of continuous, thus measurable, functions that converges to \( f \) on \([a, b]\) \setminus E. By Egoroff’s theorem, there is a subset \( A \subset [a, b] \setminus E \) such that \( \mu A < \delta/4 \) and \( h_n \) converges uniformly to \( f \) on \([a, b] \setminus (E \cup A)\). Thus \( f \) is continuous on \([a, b] \setminus (E \cup A)\) with \( \mu (E \cup A) < \delta/2 \). Now there is an open set \( O \) such that \( (E \cup A) \subset O \) and \( \mu (O \setminus (E \cup A)) < \delta/2 \). Then we have \( f \) is continuous on \([a, b] \setminus O\), which is closed. Hence there exists a \( \phi \) that is continuous on \((-\infty, \infty)\) such that \( f = \phi \) on \([a, b] \setminus O\), where \( \mu \{ x : f(x) \neq \phi(x) \} \leq \mu (O) < \delta \)

Exercise 1.12. Prove the following statement. Suppose that \( F \) is a sub-\( \sigma \)-algebra of the Borel \( \sigma \)-algebra on the real line. If \( f(x) \) and \( g(x) \) are \( F \)-measurable and if
\[
\int_A f \, dx = \int_A g \, dx, \quad \forall A \in F
\]
Then \( f(x) = g(x) \) almost everywhere.

Proof: Let \( \mu \) denote the Lebesgue measure on the Borel sets. Now since both \( f \) and \( g \) are \( F \)-measurable, for any \( n \geq 1 \), the sets
\[
A_n = \{ x : f(x) - g(x) \geq 1/n \}, \quad B_n = \{ x : g(x) - f(x) \geq 1/n \}
\]
are both measurable and contained in \( F \). Now we also have
\[
A = \{ x : f(x) - g(x) > 0 \} = \bigcap_{n=1}^\infty A_n, \quad B = \{ x : g(x) - f(x) > 0 \} = \bigcap_{n=1}^\infty B_n
\]
are both measurable and contained in \( F \) since \( F \) is a \( \sigma \)-algebra. Now using the convention that \( \infty - \infty = 0 \), we have
\[
\int_A f - g \, dx = 0
\]
If \( \mu (A) > 0 \) then as \( f - g > 0 \) implies by that \( \int_A f - g \, dx > 0 \), which is a contradiction. Hence we have \( \mu (A) = 0 \). By the same argument also have
\[
\int_B g - f \, dx = 0 \quad \Rightarrow \quad \mu (B) = 0.
\]
Now \( A \cap B = \emptyset \) and \( A \cup B \) is the set of points where \( f(x) \neq g(x) \), hence \( f = g \) almost everywhere \( \square \).

Exercise 1.13. Let \( E \subset \mathbb{R} \). Let \( E^2 = \{ e^2 : e \in E \} \)

(a) Show that if \( \mu^* (E) = 0 \), then \( \mu^* (E^2) = 0 \)
(b) Suppose \( \mu^* (E) < \infty \), it it true that \( \mu^* (E^2) < \infty \)

Proof: For part (a) consider the intervals \( I_n = [n, n+1] \) for in \( \mathbb{Z} \). Now consider the function \( f(x) = x^2 \). If \( p_n = \cup (a_k, b_k) \) is an open subset of \( I_n \) such that for \( \delta < 0 \)
\[
\mu (p_n) < \delta \quad \Rightarrow \quad \sum_{k=1}^n |f(b_k) - f(a_k)| = \sum_{k=1}^N |b_k^2 - a_k^2| \leq (2|n| + 1) \delta
\]
Hence \( f(x) \) is absolutely continuous on \( I_n \). Now a function is absolutely continuous on an interval \( I \) if and only if the following are satisfies:

\( f \) is continuous on \( I \)
Proof: For symmetry we have, by definition, equivalence relation.

For transitivity, let A, B, C \subset \mathcal{X}. Then A \sim B if \mu^*(A \Delta B) = 0. Hence A \sim B if and only if B \sim A.

For reflexivity, we have (A \Delta A) = A \setminus A = \emptyset, hence A \sim A.

For transitivity, let A, B, C \subset \mathcal{X}. First notice, by element chasing, A \Delta C \subset (A \Delta B) \cup (B \Delta C), and so we have

\[ 0 \leq \mu^*(A \Delta B) = \mu^*((A \Delta B) \cup (B \Delta C)) \leq \mu^*(A \Delta B) + \mu^*(B \Delta C) \]

Now if A \sim B and B \sim C, then \mu^*(A \Delta B) = \mu^*(B \Delta C) = 0, and so \mu^*(A \Delta B) = 0, hence A \sim C. Therefore \sim is an equivalence relation on \mathcal{X}. \qed

Exercise 1.14. Suppose a measure \mu is defined on a \sigma-algebra \mathcal{M} of subset of \mathcal{X}, and \mu^* is the corresponding outer measure. Suppose A, B \subset \mathcal{X}. Then A \sim B if \mu^*(A \Delta B) = 0. Prove that \sim is an equivalence relation.

Proof: For symmetry we have, by definition, A \Delta B = (A \cup B) \setminus (A \cap B) = (B \cup A) \setminus (B \cap A) = B \Delta A, and so if \mu^*(A \Delta B) = 0, then \mu^*(B \Delta A) = 0. Hence A \sim B if and only if B \sim A.

For reflexivity, we have (A \Delta A) = A \setminus A = \emptyset, hence A \sim A.

For transitivity, let A, B, C \subset \mathcal{X}. First notice, by element chasing, A \Delta C \subset (A \Delta B) \cup (B \Delta C), and so we have

\[ 0 \leq \mu^*(A \Delta B) = \mu^*((A \Delta B) \cup (B \Delta C)) \leq \mu^*(A \Delta B) + \mu^*(B \Delta C) \]

Now if A \sim B and B \sim C, then \mu^*(A \Delta B) = \mu^*(B \Delta C) = 0, and so \mu^*(A \Delta B) = 0, hence A \sim C. Therefore \sim is an equivalence relation on \mathcal{X}. \qed

Exercise 1.15. Let (\mathcal{X}, \mathcal{M}, \mu) be a measure space.

(a) Suppose \mu(\mathcal{X}) < \infty. If f and f_n are measurable functions with f_n \to f almost everywhere, prove that there exists sets H, E_k \in \mathcal{M} such that \mathcal{X} = H \cup \bigcup_{k=1}^{\infty} E_k, where \mu(H) = 0 and f_n \to f uniformly on each E_k.

(b) Is the result of (a) still true if (\mathcal{X}, \mathcal{M}, \mu) is \sigma-finite?

Proof: For part (a), since \mu(\mathcal{X}) < \infty and f_n \to f almost everywhere, by Egoroff’s theorem, for any
\( k \in \mathbb{N}, \) there is \( H_k \in \mathcal{M} \) such that \( \mu(H_k) < 1/k \) and \( f_n \to f \) uniformly on \( E_k = H_k^c. \) Now define \( H = \cap_{k=1}^{\infty} H_k, \) then \( H \subset H_k, \) and so \( 0 \leq \mu(H) \leq 1/k \) for all \( k, \) hence \( \mu(H) = 0. \) Now

\[
\bigcup_{k=1}^{\infty} E_k = \bigcup_{k=1}^{\infty} H_k^c = \left( \bigcap_{k=1}^{\infty} H_k \right)^c = H^c
\]

and so

\[
\mathcal{X} = H \cup \left( \bigcup_{k=1}^{\infty} E_k \right)
\]

where \( f_k \) converges uniformly to \( f \) on any \( E_k. \)

For part (b), the statement is true. Since \( \mathcal{X} \) is \( \sigma \)-finite, we can write \( \mathcal{X} \) as a disjoint union of finite sets, i.e.

\[
\mathcal{X} = \bigcup_{n=1}^{\infty} \mathcal{X}_n \text{ where } \mu(\mathcal{X}_n) < \infty \quad \forall n \quad \mathcal{X}_i \cap \mathcal{X}_j = \emptyset \text{ for } i \neq j
\]

Now for each \( \mathcal{X}_n \) apply part (a). Then we have

\[
\mathcal{X}_n = H_n \cup \bigcup_{k=1}^{\infty} E_{k,n} \text{ with } \mu(H_n) = 0
\]

Let \( H = \bigcup_{n=1}^{\infty} H_n, \) then \( \mu(H) = \sum_{n=1}^{\infty} \mu(H_n) = 0. \) So we have

\[
\mathcal{X} = \bigcup_{n=1}^{\infty} \mathcal{X}_n = \bigcup_{n=1}^{\infty} \left( H_n \cup \left( \bigcup_{k=1}^{\infty} E_{k,n} \right) \right) = \left( \bigcup_{n=1}^{\infty} H_n \right) \cup \left( \bigcup_{n,k=1}^{\infty} E_{k,n} \right) = H \cup \left( \bigcup_{n,k=1}^{\infty} E_{k,n} \right)
\]

Now \( H \) has measure zero and \( \{E_{k,n}\}_{n,k=1}^{\infty} \) is a countable collection of open sets for which \( f_n \to f \) uniformly \( \square. \)

**Exercise 1.16.** Suppose \( f_n \) is a sequence of measurable functions on \([0,1].\) For \( x \in [0,1] \) define \( h(x) = \#\{n : f_n(x) = 0\} \) (the number of indices \( n \) for which \( f_n(x) = 0. \) Assuming that \( h < \infty \) everywhere, prove that the function \( h \) is measurable.

**Proof:** First consider the measure space \(((0,1], \sigma[0,1], \mu),\) where \( \mu \) is the Lebesgue measure. Since \( f_n \) is measurable for all \( n \) we know that the set \( \{x : f_n(x) = \alpha\} \) is measurable, for \( \alpha \in \mathbb{R}. \) In particular, the set \( \{x : f_n(x) = 0\} \) is measurable. Now we have

\[
\bigcup_{n=1}^{\infty} \{x : f_n(x) = 0\}
\]

is measurable with respect to \( \mu, \) since it is the countable union of measurable sets. Now consider the measure space \((\mathbb{N}, \sigma(\mathbb{N}), \nu)\) where \( \nu \) is the counting measure. Now we know that

\[
h(x) = \{n : f_n(x) = 0\} < \infty
\]
So consider the following:

\[
\{ x : h(x) = \alpha \} = \left\{ x : \# \left( \bigcup_{n=1}^{\infty} f_n(x) = 0 \right) = \alpha \right\} \\
= \left\{ x : \sum_{n=1}^{\infty} \nu \{ n : f_n(x) = 0 \} = \alpha \right\} \\
\subset \left\{ x : \sum_{n=1}^{\infty} \nu \{ n : f_n(x) = 0 \} < \infty \right\} \\
\subset [0, 1]
\]

Hence the function \( h(x) \) is measurable \( \square \).

2. Lebesgue Integration

Exercise 2.1. Consider the Lebesgue measure space \((\mathbb{R}, \mathcal{M}, \mu)\). Let \( f \) be an extended real-valued \( \mathcal{M} \)-measurable function on \( \mathbb{R} \). For \( x \in \mathbb{R} \) and \( r > 0 \) let \( B_r(x) = \{ y \in \mathbb{R} : |y - x| < r \} \). With \( r > 0 \) fixed, define a function \( g \) on \( \mathbb{R} \) by setting

\[
g(x) = \int_{B_r(x)} f(y) \mu(dy) \quad \text{for} \quad x \in \mathbb{R}
\]

(a) Suppose \( f \) is locally \( \mu \)-integrable on \( \mathbb{R} \). Show that \( g \) is a real-valued continuous function on \( \mathbb{R} \).

(b) Show that if \( f \) is \( \mu \)-integrable on \( \mathbb{R} \) then \( g \) is uniformly continuous on \( \mathbb{R} \).

Proof: If we show part \((b)\), then part \((a)\) follows by the same argument. Let \( x \in \mathbb{R} \). Now if \( f \) is integrable on \( \mathbb{R}^2 \) so is \( |f| \). Hence if \( \epsilon > 0 \), there is \( \delta > 0 \) such that if \( \mu(A) < \delta \), then we have

\[
\int_A |f| \, dy \leq \frac{\epsilon}{2}
\]

Now as \( B(x, r) \) and \( B(y, r) \) are open balls with area \( \pi r^2 \) with centers offset by \(|y - x|\), we have that

\[
\mu(B(x, r) \setminus B(y, r)) = \mu(B(y, r) \setminus B(x, r)) \rightarrow 0 \quad \text{as} \quad y \rightarrow x
\]

Hence given \( \delta > 0 \), there is an \( \eta > 0 \) such that if \(|y - x| < \eta\), then

\[
\mu(B(x, r) \setminus B(y, r)) = \mu(B(y, r) \setminus B(x, r)) < \delta
\]

So for \(|y - x| < \eta\), we have

\[
|g(x) - g(y)| \leq \int_{B(y,r) \setminus B(x,r)} |f| \, d\mu + \int_{B(x,r) \setminus B(y,r)} |f| \, d\mu < \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon.
\]

That is, \( g(x) \) is uniformly continuous on \( \mathbb{R}^2 \) \( \square \).

Theorem (Jensen’s Inequality) If \( \phi \) is a convex function on \( \mathbb{R} \) and \( f \) an integrable function on \([0, 1]\).

\[
\int \phi(f(t)) \, dt \geq \phi \left( \int f(t) \, dt \right).
\]

Proof: Let

\[
\alpha = \int f(t) \, dt, \quad y = m(x - \alpha) + \phi(\alpha)
\]

Then \( y \) is the equation of a supporting line at \( \alpha \). Now we have

\[
\phi(f(t)) \geq m(f(t) - \alpha) + \phi(\alpha) \quad \Rightarrow \quad \int \phi(f(t)) \, dt \geq \phi(\alpha) \, dt \quad \square
\]
**Theorem (Bounded Convergence)** Let $f_n$ be a sequence of measurable functions defined on a set $E$ of finite measure, and suppose that there is a real number $M$ such that $|f_n| \leq M$ for all $N$ and all $x$. If $f(x) = \lim f_n(x)$ pointwise in $E$, then

$$\int_E f = \lim \int_E f_n.$$ 

**Proof:** Let $\epsilon > 0$, then there is an $N$ and a measurable set $A \subset E$ with $\mu A < \frac{\epsilon}{4M}$ such that for all $n \geq N$ and $x \in E \setminus A$ we have $|f_n(x) - f(x)| < \frac{\epsilon}{2}$. Now,

$$\left| \int_E f_n - \int_E f \right| = \left| \int_E f_n - f \right| \leq \int_E |f_n - f|$$

$$= \int_{E \setminus A} |f_n - f| + \int_A |f_n - f|$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Therefore we have $\int_E f_n \to \int_E f \square$.

**Exercise 2.2.** Suppose $f_n$ is a sequence of measurable functions such that $f_n$ converges to $f$ almost everywhere. If for each $\epsilon > 0$, there is a $C$ such that

$$\int_{|f_n| > C} |f_n| \, dx < \epsilon.$$ 

Show that $f$ is integrable on $[0, 2]$.

**Proof:** First the interval $[0, 2]$, is not important. The result can be shown for any finite interval. Fix $\epsilon > 0$, now if $f$ is to be integrable, then so is $|f|$. Let $C$ be such in the hypothesis, by Fatou's lemma we have

$$\int_0^2 |f| \, dx \leq \liminf \int_0^2 |f_n| \, dx$$

$$= \liminf \left( \int_{[0, 2] \cap \{|f_n| > C\}} |f_n| \, dx + \int_{[0, 2] \cap \{|f_n| \leq C\}} |f_n| \, dx \right)$$

$$\leq \epsilon + C\mu(0, 2)$$

Therefore $\int_0^2 |f| \, dx$ is bounded and hence $f$ is integrable $\square$.

**Theorem (Fatou’s Lemma)** If $f_n$ is a sequence of nonnegative measurable functions and $f_n(x) \rightarrow f(x)$ almost everywhere on a set $E$, then

$$\int_E f \leq \liminf \int_E f_n.$$ 

**Proof:** Since the integral over a set of measure zero is zero, (WLOG) we can assume that the converges is everywhere. Let $h$ be a bounded measurable function which is not greater than $f$ and which vanishes outside a set $A \subset E$ of finite measure. Define a function $h_n$, by

$$h_n(x) = \min\{h(x), f_n(x)\}.$$ 

Then $h_n$ is bounded by the bound for $h$ and vanishes outside $A$. Now $h_n \to h$ pointwise in $A$, hence we have by the bounded convergence theorem

$$\int_E h = \int_A h = \lim \int_A h_n \leq \liminf \int_E f_n.$$
Taking supremum over \( h \) gives us the result \( \Box \).

**Theorem (Monotone Convergence)** Let \( f_n \) be an increasing sequence of nonnegative measurable functions, and let \( f = \lim f \) a.e. Then

\[
\int f = \lim \int f_n.
\]

**Proof:** By Fatou’s lemma we have

\[
\int f \leq \liminf \int f_n.
\]

Now for each \( n \), since \( f \) is monotone, we have \( f_n \leq f \), and so

\[
\int f_n \leq \int f \quad \Rightarrow \quad \limsup \int f_n \leq \int f \quad \Rightarrow \quad \int f \leq \int f \lim f_n \quad \Box.
\]

**Remark:** Let the positive part of \( f \) be denoted by \( f^+(x) = \max\{f(x), 0\} \), and the negative part be denoted by \( f^-(x) = \max\{-f(x), 0\} \). If \( f \) is measurable then so are \( f^+ \) and \( f^- \). Furthermore \( f = f^+ - f^- \) and \(|f| = f^+ + f^- \).

**Exercise 2.3.** Let \( f \) be a real-valued continuous function on \([0, \infty)\) such that the improper Riemann integral \( \int_0^\infty f(x) \, dx \) converges. Is \( f \) Lebesgue integrable on \([0, \infty)\)?

**Proof:** \( f \) does not have to be Lebesgue integrable. Let \( n \geq 0 \) and define a function \( f_n \) as follows

\[
f_n(x) = \begin{cases} \frac{-4}{n+1}x & x \in [2n, 2n + \frac{1}{2}] \\ \frac{-4}{n+1}x & x \in [2n + \frac{1}{2}, 2n + \frac{3}{2}] \\ \frac{4}{n+1}x & x \in [2n + \frac{3}{2}, 2n + 2] \\ \end{cases}
\]

Now \( f_n \) is continuous on \([0, \infty)\) and when considering Riemann integration, we have

\[
\int_0^{2n+1} f_n(x) \, dx = \frac{1}{n+1} \quad \text{and} \quad \int_0^{2n+2} f_n(x) \, dx = 0 \quad \Rightarrow \quad \int_0^\infty f_n \, dx = 0
\]

for each fixed \( n \). Now define

\[
f(x) = \sum_{n=0}^{\infty} f_n(x)
\]

Then since \( f_n \) has disjoint support for any \( N \in \mathbb{N} \) and \( 2N < y < 2N + 2 \), we have

\[
\int_0^y f(x) \, dx = \int_{2N}^y f(x) \, dx,
\]

and so the Riemann integral of \( f(x) \) converges to 0 on \([0, \infty)\). Now if a measurable function \( f \) is Lebesgue integrable then so is \(|f|\). But,

\[
\int_0^\infty |f| \, dx = 2 \sum_{n=1}^{\infty} \frac{1}{n+1} = \infty.
\]

Therefore \( f \) is Riemann integrable but not Lebesgue integrable \( \Box \).

**Exercise 2.4.** Consider the real valued function \( f(x, t) \), where \( x \in \mathbb{R}^n \) and \( t \in I = (a, b) \). Suppose the following hold.

1. \( f(x, \cdot) \) is integrable over \( I \) for all \( x \in E \)
2. There exists an integrable function \( g(t) \) on \( I \) such that \(|f(x, t)| \leq g(t)\), \( \forall x \in E, t \in I \).
3. For some \( x_0 \in E \) then function \( f(\cdot, t) \) is continuous on \( I \)

Then the function \( F(x) = \int_I f(x, t) \, dt \) is continuous at \( x_0 \)

**Proof:** Let \( x_n \) be any sequence in \( E \) such that \( x_n \to x_0 \). Define a sequence of functions as \( f_n(t) = \)
$f(x_n, t)$. Then by hypothesis we have $f_n(t) \leq g(t)$, for $t \in I$ almost everywhere. Let $f(t) = f(x_0, t)$, now since $f(x, t)$ is continuous at $x_0$, we have $f_n \to f$. So by the Lebesgue Dominated Convergence theorem we have

$$\lim_{n \to \infty} \int_I |f_n(t) - f(t)| \, dt = 0$$

Hence we have

$$|F(x_n) - F(x_0)| = \left| \int_I f_n(t) - f(t) \, dt \right| \leq \int_I |f_n(t) - f(t)| \, dt \to 0$$

Or $F(x)$ is continuous at $x_0 \square$

**Theorem (Lebesgue Dominated Convergence)** Let $g$ be integrable over $E$ and let $f_n$ be a sequence of measurable functions such that $|f_n| \leq g$ on $E$ and for almost all $x \in E$ we have $f(x) = \lim f_n(x)$. Then

$$\int_E f = \lim \int_E f_n.$$  

**Proof:** Assuming the hypothesis, the function $g - f_n$ is nonnegative, so by Fatou’s lemma we have

$$\int_E (g - f) \leq \liminf \int_E (g - f_n)$$

Now since $|f| \leq g$, $f$ is integrable and we have

$$\int_E g - \int_E f \leq \int_E g - \limsup \int_E f_n$$

Hence we have

$$\int_E f \geq \limsup \int_E f_n$$

Considering $g + f_n$, we have the result

$$\int_E f \leq \liminf \int_E f_n$$

and so the result follows $\square$.

**Exercise 2.5.** Show that the Lebesgue Dominated Convergence theorem holds if almost everywhere convergence is replaced by convergence in measure.

**Proof:** Suppose that $f_n \to f$ in measure, and there is an integrable function $g$ such that $f_n \leq g$ almost everywhere. Now $|f_n - f|$ is integrable for each $n$, and $|f_n - f|\chi_{[-k,k]}$ converges to $|f_n - f|$. By the Lebesgue Dominated Convergence theorem we have

$$\int_{-k}^k |f_n - f| \to \int_{\mathbb{R}} |f_n - f|$$

Let $\epsilon > 0$, then there exists an $N_0$ such that

$$\int_{|x|>N_0} |f_n - f| < \frac{\epsilon}{3}$$

also for each $n$, given $\epsilon > 0$, there exists $\delta > 0$ such that for any set $A$ with $\mu(A) < \delta$ we have

$$\int_A |f_n - f| < \frac{\epsilon}{3}$$

Let $A = \{|f_n - f| \geq \delta\}$. Then there exists an $N_1$, such that for all $n \geq N_1$, we have $A = \{|f_n - f| \geq \delta\} \leq \delta$. Let $N = \max\{N_0, N_1\}$

$$\int_X |f_n - f| = \int_{|x|>N} |f_n - f| + \int_{[-N,N]\setminus A} |f_n - f| + \int_{[-N,N]\cap A} |f_n - f| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + 2N\delta < \epsilon$$

Let $\delta = \frac{\epsilon}{6N}$, therefore we have $\int_X |f_n - f| \to 0$, as $n \to \infty \square$.
**Exercise 2.6.** Show that an extended real valued integrable function is finite almost everywhere.

**Proof:** Consider the measur space \((X, \mathcal{M}, \mu)\). Let \(E = \{ x \in C : |f| = \infty \}\). Now since \(f\) is integrable, it is measurable hence the set \(E\) is measurable. Now suppose \(\mu(E) > 0\), then as \(|f| > 0\) on \(E\) we have

\[
\infty > \int_X |f| \, d\mu \geq \int_E |f| \, d\mu = \infty
\]

This contradicts to the integrability of \(f\), thus \(\mu(E) = 0\). Therefore \(f\) is finite almost everywhere \(\square\).

**Exercise 2.7.** If \(f_n\) is a sequence of measurable functions such that

\[
\sum_{n=1}^{\infty} \int |f_n| < \infty
\]

Show that \(\sum_{n=1}^{\infty} f_n\) converges almost everywhere to an integrable function \(f\) and that

\[
\int f = \sum_{n=1}^{\infty} \int f_n < \infty
\]

**Proof:** Define \(g_N\) to be the partial sums of \(|f_n|\). Then \(g_N\) is measurable since each \(f_n\), and hence \(|f_n|\) is measurable. Let \(g = \lim g_n\), then \(g\) is measurable as it is the limit of measurable functions. Now

\[
\int f = \int \sum_{n=1}^{\infty} |f_n| = \sum_{n=1}^{\infty} \int |f_n| < \infty
\]

So \(g\) is integrable, and hence \(g\) is finite almost everywhere. Define \(f(x)\) as follows

\[
f(x) = \begin{cases} \sum_{n=1}^{\infty} f_n & \text{if } |g(x)| < \infty \\ 0 & \text{otherwise} \end{cases}
\]

Then \(g_N \to f\) as \(N \to \infty\) almost everywhere. We also have

\[
\left| \int f \right| \leq \int |f| \\
= \int \sum_{n=1}^{\infty} f_n \\
\leq \sum_{n=1}^{\infty} |f_n| \\
= \int g < \infty
\]

We also have that

\[
|g_N| = \left| \sum_{n=1}^{N} f_n \right| \leq \sum_{n=1}^{N} |f_n| \leq \sum_{n=1}^{\infty} |f_n| = g
\]

almost everywhere. Now by the Lebesgue Dominated Convergence theorem, we have

\[
\int f = \int \lim g_N = \lim \int g_N = \lim \sum_{n=1}^{N} \int f_N = \sum_{n=1}^{\infty} \int f_n \square
\]
**Exercise 2.8.** Let \((X, \mathcal{M}, \mu)\) be a measure space, and let \(f_n\) be a sequences of nonnegative extended real-valued \(\mathcal{M}\)-measurable functions on \(X\). Suppose \(\lim_{n \to \infty} f_n = f\) exists almost everywhere on \(X\) and \(f_n \leq f\) almost everywhere. For \(n \in \mathbb{N}\), show that

\[
\int_X f \, d\mu = \lim_{n \to \infty} \int_X f_n \, d\mu
\]

**Proof:** First if \(\int f \, dx = \infty\), applying Fatou’s lemma we have

\[
\int_X \lim_{n \to \infty} \inf f_n \, d\mu \leq \lim_{n \to \infty} \inf \int_X f_n \, d\mu \leq \lim_{n \to \infty} \int_X f_n \, d\mu \leq \infty.
\]

And so \(\lim_{n \to \infty} \int_X f_n \, d\mu = \int f \, dx = \infty\).

Now if \(\int f \, dx < \infty\), since \(f_n \leq f\) almost everywhere, we have \(|f_n| \leq |f|\) almost everywhere, and we have \(\lim_{n \to \infty} f_n = f\) exists almost everywhere, we have by the Lebesgue Dominated Convergence theorem

\[
\int_X ||f_n| - |f|| \, d\mu \leq \int_X |f_n - f| \, d\mu = 0 \Rightarrow \lim_{n \to \infty} \int_X f_n \, d\mu = \int_X f \, d\mu \square
\]

**Exercise 2.9.** Let \(f\) be a nonnegative Lebesgue measurable function on \([0,1]\). Suppose \(f\) is bounded above by \(1\) and \(\int_0^1 f \, dx = 1\). Show that \(f = 1\) almost everywhere on \([0,1]\)

**Proof:** let \(1 > \epsilon > 0\) and define the set \(E\) as

\[
E = \{x \in [0,1] : 0 \leq f \leq 1 - \epsilon\}
\]

Now we have

\[
1 = \int_0^1 f \, dx = \int_{E^c} f \, dx + \int_E f \, dx \leq \int_{E^c} f \, dx + \int_E 1 - \epsilon \, dx \leq \mu(E^c) + \mu(E) - \epsilon \mu(E) = 1 - \epsilon \mu(E)
\]

Hence since this holds for any \(\epsilon \in (0,1)\), we must have \(\mu(E) = 0\). Therefore \(f = 1\) almost everywhere on \([0,1]\) \(\square\).

**Exercise 2.10.** Let \(f\) be a real-valued Lebesgue measurable function on \([0, \infty)\) such that:

1. \(f\) is locally integrable
2. \(\lim_{x \to \infty} f = c\)

Show that \(\lim_{a \to \infty} \frac{1}{a} \int_0^a f \, dx = c\).
Proof: Let $\epsilon > 0$, then there is an $M > 0$ such that if $x > M$, then $|f(x) - c| < \epsilon$. Let $a > M$, now
\[
\left| \frac{1}{a} \int_0^a f \, dx - c \right| = \frac{1}{a} \left| \int_0^a f - c \, dx \right|
\leq \frac{1}{a} \int_0^a |f - c| \, dx
= \frac{1}{a} \int_0^M |f - c| \, dx + \frac{1}{a} \int_M^a |f - c| \, dx
< \frac{1}{a} \int_0^M |f - c| \, dx + \epsilon \frac{1}{a} (a - M)
= \frac{1}{a} \int_0^M |f - c| \, dx + \epsilon (1 - \frac{M}{a})
\]
Now since $M$ is fixed, and by the integrability of $f$, we have
\[
\left| \frac{1}{a} \int_0^a f \, dx - c \right| < \epsilon
\]
and since this is for any $\epsilon > 0$ and all $a > M$, we have
\[
\lim_{a \to \infty} \frac{1}{a} \int_0^a f \, dx - c = 0 \iff \lim_{a \to \infty} \frac{1}{a} \int_0^a f \, dx = c \, \square.
\]

Exercise 2.11. Let $f$ be a real-valued uniformly continuous function on $[0, \infty)$. Show that if $f$ is Lebesgue integrable on $[0, \infty)$, then $\lim_{x \to \infty} f(x) = 0$.

Proof: First if $f$ is Lebesgue integrable, then so is $|f|$. Now decompose the integral as follows
\[
\infty > \int_0^\infty |f(x)| \, dx = \sum_{k=1}^{\infty} \int_k^{k+1} |f(x)| \, dx,
\]
denote $a_k = \int_k^{k+1} |f(x)| \, dx$.

Now $a_k > 0$, and since the integral is convergent this implies that $a_k \to 0$ as $k \to \infty$, which inturn implies that $a_k$ is Cauchy. So we have
\[
\forall \epsilon > 0, \exists N \text{ s.t. } \sum_{k=n}^m a_k < \epsilon, \forall n, m > N \Rightarrow \int_{N+1}^\infty |f(x)| \, dx < \epsilon
\]
Since $|f(x)|$ is positive and $\epsilon$ is arbitrary this implies that $f(x) \to 0$ as $N \to \infty \, \square$.

Exercise 2.12. Let $f \in L_1(\mathbb{R})$. With $h > 0$ fixed, define a function $\phi_h$ on $\mathbb{R}$ by setting
\[
\phi_h(x) = \frac{1}{2h} \int_{x-h}^{x+h} f(t) \, \mu(dt), \text{ for } x \in \mathbb{R}
\]
(a) Show that $\phi_h$ is measurable on $\mathbb{R}$.
(b) Show that $\phi_h \in L_1(\mathbb{R})$ and $\|\phi_h\|_1 \leq \|f\|_1$.

For part (a) since $f$ is integrable, then $f$ is measurable. So the integral of a measurable function is measurable, thus $\phi_h(x)$ is measurable.

For part (b) First apply the change of variable $y = x - t$, then we have
\[
\int_{x-h}^{x+h} f(t) \, \mu(dt) = - \int_{-h}^{h} f(x - y) \, \mu(dy) = \int_{-h}^{h} f(x - y) \, \mu(dy) = \int_{-\infty}^{\infty} f(x - y) \chi_{[-h, h]}(y) \, \mu(dy)
\]
Where $\chi_{[-h, h]}(y)$ is the charactistic function on $[-h, h]$. So we have
\[
\phi_h(x) = \frac{1}{2h} f * \chi_{[-h, h]} \Rightarrow \|\phi_h(x)\|_1 = \frac{1}{2h} \|f\|_1 \chi_{[-h, h]} \leq \frac{1}{2h} \|f\|_1 \|\chi_{[-h, h]}\|_1 = \|f\|_1 \, \square.
\]
Exercise 2.13. Let $f$ be a Lebesgue integrable function of the real line. Prove that
\[ \lim_{n \to \infty} \int_{\mathbb{R}} f(x) \sin(nx) \, dx = 0. \]

Proof: If $f$ is integrable, then there exists a sequences of step function $\phi_n$ such that
\[ \forall \epsilon > 0 \exists N \text{ s.t. } \int |f - \phi_n| < \frac{\epsilon}{2} \]
Now we have
\[ \left| \int_{\mathbb{R}} f(x) \sin(nx) \, dx \right| \leq \int_{\mathbb{R}} |f(x)\sin(nx)| \, dx \]
\[ \leq \int_{\mathbb{R}} |(f(x) - \phi_n(x))\sin(nx)| \, dx + \int_{\mathbb{R}} |\phi_n(x)\sin(nx)| \, dx \]
\[ < \frac{\epsilon}{2} + \int_{\mathbb{R}} |\phi_n(x)\sin(nx)| \, dx \]
Now $\phi_n$ being a step function we have it as the sum of simple functions over disjoint interval $I_n$, where
\[ \bigcup_{n=1}^{\infty} I_n = \mathbb{R}, \text{ i.e.} \]
\[ \phi_n = \sum_{k=1}^{\infty} a_{k,n} \chi_{I_{k,n}} \]
and so we have
\[ \int_{\mathbb{R}} |\phi_n(x)\sin(nx)| \, dx = |a_{k,n}| \int_{I_{k,n}} |\chi_{I_{k,n}} \sin(nx)| \, dx \]
\[ = \sum_{k=1}^{\infty} |a_{k,n}| \int_{I_{k,n}} |\sin(nx)| \, dx \to 0 \text{ as } n \to \infty \]
Hence for some $N$ large enough and all $n > N$ we have
\[ \left| \int_{\mathbb{R}} f(x) \sin(nx) \, dx \right| < \frac{\epsilon}{2} + \int_{\mathbb{R}} |\phi_n(x)\sin(nx)| \, dx < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \]

3. Convergence

Exercise 3.1. Consider the Lebesgue measure space $(\mathbb{R}, M, \mu)$ on $\mathbb{R}$. Let $f$ be a $\mu$-integrable extended real-valued $M$-measurable function on $\mathbb{R}$. Show that
\[ \lim_{h \to 0} \int_{\mathbb{R}} |f(x+h) - f(x)| \, \mu(dx) = 0. \]

Proof: First since $f(x)$ is integrable, we have
\[ \int_{\mathbb{R}} f(x+h) \, \mu(dx) = \int_{\mathbb{R}} f(x) \, \mu(dx) \quad \forall h \in \mathbb{R} \]
Also since $f$ is integrable, there exists a sequence of continuous function $\phi_n$, such that
\[ \int |f(x) - \phi_n(x)| \, \mu(dx) < \frac{\epsilon}{3} \]
Now $|\phi_n(x+h) - \phi_n(x)| < \frac{\epsilon}{3}$ if $|h| < \delta$. Let $N$ be large enough, then
\[ \int |f - f(x+h)| \, \mu(dx) \leq \int |f - \phi_n(x)| + |\phi_n(x+h) - f(x+h)| + |\phi_n(x+h) - \phi_n(x)| \, \mu(dx) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \]
Therefore we have
\[ \lim_{h \to 0} \int_{\mathbb{R}} |f(x+h) - f(x)| \, \mu(dx) = 0. \]
Exercise 3.2. Let \((\mathcal{X}, \mathcal{M}, \mu)\) be a measure space. Let \(f_n\) and \(f\) be an extended real-valued functions on a set \(E \in \mathcal{X}\) such that \(\lim_{n \to \infty} f_n = f\) on \(E\). Then for every \(\alpha \in \mathbb{R}\) we have

\[
\mu\{E : f > \alpha\} \leq \lim_{n \to \infty} \inf \mu\{E : f_n \geq \alpha\} \quad \text{and} \quad \mu\{E : f < \alpha\} \leq \lim_{n \to \infty} \inf \mu\{E : f_n \leq \alpha\}
\]

Proof: I will only show the first inequality since the proofs are identical. Let \(A_\alpha = \{x \in E : f(x) \geq \alpha\}\), \(A_{\alpha,n} = \{x \in E : f_n(x) \geq \alpha\}\) and let \(\chi_A\) denote the characteristic function of \(A\). First we need to show \(\chi_{A_{\alpha,n}} \to \chi_A\) in measure. Let \(\epsilon > 0\), denote the set \(F_{\alpha-\epsilon,n}\) by

\[
F_{\alpha-\epsilon,n} = \{x \in E : |\chi_{A_{\alpha,n}} - \chi_A| \geq \epsilon\}.
\]

Now we want to show that the measure of this set is small. First notice that

\[
F_{\alpha-\epsilon,n}^c \supset \{x \in A_\alpha : |f - f_n| < \epsilon\}
\]

Let \(x\) be in this subset, then this implies two things. First if \(f(x) > \alpha > \alpha - \epsilon\), and \(f_n(x) > f(x) - \epsilon > \alpha - \epsilon\). So we must have

\[
F_{\alpha-\epsilon,n} \subset \{x \in A_\alpha : |f - f_n| \leq \epsilon\}
\]

Now since \(f_n\) converges to \(f\) almost everywhere in \(E\), it converges in measure, and hence the measure of the set \(\mu(F_{\alpha-\epsilon,n}) < \epsilon\). This implies that \(\chi_{A_{\alpha,n}}\) converges to \(\chi_A\) in measure. Now Fatou’s lemma holds for a sequence of functions converging in measure, so we have

\[
\int_E \chi_{A_n} \, d\mu \leq \liminf \int_E \chi_{A_{\alpha,n}} \, d\mu \quad \Rightarrow \quad \mu\{E : f > \alpha\} \leq \lim_{n \to \infty} \inf \mu\{E : f_n \geq \alpha\} \quad \square
\]

Exercise 3.3. Let \(g(x)\) be a real-valued function of bounded variation on an interval \([a, b]\). Suppose that \(f\) is a real-valued decreasing function on \([a, b]\). Show that \(g(f(x))\) is also of bounded variation. If \(f\) is just a bounded continuous function is \(g(f(x))\) still of bounded variation.

Proof: Since \(g\) is of bounded variation we have, let \(\mathcal{P}\) be all the possible partitions of \([a, b]\)

\[
V^b_a(g) = \sup_{\{x_i\} \in \mathcal{P}} \sum_{i=1}^{n} |g(x_i) - g(x_{i-1})|
\]

Now fix \(\epsilon > 0\) and pick an \(\{x_i\}\) such that

\[
V^b_a(g) < \sum_{i=1}^{N} |g(x_i) - g(x_{i-1})| + \epsilon = \epsilon
\]

Now since \(f\) is decreasing on we have that \(f(x_{i+1}) < f(x_i)\). Now call \(y_i = f(x_i), \{y_i\} \cup \{a, b\}\) then is a partition of \([a, b]\), and so we have

\[
\sum_{i=1}^{N} |g(y_i) - g(y_{i-1})| < \sum_{\{x_i\} \in \mathcal{P}} \sup_{i=1}^{n} |g(x_i) - g(x_{i-1})| = V^b_a(g)
\]

This can be done for any partition of \([a, b]\). Therefore \(g(f(x))\) is also of bounded variation.

For the second part, no. Consider the function

\[
f(x) = \begin{cases} 
  x \sin \left(\frac{x}{\epsilon}\right) & x \neq 0 \\
  1 & x = 0
\end{cases}
\]

and let \(g(x) = x\). Now \(g(x)\) is a function of bounded variation on \([-1, 1]\) and \(f(x)\) is a bounded and continuous on \([-1, 1]\), but \(g(f(x)) = f(x)\), which not a function of bounded variation on \([-1, 1]\).

Exercise 3.4. Let \((\mathcal{X}, \mathcal{M}, \mu)\) be a measure space. Let \(f_n\) and \(f\) be an extended real-valued \(\mathcal{M}\)-measurable functions on a set \(E \in \mathcal{X}\) with \(\mu(E) < \infty\). Show that \(f_n\) converges to 0 in measure on \(E\) if and only if \(\lim_{n \to \infty} \int_E \frac{|f_n|}{1 + |f_n|} \, d\mu = 0\).
Proof: $(\Rightarrow)$ If $f_n$ converges to 0 in measure then we have
\[ \mu\{x \in E : |f_n| \geq \epsilon\} < \epsilon. \]
Call this set $A$. Now
\[ \int_E \frac{|f_n|}{1 + |f_n|} \, d\mu = \int_{A^c} \frac{|f_n|}{1 + |f_n|} \, d\mu + \int_{A^c} \frac{|f_n|}{1 + |f_n|} \, d\mu \]
Now the function $\frac{1}{1+x}$, $x \geq 0$ is monotone, and uniformly continuous on any bounded interval. Now $\mu(A) < \epsilon$, and so there is a $\delta$ such that $\mu(A^c) + \mu(e(1 + \mu(E)) < \epsilon(1 + \mu(E))$
Hence we have
\[ \int_{A^c} \frac{|f_n|}{1 + |f_n|} \, d\mu \to 0 \text{ as } \epsilon \to 0 \]
$(\Leftarrow)$ Now suppose that
\[ \int_E \frac{|f_n|}{1 + |f_n|} \, d\mu \to 0 \text{ as } \epsilon \to 0 \]
and suppose that there exists $\epsilon_0$ such that $\mu\{x \in E : |f_n| \geq \epsilon_0\} \geq \epsilon_0$. Then we have
\[ \int_{A^c} \frac{|f_n|}{1 + |f_n|} \, d\mu \geq \frac{\epsilon_0^2}{1 + \epsilon_0^2} \int_{A^c} \frac{|f_n|}{1 + |f_n|} \, d\mu \]
But this implies that
\[ \frac{\epsilon_0^2}{1 + \epsilon_0^2} \leq \int_{A^c} \frac{|f_n|}{1 + |f_n|} \, d\mu \to 0 \]
let $\epsilon = \frac{\epsilon_0^2}{2(1 + \epsilon_0^2)}$, then we have a contradiction □.

Exercise 3.5. Suppose $\mu(E) < \infty$ and $f_n$ converges to $f$ in measure on $E$ and $g_n$ converges to measure on $E$. Prove that $f_n g_n$ converges to $fg$ in measure on $E$.

Proof: Let $h_n = f_n g_n$ and let $h = fg$. Now $h$ and $h_n$ are measurable since $f_n$ and $g_n$ are. For each $\delta > 0$ define
\[ A_n(\delta) = \{x : |h_n(x) - h(x)| \geq \delta\} \]
and let $a_n(\delta) = \mu(A_n(\delta))$. Now because $f_n$ and $g_n$ converge in measure, for any subsequences $f_{n_k}$, $g_{n_k}$ there are subsequences $f_{n_{k_j}}$ and $g_{n_{k_j}}$ such that both $f_{n_{k_j}}$ and $g_{n_{k_j}}$ converge almost everywhere to $f$ and $g$ respectively. Hence we have $h_{n_{k_j}} = f_{n_{k_j}} g_{n_{k_j}}$, which converges to $h = fg$ almost everywhere on $E$. Now since $h_{n_{k_j}}$ converges almost everywhere and $\mu(E)$ is finite we have that $h_{n_{k_j}}$ converges in measure. Now
\[ \lim_{n \to \infty} |h - h_n| \leq \lim_{n \to \infty} \sup_{n \to \infty} |h - h_{n_k}| \to 0 \]
Hence $\lim_{n \to \infty} a_n(\delta) = \lim_{k \to \infty} a_{n_{k_j}}(\delta) = 0$, or $h_n$ converges in measure □.

(Convergence in measure) A sequences $f_n$ of measurable functions is said to converge to $f$ in measure if, given $\epsilon > 0$, there is an $N$ such that for all $n \geq N$ we have
\[ \mu\{x : |f(x) - f_n(x)| \geq \epsilon\} \leq \epsilon \]
Remark: Let $a_n$ be a sequence of real numbers. If there is an $a \in \mathbb{R}$, such that for every subsequence $a_{n_k}$, there is a subsequences for which $a_{n_{k_i}} \to a$, then $a_n \to a$. 

Exercise 3.6. If $f_n$, $f \in L_2$ and $f_n \to f$ almost everywhere, then $\|f_n - f\|_2 \to 0$ if and only if $\|f_n\|_2 \to \|f\|_2$.

Proof: ($\Rightarrow$) Suppose $\|f_n - f\|_2 \to 0$, now

$$\|f_n - f\|_2^2 = \int f_n^2 - 2f_n f + f^2 \geq \|f\|_2^2 - 2 \int |f_n f| + \|f\|_2^2 \geq \|f\|_2^2 - 2\|f_n\|_2 \|f\|_2 + \|f\|_2^2 = \|f\|_2^2 - \|f_n\|_2^2.$$ 

Therefore as $\|f_n - f\|_2 \to 0$ we have $\|f_n\|_2 \to \|f\|_2$.

($\Leftarrow$) Now suppose $\|f_n\|_2 \to \|f\|_2$ and $f_n \to f$ almost everywhere. Now for $p \geq 1$, and for finite $a,b$, we have

$$|a + b|^p \leq 2^p(|a|^p + |b|^p)$$

For each $n$, let

$$g_n = 4(|f_n|^2 + |f|^2) - |f_n - f|^2.$$ 

Now $g_n \geq 0$ almost everywhere. Since $f_n$ and $f$ are finite almost everywhere, by Fatou’s lemma we have

$$\int \liminf g_n \leq \liminf \int g_n.$$ 

Now since $f_n \to f$ almost everywhere we have $\liminf g_n = 8|f|^2$ almost everywhere. So we have

$$8\|f\|_2^2 \leq \liminf \int g_n.$$ 

Now

$$\liminf \int g_n = 4 \liminf \int |f_n|^2 + 4 \liminf \int |f|^2 = \limsup \int |f_n - f|^2$$

$$= 4 \liminf \|f_n\|_2^2 + 4\|f\|_2^2 - \limsup \|f_n - f\|_2^2$$

$$= 8\|f\|_2^2 - \limsup \|f_n - f\|_2^2$$

so we have $0 \leq - \limsup \|f_n - f\|_2^2$, hence $0 \leq \limsup \|f_n - f\|_2^2 \leq 0$. Therefore we have

$$\limsup \|f_n - f\|_2 = \liminf \|f_n - f\|_2 = 0 \Rightarrow \|f_n - f\|_2 \to 0 \square.$$ 

Remark: A sequences of functions $f_n$ converges in measure to $f$ if and only if for every sequences $f_{n_k}$, there is a subsequence $f_{n_{k_j}}$ that converges almost everywhere to $f$.

Exercise 3.7. If $f_n \geq 0$ and $f_n(x) \to f(x)$, in measure then

$$\int f(x) \, dx \leq \liminf \int f_n(x) \, dx$$

Proof: Let $f_{n_k}$ be any subsequence of $f_n$, then there exists an $f_{n_{k_j}}$ such that $f_{n_{k_j}}$ converges to $f$ almost everywhere. By Fatou’s Lemma we have

$$\int f \leq \liminf \int f_{n_{k_j}} = \lim \int f_{n_{k}} \leq \liminf \int f_n.$$ 

Exercise 3.8. Suppose $f_n$ converges to two functions $f$ and $g$ in measure on $D$. Show that $f = g$ almost everywhere on $D$.

Proof: Define the set $E$ as $E = \{x \in D : |f_n(x) - f(x)| > 0\}$. Then if $E_n = \{x \in D : |f_n(x) - f(x)| \geq 1/m\}$, we have $E = \lim E_n$. Now if for some $n$ we have $|f_n(x) - f(x)| < 1/m$ and $|f_n(x) - g(x)| < 1/m$, then we have

$$|f - g| \leq |f_n - f| + |f_n - g| < \frac{1}{m}.$$
And so
\[ \{ x : |f_n(x) - f(x)| < \frac{1}{2m} \} \cap \{ x : |f_n(x) - g(x)| < \frac{1}{2m} \} \subset \{ x : |f(x) - g(x)| < \frac{1}{2m} \} \]
which implies that
\[ \{ x : |f_n(x) - f(x)| \geq \frac{1}{2m} \} \cap \{ x : |f_n(x) - g(x)| \geq \frac{1}{2m} \} \supset \{ x : |f(x) - g(x)| \geq \frac{1}{2m} \} \]
This implies that \( \mu \left\{ x : |f(x) - g(x)| \geq \frac{1}{2m} \right\} < \frac{2}{m} \). Now as \( n \to \infty \), we have \( \frac{2}{m} \to 0 \). Hence \( \mu \left\{ x : |f(x) - g(x)| > 0 \right\} = 0 \). □.

**Exercise 3.9.** Let \( f_n \to f \) in \( L_p(X, \mathcal{M}, \mu) \), with \( 1 \leq p < \infty \), and let \( g_n \) be a sequence of measurable functions such that \( |g_n| \leq M < \infty \) for all \( n \), and \( g_n \to g \) almost everywhere. Prove that \( g_n f_n \to gf \) in \( L_p(X, \mathcal{M}, \mu) \).

**Proof:** Since \( f_n \to f \) in \( L_p \), since \( L_p \) is complete we have \( f \in L_p \). Also since \( |g_n| \leq M \), for all \( n \) this implies that \( |g| \leq M \). Now
\[
\|f_n g_n - g_n f\|_p^p = \int (f_n g_n - g_n f)^p \leq M^p \int |f_n - f|^p \to M^p \|f_n - f\|_p^p
\]
So we have \( \|f_n g_n - g_n f\|_p \leq M \|f_n - f\|_p \), and so
\[
\|f_n g_n - g f\|_p \leq M \|f_n - f\|_p \to 0 \text{ as } n \to \infty
\]
Therefore \( g_n f_n \to gf \) in \( L_p(X, \mathcal{M}, \mu) \). □.

**Exercise 3.10.** Suppose \( f \) is differentiable everywhere on \((a, b)\). Prove that \( f' \) is a Borel measurable function on \((a, b)\).

**Proof:** \( f' \) is Borel measurable if \( \{ x : f'(x) \leq \alpha \} \) is a Borel set. So
\[
f'(x) \leq \alpha \iff \lim_{n \to \infty} n \left( f \left( x + \frac{1}{n} \right) - f(x) \right) \leq \alpha
\]
\[
\iff \lim_{n \to \infty} \left( f \left( x + \frac{1}{n} \right) - f(x) \right) - \frac{\alpha}{n} \leq 0
\]
for all but finitely many \( n \)
\[
\iff \left( f \left( x + \frac{1}{n} \right) - f(x) \right) - \frac{\alpha}{n} \leq \frac{1}{m} \forall m
\]
\[
\iff x \in \lim \inf \left\{ x : f \left( x + \frac{1}{n} \right) - f(x) - \frac{\alpha}{n} \leq \frac{1}{m} \right\} \forall m
\]
\[
\iff x \in \bigcup_{n \geq 1} \bigcap_{k \geq n} \left\{ x : f \left( x + \frac{1}{k} \right) - f(x) - \frac{\alpha}{k} \leq \frac{1}{m} \right\} \forall m
\]
\[
\iff x \in \bigcup_{m \geq 1} \bigcup_{n \geq 1} \bigcap_{k \geq n} \left\{ x : f \left( x + \frac{1}{k} \right) - f(x) - \frac{\alpha}{k} \leq \frac{1}{m} \right\}
\]
Now since \( f(x) \) is differentiable almost everywhere, it is continuous almost everywhere and so the \( f(x) \), and \( f(x + 1/k) \) are measurable. Any linear combination of them is measurable, and so the set
\[
\left\{ x : f \left( x + \frac{1}{k} \right) - f(x) - \frac{\alpha}{k} \leq \frac{1}{m} \right\}
\]
is measurable. Now the collection of all such sets form a \( \sigma \)-algebra, and hence the countable union and intersection of these sets are measurable. Therefore \( f'(x) \) is measurable □.
Exercise 3.11. Let \(c_{n,i}\) be an array of nonnegative extended real numbers for \(n, i \in \mathbb{N}\).

(a) Show that
\[
\lim_{n \to \infty} \inf_{i \in \mathbb{N}} \sum_{i \in \mathbb{N}} c_{n,i} \geq \sum_{i \in \mathbb{N}} \lim_{n \to \infty} \inf_{i \in \mathbb{N}} c_{n,i}
\]

(b) If \(c_{n,i}\) is an increasing sequences for each \(i \in \mathbb{N}\) then
\[
\lim_{n \to \infty} \sum_{i \in \mathbb{N}} c_{n,i} = \sum_{i \in \mathbb{N}} \lim_{n \to \infty} c_{n,i}
\]

Proof: For part (a) first let \(\nu\) denote the counting measure. Now if \(\mathcal{M} = \mathcal{P}(\mathbb{N})\), then \((\mathbb{N}, \mathcal{M}, \nu)\) forms a measure space. Now let \(a_n\) be sequences with \(c_n \in [0, \infty]\). Then the function \(a_n(n) = a_n\) is \(\mathcal{M}\)-measurable, and so
\[
\int_{\mathbb{N}} c \, d\nu = \sum_{n \in \mathbb{N}} c_n
\]

Then by Fatou’s lemma we have
\[
\int_{\mathbb{N}} \lim_{n \to \infty} \inf_{i \in \mathbb{N}} c_n \leq \lim_{n \to \infty} \inf_{i \in \mathbb{N}} \int_{\mathbb{N}} c_n \Rightarrow \sum_{i \in \mathbb{N}} \lim_{n \to \infty} \inf_{i \in \mathbb{N}} c_{n,i} \leq \sum_{i \in \mathbb{N}} \lim_{n \to \infty} c_{n,i} \square
\]

For part (b) using the same measure space \((\mathbb{N}, \mathcal{M}, \nu)\), we know that \(c_n(i) \leq c_n(i+1)\), so by the Monotone convergence theorem we have
\[
\int_{\mathcal{N}} \lim_{n \to \infty} c_n = \lim_{n \to \infty} \int_{\mathcal{N}} c_n \Rightarrow \sum_{i \in \mathbb{N}} \lim_{n \to \infty} c_{n,i} \leq \lim_{n \to \infty} \sum_{i \in \mathbb{N}} c_{n,i} \square
\]

Theorem (Ascoli-Arzela) Let \(\mathcal{F}\) be an equicontinuous family of functions from a separable space \(X\) to a metric space \(Y\). Let \(f_n\) be a sequence in \(\mathcal{F}\) such that for each \(x \in X\) the closure of the set \(\{f_n(x) : 0 \leq n < \infty\}\) is compact. Then there is a subsequence \(f_{n_k}\) that converges pointwise to a continuous function \(f\), and the convergence is uniform on each compact subset \(X\).

Exercise 3.12. Let \(\{q_k\}\) be all the rational numbers in \([0, 1]\). Show that
\[
\sum_{k=1}^{\infty} \frac{1}{k^2} \frac{1}{\sqrt{|x-q_k|}} \text{ converges a.e. in } [0, 1]
\]

Proof: Fix \(\epsilon_0 > 0\), consider the two sets
\[
E_1 = \frac{1}{\sqrt{|x-q_k|}} \leq \frac{1}{\epsilon_0} \quad \text{and} \quad E_2 = \frac{1}{\sqrt{|x-q_k|}} > \frac{1}{\epsilon_0}
\]

Now for each fixed \(x \in [0, 1]\) we can enumerate the rationals however we want (Zorn’s Lemma). Choose such an ordering so that
\[
x \in E_2 \quad \Rightarrow \quad \frac{1}{\sqrt{|x-q_k|}} < k^{1-\epsilon_0}
\]

That is the closer \(q_k\) gets to \(x\), the large the index. Now let \(\nu\) be the counting measure, then we have
\[
\sum_{k=1}^{\infty} \frac{1}{k^2} \frac{1}{\sqrt{|x-q_k|}} = \int_{E_1} \frac{1}{k^2} \frac{1}{\sqrt{|x-q_k|}} \, d\nu + \int_{E_2} \frac{1}{k^2} \frac{1}{\sqrt{|x-q_k|}} \, d\nu
\]
\[
< \int_{E_1} \frac{1}{k^2} \frac{1}{\epsilon_0} \, d\nu + \int_{E_2} \frac{1}{k^{1+\epsilon_0}} \, d\nu
\]
\[
< \int_{[0,1]} \frac{1}{k^2} \frac{1}{\epsilon_0} \, d\nu + \int_{[0,1]} \frac{1}{k^{1+\epsilon_0}} \, d\nu < \infty
\]

This can be done for all \(x \in [0, 1]\). Therefore the series converges almost everywhere in \([0, 1]\) \(\square\).
Exercise 3.13. Let \((\mathcal{X}, \mathcal{M}, \mu)\) be a finite measure space. Let \(f_n\) be an arbitrary sequence of real-valued measurable functions on \(\mathcal{X}\). Show that for every \(\epsilon > 0\) there exists \(E \subset \mathcal{M}\) with \(\mu(E) < \epsilon\) and a sequence of positive real numbers \(a_n\) such that \(a_n f_n \to 0\) for \(x \in \mathcal{X}\setminus E\).

**Proof:** First denote the set \(E_m = \{x : m - 1 \leq |f_n| < m\}\), then the sets \(E_m\) are disjoint and cover \(\mathcal{X}\). Now define \(\alpha\) such as

\[
\sum_{k=1}^{\infty} \frac{1}{k^2} = \alpha
\]

Since \(\mu(\mathcal{X}) < \infty\), if \(\epsilon > 0\), there is an \(M_n\) such that

\[
\frac{\epsilon}{\alpha n^2} > \sum_{m \geq M_n} \mu(E_n) = \mu\{x : |f_n| \geq M_n\}
\]

Now choose these \(M_n\) such that \(M_n > M_{n-1}\) for all \(n\). Define the sets \(F_n = \{x : |f_n| \geq M_n\}\), then we have \(\mu(F_n) < \frac{\epsilon}{\alpha n^2}\). Now if \(E = \bigcup E_n\), then

\[
\mu(E) \leq \sum_{n=1}^{\infty} \mu(E_n) < \frac{\epsilon}{\alpha} \sum_{n=1}^{\infty} \frac{1}{n^2} = \epsilon
\]

Let \(a_n = \frac{1}{M_n^3}\), then if \(x \in \mathcal{X}\setminus E\), then we have

\[
a_n |f_n(x)| < \frac{1}{M_n^3} M_n = \frac{1}{M_n^2} \quad \forall n
\]

And so we have

\[
\left| \sum_{n=1}^{\infty} a_n f_n(x) \right| \leq \sum_{n=1}^{\infty} |a_n f_n(x)| \leq \sum_{n=1}^{\infty} \frac{1}{M_n^2} \leq \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty
\]

Therefore we must have \(a_n f_n(x) \to 0\) on \(\mathcal{X}\setminus E\). \(\square\)

Exercise 3.14. Prove that the gamma function

\[
\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t}
\]

is well defined and continuous for \(x > 0\).

**Proof:** Let \(f(t, x) = t^{x+1} e^{-t}\), and \(x > 0\) and decompose the integral into two integrals \((0, 1]\) and \((1, \infty)\). For the first we have

\[
\int_0^1 t^{x-1} e^{-t} \, dt \leq \int_0^1 t^{x-1} \, dt = \frac{t^x}{x} \bigg|_0^1 < \infty
\]

Now \(f(t, x)\) is continuous on \((1, \infty)\), and also \(t^2 f(t, x) \to 0\) as \(t \to \infty\), so there is an \(M\) such that \(M\) bounds \(t^2 f(t, x)\) on \((1, \infty)\). Now

\[
\int_1^{\infty} t^{x-1} e^{-t} \, dt = \int_0^1 t^{x+1} e^{-t} \, dt = M \int_0^\infty \frac{1}{t^2} \, dt = M
\]

And so \(\Gamma(x)\) is well defined on \((0, \infty)\).

To show continuity, let \(x_n\) be a Cauchy sequence, and define \(f_n(t) = f(t, x_n)\). Now by continuity of \(f(t, x)\) on \((0, \infty) \times (0, \infty)\), we have that for each \(x\), \(f_n \to f\) on \(t \in (0, \infty)\). Now \(f(t, x)\) is bounded on \((1, \infty)\), call this bound \(M > 1\). Define a function \(g(t)\) by

\[
g(t) = \begin{cases} 
  t^{x-1} & 0 < t \leq 1 \\
  t^M e^{-t} & 1 < t \leq \infty
\end{cases}
\]

Now \(f_n, \ f \leq g\) on \((0, \infty)\), so by the Lebesgue Dominated Convergence theorem we have

\[
\int_0^{\infty} |f_n - f| \to 0 \text{ as } n \to \infty
\]
and so we have
\[ |\Gamma(x_n) - \Gamma(x)| = \left| \int_0^\infty f_n(t) - f(t,x) \, dt \right| \leq \int_0^\infty |f_n(t) - f(t,x)| \, dt \to 0 \text{ as } n \to \infty \]
This holds for any sequence such that \( x_n \to x \in (0, \infty) \), therefore \( \Gamma(x) \) is continuous on \((0, \infty) \). □

4. **Lp spaces**

**Exercise 4.1.** Let \( 1 \leq p < q < \infty \). Which of the following statements are true and which are false?

(a) \( L_p(\mathbb{R}) \subset L_q(\mathbb{R}) \)
(b) \( L_q(\mathbb{R}) \subset L_p(\mathbb{R}) \)
(c) \( L_p([2,5]) \subset L_q([2,5]) \)
(d) \( L_q([2,5]) \subset L_p([2,5]) \)

**Proof:** Only part (d) is true. This can easily be shown for any finite interval, let \( I = [a,b] \) Let \( f \in L_q(I) \). Then \( |f|^p \in L_{q/p}(I) \). Now by Holder’s inequality we have
\[
\int_I |f|^p = \| |f|^p \|_{q/p} \| 1 \|_r
\]
where \( r \) is conjugate to \( \frac{q}{p} \). Now
\[
\| f \|^p \leq \| |f|^p \|_{q/p} \| 1 \|_r = \left( \int_I (|f|^p)^{q/p} \right)^{p/q} \mu(I)^{\frac{q-p}{q}} = \| |f|^p \|_{q/p} \mu(I)^{\frac{q-p}{q}}
\]
Hence we have \( \| f \|^p \leq \| f \|_{q/p} \mu(I)^{\frac{q-p}{q}} \), therefore \( f \in L_p(I) \). □

For a counterexample to part (c) consider the function \( f(x) = (x - 2)^{-1/2} \), and let \( p = 1 \) and \( q = 2 \), then \( f \in L_q([2,5]) \), but \( f \notin L_q([2,5]) \).

For a counterexample to part (b) consider the function \( f(x) = (1 + x^2)^{-1/2} \), and let \( p = 1, q = 2 \), then \( f \in L_q(\mathbb{R}) \) but \( f \notin L_p(\mathbb{R}) \).

For a counterexample to part (a) consider the counterexample to part (c) with the zero extension.

**Theorem (Holder Inequality)** If \( p \) and \( q \) are nonnegative extended real numbers such that
\[
\frac{1}{p} + \frac{1}{q} = 1
\]
and if \( f \in L_p \) and \( g \in L_q \) then \( fg \in L_1 \) and
\[
\int |fg| \leq \| f \|_p \| g \|_q
\]

**Proof:** Assume \( 1 < p < \infty \), and suppose that \( f, g \geq 0 \). Let \( h = g^{q-1} \), then \( g = h^{p-1} \). Now
\[
pt f(x) h(x) = pt f(x) h^{p-1} \leq (h(x) + tf(x))^p - h(x)^p
\]
so we have
\[
pt \int fg \leq \int (h + tf)^p - h^p
\]
and we also have
\[
pt \int fg \leq \| h \|_p^p + \| tf \|_p^p - \| h \|_p^p
\]
now differentiating bothsides with respect to \( t \) at \( t = 0 \), we have
\[
p \int fg \leq p \| f \|_p \| h \|_{p-1}^p = p \| f \|_p \| g \|_q \). □
Exercise 4.2. Let $f \in \mathcal{L}_{3/2}([0,5])$. Prove that

$$\lim_{t \to 0^+} \frac{1}{t^{1/3}} \int_0^t f(s) \, ds = 0.$$ 

Proof: Applying Holder's inequality we have

$$\left| \frac{1}{t^{1/3}} \int_0^t f(s) \, ds \right| \leq \frac{1}{t^{1/3}} \int_0^t |f(s)| \, ds \leq \frac{1}{t^{1/3}} \left( \int_0^t |f(s)| \, ds \right)^{2/3} \left( \int_0^t ds \right)^{1/3} \leq \frac{1}{t^{1/3}} \|f(s)\|_{3/2}^{1/3} \leq \left( \int_0^t |f(s)|^{3/2} \, ds \right)^{2/3} \to 0 \quad \text{as} \quad t \to 0^+ \quad \Box$$

Exercise 4.3. Suppose $f \in C^1[0,1]$, $f(0) = f(1)$, and $f > f'$ everywhere.

(1) Prove that $f > 0$ everywhere.

(2) Prove that

$$\int_0^1 \frac{f^2}{f - f'} \, d\mu \geq \int_0^1 f \, d\mu$$

Proof: For (1), if $f(x) = \alpha \in \mathbb{R}^+$ then everything holds. So suppose that, there exists an $c \in (a,b) \subset (0,1)$ such that $f'(c) = 0$. (WLOG) suppose that this $c$ is not a saddle point for $f(x)$, also suppose that $f(c) < 0$. Now if there is a $\delta > 0$ such that $f(c) > f(x)$, for all $x \in B(c,\delta)$, then we have $f'(x) > 0$ for $x \in (c-\delta,c)$. This implies that $f'(x) > f(x)$ for $x \in (c-\delta,c)$. If there is a $\delta > 0$ such that $f(c) < f(x)$, for all $x \in B(c,\delta)$, then we have $f'(x) > 0$ for $x \in (c,c+\delta)$, which implies that $f'(x) > f(x)$ for $x \in (c,c+\delta)$. For both cases we have a contradiction. Therefore $f(x) > 0$ for all $x \in (0,1)$. Now if $f(0) = 0$, $f$ cannot be constant since $0 \not\geq 0$. This implies that, for some $\delta > 0$, $f'(x) > 0$ for $x \in [0,\delta)$, which is a contradiction. Therefore $f(x) > 0$ for all $x \in [0,1]$.

For (2) since $f > f'$ we have that $\sqrt{f-f'}$ is well defined on $[0,1]$. So,

$$\left( \int_0^1 f \right)^2 \leq \left( \int_0^1 \frac{f}{\sqrt{f-f'}} \sqrt{f-f'} \, d\mu \right)^2$$

Hölder's inequality \leq \int_0^1 \frac{f^2}{f-f'} \, d\mu \int_0^1 f-f' \, d\mu \leq \int_0^1 \frac{f^2}{f-f'} \, d\mu \int_0^1 f \, d\mu$$

The last line holds since $f-f' > 0$. This implies that:

$$\int_0^1 f \, d\mu \leq \int_0^1 \frac{f^2}{f-f'} \, d\mu \quad \Box$$

Exercise 4.4. If $f(x) \in \mathcal{L}_p \cap \mathcal{L}_\infty$ for some $p < \infty$. Show that

(a) $f(x) \in \mathcal{L}_q$ for $q > p$.

(b) $\lim_{q \to \infty} \|f\|_q = \|f\|_\infty$.

Proof: For part (a) Let $0 < p < q < \infty$ and let $f \in \mathcal{L}_p \cap \mathcal{L}_\infty$. Then if $\alpha = \frac{q}{p}$ and if $\beta = \frac{q}{q-p}$, then we
have $\frac{1}{a} + \frac{1}{b} = 1$. Now applying Hölder’s inequality we have

$$
\|f\|_q^q = \int |f|^q = \int |f|^q(\frac{1}{a} + \frac{1}{b}) = \int |f|^p|f|^{q-p} = \int |f|^p|f|^{q-p} \leq \|f\|_1 \|f\|_{q-p} \leq \|f\|_1 \|f\|_{q-p}.
$$

Now since $|f| \leq \|f\|_\infty$ almost everywhere and $q - p > 0$ we have $|f|^{q-p} \leq \|f\|^{q-p} \|f\|_\infty$ almost everywhere, and so $\|f\|_{q-p} \|f\|_\infty < \infty$. Also since $f$ is monotone increasing, we have $\|f\|_{q-p} \|f\|_\infty = \|f\|_{q-p}$. We also have $\|f\|_1 \|f\|_{q-p} < \infty$. Therefore $f \in L_q \implies f \in L_p$. □.

For part (b), first suppose that $\|f\|_\infty = 0$. This implies that $f = 0$ almost everywhere and hence $\|f\|_q = 0$ for all $q$. Hence $\lim q \|f\|_q \to \|f\|_\infty$ trivially.

Now suppose that $f \in L_p \cap L_\infty$ and $\|f\| \neq 0$. From part (a) we have

$$
\|f\|_q \leq (\|f\|_p)^{1/q} (\|f\|_\infty)^{1-\frac{q}{p}}
$$

Now let $\epsilon > 0$, then on a set $E$ of nonzero measure, $|f| > \|f\|_\infty - \epsilon$. If $\mu(E) = \infty$, choose a subset of $E$ with finite measure. Then we have

$$
\|f\|_q^q = \int_E |f|^q d\mu 
$$

Now this is for all $q > p$. Let $q_n$ be a sequence of numbers greater than $p$ that converges to $\infty$. Then

$$
\lim_{n \to \infty} \mu(E)^{\frac{q_n}{p}} \|f\|_q^q - \epsilon \leq \lim \inf \|f\|_{q_n}
$$

Now since this holds for all $\epsilon > 0$ we have $\lim_{n \to \infty} \|f\|_{q_n} = \|f\|_\infty$.

Exercise 4.5. Suppose that $f \in L_p([0,1])$ for some $p > 2$. Prove that $g(x) = f(x^2) \in L_1([0,1])$

Proof: $f \in L_p([0,1])$ implies that $\|f\|_p < \infty$. In particular this implies that $\|g\|_p = \|f(x^2)\|_p < \infty$.

Now

$$
\int_0^1 |g(x)| \, dx = \int_0^1 |f(x^2)| \, dx
$$

change of variables ($y = x^2$) = $\int_0^1 |f(y)| \frac{1}{2\sqrt{y}} \, dy$

hölder’s inequality $\leq \frac{1}{2} \|f\|_p \left( \frac{1}{\sqrt{y}} \right)^{\frac{1}{p-1}}$
Now $f \in L_p([0,1])$ and since $p > 2$ we have $\left\| \frac{1}{\sqrt{p}} f \right\|_C < \infty$, therefore $g(x) \in L_1([0,1])$. □.

**Exercise 4.6.** Let $f \in L_p(\mathcal{X}) \cap L_q(\mathcal{X})$ with $1 \leq p < q < \infty$. Prove that $f \in L_r(\mathcal{X})$ for all $p \leq r \leq q$.

**Proof:** Let $E_1 = \{x : 0 \leq |f(x)| \leq 1\}$, and $E_2 = \{x : 1 > |f(x)|\}$, then $E_1, E_2$ are a Hahn decomposition for $\mathcal{X}$. Now suppose $f \in L_p \cap L_q$. Now

$$
\|f\|_r = \int_{E_1} |f|^r + \int_{E_2} |f|^r \\
\leq \int_{E_1} |f|^p + \int_{E_2} |f|^q \\
\leq \int_{\mathcal{X}} |f|^p + \int_{\mathcal{X}} |f|^q \\
= \|f\|_p + \|f\|_q \\
\therefore f \in L_r(\mathcal{X})
$$

**Exercise 4.7.** Suppose $f$ and $g$ are real-valued $\mu$-measurable functions on $\mathbb{R}$, such that

1. $f$ is $\mu$-integrable.
2. $g \in C_0(\mathbb{R})$.

For $c > 0$ define $g_c(t) = g(ct)$. Prove that:

(a) $\lim_{c \to \infty} \int_{\mathbb{R}} f g_c \, d\mu = 0$,

(b) $\lim_{c \to 0} \int_{\mathbb{R}} f g_c \, d\mu = g(0) \int_{\mathbb{R}} f \, d\mu$.

**Proof:** For part (a) define $h_n(x) = f(x)g_n(x)$. Now since $f \in L_1(\mathbb{R})$ we know that $f(x) < \infty$ a.e., and since $g \in C_0(\mathbb{R})$ we know that $g_n(x) \to 0$ as $n \to \infty$.

For a fixed $x$ such that $f(x) < \infty$ we have

$$
h_n(x) \to 0 \text{ as } n \to \infty
$$

Hence $h_n \to 0$ a.e.. Also since $g \in C_0(\mathbb{R})$ we have that there is some $M$ such that $|g(x)| < M$. So we have

$$
\left| \int_{\mathbb{R}} h_n(x) \, d\mu \right| \leq \int_{\mathbb{R}} |f(x)g_n(x)| \, d\mu \leq M \int_{\mathbb{R}} |f(x)| \, d\mu < \infty
$$

since $f \in L_1(\mathbb{R})$. Hence by the Lebesgue Dominated Convergence theorem we have

$$
\lim_{n \to \infty} \int_{\mathbb{R}} f g_n \, d\mu = \lim_{n \to \infty} \int_{\mathbb{R}} h_n \, d\mu = \int_{\mathbb{R}} \lim_{n \to \infty} h_n \, d\mu = 0
$$

**Proof:** For part (b) we know that for all $n > 0$, $f g_n \in L_1(\mathbb{R})$. Define $h_n(x) = |f(x)g(xn^{-1})|$, again since $g \in C_0(\mathbb{R})$ we have that there is some $M$ such that $|g(x)| < M$. So

$$
\left| \int_{\mathbb{R}} h_n(x) \, d\mu \right| \leq \int_{\mathbb{R}} |f(x)g_n(x)| \, d\mu \leq M \int_{\mathbb{R}} |f(x)| \, d\mu < \infty
$$

Hence by the Lebesgue Dominated Convergence theorem we have

$$
\lim_{n \to \infty} \int_{\mathbb{R}} f g_{1/n} \, d\mu = \lim_{n \to \infty} \int_{\mathbb{R}} h_n \, d\mu \\
= \int_{\mathbb{R}} \lim_{n \to \infty} h_n \, d\mu \\
= \int_{\mathbb{R}} \lim_{n \to \infty} f g_{1/n} \, d\mu \\
= g(0) \int_{\mathbb{R}} f \, d\mu
$$
Exercise 4.8. Let $E$ be a measurable subset of the real line. Prove that $L_\infty(E)$ is complete.

Proof: Let $f_n$ be a Cauchy sequence of measurable functions in $L_\infty$. Then there exists $k \in \mathbb{N}$ such that if $m, n \geq N_k$, then

$$\|f_n - f_m\|_\infty < \frac{1}{k}, \forall n, m > N_k \quad \Rightarrow \quad |f_n - f_m| < \frac{1}{k} \text{ a.e.}$$

Now define the sets $E_{n,m,k}$ by

$$E_{n,m,k} = \{ x \in E : |f_n(x) - f_m(x)| \geq \frac{1}{k} \}$$

then for each $n, m > N_k$, the set $E_{n,m,k}$ is empty. Let $F$ be defined by

$$F = \bigcup_{k \geq m,n,N_k} E_{n,m,k}$$

Now $F$ is a countable union of empty sets, and therefore is empty. Now for any $x \in E \setminus F$ we have

$$|f_n(x) - f_m(x)| < \frac{1}{k}$$

and so $f_n(x)$ is a Cauchy sequence in $\mathbb{R}$. Now

$$|f_m(x)| \leq |f_m(x) - f_n(x)| + |f_n(x)| < \frac{1}{k} + |f_n(x)|.$$ 

Taking $m \to \infty$, we have

$$|f(x)| \leq \frac{1}{k} + |f_n(x)| < \frac{1}{k} + \|f_n(x)\|_\infty \quad \text{a.e.}$$

Hence for each $n$ we have $|f| \leq \frac{1}{k} + \|f_n\|_\infty$ almost everywhere so $f \in L_\infty$. Therefore $L_\infty$ is complete.

Theorem (Riesz-Fischer) The $L_p(E)$ spaces are complete.

Proof: For $1 \leq p < \infty$, let $f_n$ be a Cauchy sequence on $L_p$.

$$\forall \epsilon > 0 \exists N \quad \text{s.t.} \quad \|f_m - f_n\|_p < \epsilon \quad \forall n, m > N$$

Now let $n_k = N 2^{-k}$, then the subsequence $f_{n_k}$ satisfies

$$\|f_{n_{k+1}} - f_{n_k}\|_p < \frac{1}{2k}$$

Define the function $f$ by

$$f(x) = f_{n_1} + \sum_{k=1}^{\infty} (f_{n_{k+1}} - f_{n_k}) \quad \text{for} \quad x \in E$$

Now the partial sums $S_N(f)$ is just

$$S_N(f) = f_{n_1} + \sum_{k=N}^{\infty} (f_{n_{k+1}} - f_{n_k}) = f_{n_N}$$

Define the function $g(x)$ by,

$$f(x) = |f_{n_1}| + \sum_{k=1}^{\infty} |f_{n_{k+1}} - f_{n_k}| \quad \text{for} \quad x \in E$$

Now by Minikowski’s inequality we have

$$\|S_N(g)\|_p \leq \|f_{n_1}\|_p + \sum_{k=1}^{N-1} \|f_{n_{k+1}} - f_{n_k}\|_p \leq \|f_{n_1}\|_p + \sum_{k=1}^{N-1} \frac{1}{2k}$$

So the increasing sequences of partial sums $\|S_N(g)\|_p$ is bounded above by $\|f_{n_1}\|_p + 1$. Hence we have

$$\int_E g^p < \infty \quad \Rightarrow \quad \int_E |f|^p < \infty \quad \Rightarrow \quad \int_E f^p < \infty$$
This implies that the series \( f_{n_k} \) converges almost everywhere. Now
\[
|f - f_{n_N}| = |S_\infty(f) - S_{N-1}(f)| = \left| \sum_{k=1}^{N} f_{n_k+1} - f_{n_k} \right| \leq g
\]
Hence by the Lebesgue dominated convergence theorem we have
\[
\lim_{k \to \infty} \|f - f_{n_k}\|_p = \int_E \lim_{k \to \infty} (f(x) - f_{n_k})^p = 0
\]
Hence \( f_{n_k} \) converges to \( f \) in \( L_p(E) \). Now \( f_n \) is itself Cauchy, hence \( f_n \) converges to \( f \) is in \( L_p(E) \).

**Exercise 4.9.** Let \( g(x) \) be measurable and suppose \( \int_a^b f(x)g(x) \, dx \) is finite for any \( f(x) \in L_2 \). Prove that \( g(x) \in L_2 \).

**Proof:** If \( f = 1 \), then \( f \in L_2([a,b]) \) so \( \int_a^b g \, dx < \infty \) which implies that \( g \in L_1[a,b] \). Let \( F = \int_a^b g \, dx \), then \( F \) is a bounded linear functional from \( L_2([a,b]) \) to \( \mathbb{R} \). So there exists an \( M \) such that
\[
\|F(f)\| = \sup_{\|f\|_2 = 1} \left\{ \int_a^b fg \right\} < M, \quad f \in L^2([0,1])
\]
Then by the Riesz Representation Theorem \( g \) must be in \( L_2([0,1]) \).

**Theorem (Riesz Representation)** Let \( F \) be a bounded linear functional on \( L_p \) for \( 1 \leq p < \infty \). Then there exists a function \( g \in L_q \) such that
\[
F(f) = \int f g.
\]
We also have \( \|F\| = \|g\|_q \).

**Proof:** Just considering the finite dimensional case. Let \( \mu \) be of finite measure. Then every bounded measurable function is in \( L_p(\mu) \). Define a set function \( \nu \) on the measurable sets by \( \nu(E) = F(\chi_E) \). If \( E \) is the union of a sequence \( E_n \) of disjoint measurable sets, define a sequence \( \alpha_n = \text{sgn} F\chi_{E_n} \) and set
\[
f = \sum \alpha_n \chi_{E_n}
\]
Then \( F \) is bounded and we have
\[
\sum_{n=1}^{\infty} |\nu(E_n)| = F(f) < \infty, \quad \sum_{n=1}^{\infty} \nu(E_n) = F(f) = \nu(E)
\]
Hence \( \nu \) is a signed measure, and by construction it is absolutely continuous with respect to \( \mu \). By the Radon-Nikodym Theorem, there is a measurable function \( g \) such that for each measurable set \( E \) we have
\[
\nu(E) = \int_E g \, d\mu
\]
Since \( \nu \) is always finite implies that \( g \) integrable. Now if \( \phi \) is a simple function, the linearity of \( F \) and of the integral imply that
\[
F(\phi) = \int \phi g \, d\mu
\]
Since the left-hand side is bounded by \( \|F\| \|\phi\|_p \) we have \( g \in L^q \). Now let \( G \) be the bounded linear functional defined on \( L_p \) by
\[
G(f) = \int fg \, d\mu
\]
Then $G - F$ is a bounded linear function which vanishes on the subspace of simple functions, which are dense in $\mathcal{L}_p$. Hence we must have $G - F = 0$ in $\mathcal{L}_p$. So for all $f \in \mathcal{L}_p$, we have

$$F(f) = \int fg \, d\mu$$

and by construction $\|F\| = \|G\| = \|g\|_p$. □

**Exercise 4.10.** Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f$ be an extended real-valued $\mathcal{M}$-measurable function on $X$ such that

$$\int_X |f|^p \, d\mu < \infty \text{ for } p \in (0, \infty).$$

Show that $\lim_{\lambda \to \infty} \lambda^p \mu\{x : |f(x)| \geq \lambda\} = 0$

**Proof:** First define the set $E_\lambda = \{x \in X : f(x) \geq \lambda\}$. Now notice that $E_\nu \subset E_\lambda$ if $\nu > \lambda$, also because $f \in \mathcal{L}_p$ we have $\mu(E_\lambda) < \lambda^{-1}$ if $\lambda$ is large enough, in particular $\mu(E_\infty) = 0$. Now

$$\lambda^p \mu\{x : |f(x)| \geq \lambda\} = \lambda^p \int_{E_\lambda} \, d\mu \leq \int_{E_\lambda} |f|^p \, d\mu$$

Hence we have

$$\lim_{\lambda \to \infty} \lambda^p \mu\{x : |f(x)| \geq \lambda\} \leq \int_{E_\infty} |f|^p \, d\mu = 0$$

Since $f \in \mathcal{L}_p$, then $|f|^p \in \mathcal{L}_1(X)$, $\mu(E_\infty) = 0$ and the integral of an Lebesgue integrable function over a set of measure zero is zero. □

5. **Signed Measures**

**Remark:** If $(X, \mathcal{M})$ is a measure space, and if $\mu, \nu$ are two measure defined on $(X, \mathcal{M})$. $\mu$ and $\nu$ are said to be mutually singular ($\mu \perp \nu$), if there are disjoint stes $A$ and $B$, in $\mathcal{M}$ such that $X = A \cup B$ and $\nu(A) = \mu(B) = 0$. A measure $\nu$ is said to be absolutely continuous with respect to the measure $\mu$, $(\nu \ll \mu)$, if $\nu(A) = 0$ for each set $A$ for which $\mu(A) = 0$.

**Exercise 5.1.** Let $\mu$ be a measure and let $\lambda, \lambda_1, \lambda_2$ be signed measure on the measurable space $(X, \mathcal{A})$.

Prove:

(a) If $\lambda \perp \mu$ and $\lambda \ll \mu$, then $\lambda = 0$

(b) If $\lambda_1 \perp \mu$ and $\lambda_2 \perp \mu$, then, if we set $\lambda = c_1 \lambda_1 + c_2 \lambda_2$ with $c_1, c_2 \in \mathbb{R}$ such that $\lambda$ is a signed measure, than we have $\lambda \perp \mu$.

(c) If $\lambda_1 \ll \mu$ and $\lambda_2 \ll \mu$, then, if we set $\lambda = c_1 \lambda_1 + c_2 \lambda_2$ with $c_1, c_2 \in \mathbb{R}$ such that $\lambda$ is a signed measure, than we have $\lambda \ll \mu$.

**Proof:** For part (a), if $\nu$ is a signed measure such that $\nu \perp \mu$ and $\nu \ll \mu$. There are disjoint measurable sets $A$ and $B$ such that $X = A \cup B$ and $|\nu|(B) = |\mu|(A) = 0$. Then $|\nu|(A)| = 0$ so $|\nu|(X) = |\nu|(A) + |\nu|(B) = 0$. Hence we have $\nu^+ = \nu^- = 0$ i.e. $\nu = 0$.

For part (b), there are disjoint measurable sets $A_i$ and $B_i$ such that $X = A_i \cup B_i$ and $\mu(B_i) = \nu_i(A_i) = 0$, for $i = 1, 2$. Now $X = (A_1 \cap A_2) \cup (B_1 \cup B_2)$ and $(A_1 \cap A_2) \cap (B_1 \cup B_2) = \emptyset$. Now we have

$$(c_1 \nu_1 + c_2 \nu_2)(A_1 \cap A_2) = \mu(B_1 \cup B_2) = 0 \Rightarrow (c_1 \nu_1 + c_2 \nu_2) \perp \mu$$

For part (c), suppose $\nu_1 \ll \mu$ and $\nu_2 \ll \mu$. If $\mu(E) = 0$, then $\nu_1(E) = \nu_2(E) = 0$. Hence

$$(c_1 \nu_1 + c_2 \nu_2)(E) = 0 \Rightarrow (c_1 \nu_1 + c_2 \nu_2) \ll \mu$$
Exercise 5.2. Let $\mu$ be a positive measure and $\nu$ be a finite positive measure on a measurable space $(\mathcal{X}, \mathcal{M})$. Show that if $\nu << \mu$, then for every $\epsilon > 0$ there is a $\delta > 0$, such that for every $E \subset \mathcal{M}$ with $\mu(E) < \delta$, we have $\nu(E) < \epsilon$.

Proof: Suppose not. Then there is an $\epsilon > 0$ such that for every $\delta > 0$, there is $E_\delta \subset \mathcal{M}$, such that $\mu(E_\delta) < \delta$, and $\nu(E_\delta) \geq \epsilon$. In particular, for every $n \geq 1$, there is an $E_n$ such that $\mu(E_n) < \frac{1}{n^2}$ and $\nu(E_n) \geq \epsilon$. Now we have

$$\sum_{n=1}^{\infty} \mu(E_n) = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

Let $E = \limsup E_n$, then $\mu(E) = 0$. Now since $\nu << \mu$, we have $\nu(E) = 0$. Now

$$\nu(E) = \nu(\limsup E_n) \geq \limsup \nu(E_n) \geq \epsilon$$

But this implies that $\nu(E_n) \geq \epsilon > 0$, and hence $\nu(E) > 0$, which is a contradiction. Therefore given $\epsilon > 0$ there is a $\delta > 0$, such that for every $E \subset \mathcal{M}$ with $\mu(E) < \delta$, we have $\nu(E) < \epsilon$ \Box.

Theorem (Hahn Decomposition) Let $\nu$ be a signed measure on the measurable space $(\mathcal{X}, \mathcal{M})$. Then there is a positive set $A$ and a negative set $B$ such that $X = A \cup B$ and $A \cap B = \emptyset$.

Theorem (Jordan Decomposition) Let $\nu$ be a signed measure on the measurable space $(\mathcal{X}, \mathcal{M})$. Then there are two mutually singular measure $\nu^+$ and $\nu^-$ on $(\mathcal{X}, \mathcal{M})$ such that $\nu = \nu^+ - \nu^-$. Moreover, there is only one such pair of mutually singular measures.

Exercise 5.3. Suppose $(\mathcal{X}, \mathcal{M})$ is a measurable space, and $Y$ is the set of all signed measure $\nu$ on $\mathcal{M}$ for which $\nu(E) < \infty$, whenever $E \in \mathcal{M}$. For $\nu_1, \nu_2 \in Y$, define

$$d(\nu_1, \nu_2) = \sup_{E \in \mathcal{M}} |\nu_1(E) - \nu_2(E)|$$

Show that $d$ is a metric on $Y$ and that $Y$ equipped with $d$ is a complete metric space.

Proof: Since $\nu_i$ are choosen such that $\nu_i(E) < \infty$, then for any $\nu_1, \nu_2 \in Y$ and $E \in \mathcal{M}$, we have $|\nu_1(E) - \nu_2(E)| < \infty$. So we have $d : Y \times Y \to [0, \infty)$. Now to show $d$ is a metric on $Y$ we need to show symmetry, positive definiteness and the triangle inequality. Clearly $d(\nu_1, \nu_2) = d(\nu_2, \nu_1)$ by definition of $d$. For the triangle inequality we have

$$d(\mu, \nu) = \sup_{E \in \mathcal{M}} |\mu(E) - \nu(E)|$$

$$\leq \sup_{E \in \mathcal{M}} \{|\nu(E) - \sigma(E)| + |\mu(E) - \sigma(E)|\}$$

$$\leq \sup_{E \in \mathcal{M}} \{|\nu(E) - \sigma(E)|\} + \left\{\sup_{F \in \mathcal{M}} |\mu(F) - \sigma(F)|\right\}$$

$$= d(\mu, \sigma) + d(\sigma, \nu)$$

Now to show definiteness, if $\mu = \nu$, then $|\mu(E) - \nu(E)| = 0$ for any $E \in \mathcal{M}$, and so $d(\mu, \nu) = 0$. On the other hand if $d(\mu, \nu) = 0$, then we have $|\mu(E) - \nu(E)| = 0$. Let $(A_1, B_1), (A_2, B_2)$ be Hahn decompositions of $\mu$, and $\nu$ respectively.

Case 1: If $E \subset A_1 \cap A_2$, then $\mu(E) = \mu^+(E)$, and $\nu(Y) = \nu^+(Y)$, hence $|\mu(E) - \nu(E)| = |\mu^+(E) - \nu^+(E)|$. So we have $\mu^+ = \nu^+$ on $A_1 \cap A_2$.

Case 2, 3: If $E \subset A_1 \cap B_2$, then we have $\mu(E) = -\mu^-(E)$ and $\nu(E) = \nu^+(E)$, hence

$$0 = |\mu(E) - \nu(E)| = |-\mu^-(E) - \nu^+(E)| = \mu^-(E) + \nu^+(E)$$

Hence $\mu^- = \nu^+ = 0$ on $E \subset A_1 \cap B_2$. If $E \subset A_2 \cap B_1$, by the same proof we have the result $\mu^+ = \nu^- = 0$ on $E \subset A_2 \cap B_1$.
Case 3: If $E \subset B_1 \cap B_2$, then $\mu(E) = -\mu^-(E)$ and $\nu(E) = -\nu^-(E)$. So
\[
0 = |\mu(E) - \nu(E)| = | - \mu^-(E) + \nu^+(E) |
\]
and so $\mu^- = \nu^- = 0$ on $E \subset B_1 \cap B_2$. So definiteness holds, therefore $d$ is a metric on $Y$.

Now to show the metric space is complete. Let $\nu_n$ be a Cauchy sequence. Then for any $\epsilon > 0$, there is an $N$ such that if $m, n > N$, we have
\[
\sup_{E \in \mathcal{M}} |\nu_n(E) - \nu_m(E)| < \epsilon
\]
If particular, for a fixed set $E$, we have $\nu_n$ is a Cauchy sequence in $\mathbb{R}$. Hence there exists some $\mu(E) \in \mathbb{R}$, such that $\nu_n \to \mu$. By the uniform boundedness principal we know that $\mu$ is bounded, and hence
\[
\nu_n \to \mu \text{ in the metric } d \quad □
\]
Remark: The measure $|\nu|$ is defined from the Jordan decomposition by, $|\nu|(E) = \nu^+ E + \nu^- E$.

**Theorem (Radon-Nikodym)** let $(\mathcal{X}, \mathcal{M}, \mu)$ be a $\sigma$-finite measure space, and let $\nu$ be a measure defined on $\mathcal{M}$ which is absolutely continuous with respect to $\mu$. Then there is a nonnegative measurable function $f$ such that for each set $E$ on $\mathcal{M}$ we have
\[
\nu(E) = \int_E f \, d\mu
\]
The function $f$ is unique in the sense that if $g$ is any measurable function with this property then $g = f$ almost everywhere.

**Proof:** Only the finite case is considered. Let $\mu$ be finite then $\nu - \alpha \mu$ is a signed measure for each rational number $\alpha$. Let $(A_\alpha, B_\alpha)$ be a Hahn decomposition for $\nu - \alpha \mu$, and take $A_0 = \mathcal{X}$ and $B_0 = \emptyset$. Now $B_\alpha \sim B_\beta = B_\alpha \cap A_\beta$. So we have
\[
(\nu - \alpha \mu)(B_\alpha \sim B_\beta) \leq 0 \quad (\nu - \beta \mu)(B_\alpha \sim B_\beta) \geq 0
\]
hence we must have $\mu(B_\alpha \sim B_\beta) = 0$. Now there exists a measurable function $f$ such that for each rational $\alpha$ we have $f \geq \alpha$ almost everywhere on $A_\alpha$ and $f \leq \alpha$ almost everywhere on $B_\alpha$. Since $B_0 = \emptyset$ be an arbitrary set in $\mathcal{M}$, and set
\[
E_k = E \cap (B_{(k+1)/N} \sim B_{k/N})
\]
Then $E = \bigcup_{k=1}^{\infty} E_k$, and this union is disjoint modulo null sets. Hence we have
\[
\nu(E) = \nu(E_\infty) + \sum_{k=0}^{\infty} \nu(E_k).
\]
Since $E_k \subset B_{(k+1)/N} \cap A_{k/N}$, we have
\[
\frac{k}{N} \leq f \leq \frac{k+1}{N} \quad \text{on } E_k, \quad \text{and so}
\]
\[
\frac{k}{N} \mu(E_k) \leq \int_{E_k} f \, d\mu \leq \frac{k+1}{N} \mu(E_k).
\]
Now since $\frac{k}{N} \mu(E_k) \leq \nu(E_k) \leq \frac{k+1}{N} \mu(E_k)$, we have
\[
\nu(E_k) - \frac{1}{N} \mu(E_k) \leq \int_{E_k} f \, d\mu \leq \nu(E_k) + \frac{1}{N} \mu(E_k).
\]
) On $E_\infty$ we have $f = \infty$ almost everywhere. If $\mu(E_\infty) > 0$, we must have $\nu(E_\infty) > 0$, since $(\nu - \alpha \mu)(E_\infty)$ is positive for each $\alpha$. If $\mu(E_\infty) = 0$, we have $\nu(E_\infty) = 0$. Since $\nu << \mu$, for either case we have
\[
\nu(E_\infty) = \int_{E_\infty} f \, d\mu.
\]
Hence we have
\[
\nu(E) - \frac{1}{N} \mu(E) \leq \int_{E} f \, d\mu \leq \nu(E) + \frac{1}{N} \mu(E).
\]
Since $\mu(E)$ is finite and $N$ arbitrary, we must have $\nu(E) = \int_E f \, d\mu$.

The function $f = \left[\frac{d\nu}{d\mu}\right]$ above is called the Radon-Nikodym derivative of $\nu$ with respect to $\mu$.

**Exercise 5.4.** Suppose $\nu$ and $\mu$ are $\sigma$-finite measures on a measurable space $(\mathcal{X}, \mathcal{A})$, such that $\nu << \mu$, and $\nu << \mu - \nu$. Prove that

$$\mu\left(\left\{x \in \mathcal{X} : \frac{d\nu}{d\mu} = 1\right\}\right) = 0.$$ 

**Proof:** First notice that if $E \subset \mathcal{X}$, such that $(\mu - \nu)E = 0$, then we have $\mu(E) = \nu(E)$. But we have $\nu << \mu - \nu$, hence if $(\mu - \nu)E = 0$, then $\nu(E) = 0 = \mu(E)$. Conversely if $\nu(E) = \mu(E)$ and $\nu << \mu - \nu$, then $\mu(E) - \nu(E) = 0$, and so $\nu = 0$ thus $\mu(E) = 0$. So if $\nu(E) = \mu(E)$, then $\nu(E) = 0$.

Now let $E = \left\{x \in \mathcal{X} : \frac{d\nu}{d\mu} = 1\right\}$ and consider $\nu(E)$. By the Radon-Nikodym theorem we have

$$\nu(E) = \int_E d\nu = \int_E \frac{d\nu}{d\mu} \, d\mu$$

but $\frac{d\nu}{d\mu} = 1$ on $E$, and so

$$\nu(E) = \int_E \frac{d\nu}{d\mu} \, d\mu = \int_E d\mu = \mu(E)$$

Hence $\mu(E) = \mu\left(\left\{x \in \mathcal{X} : \frac{d\nu}{d\mu} = 1\right\}\right) = 0$ □.

**Exercise 5.5.** Let $\mu$ and $\nu$ be two measure on the same measurable space, such that $\mu$ is $\sigma$-finite and $\nu$ is absolutely continuous with respect to $\mu$.

(a) If $f$ is a nonnegative measurable function, show that

$$\int f \, d\nu = \int f \left[\frac{d\nu}{d\mu}\right] \, d\mu$$

(b) If $f$ is a measurable function, prove that $f$ is integrable with respect to $\nu$, if and only if $f \left[\frac{d\nu}{d\mu}\right]$ is integrable with respect to $\mu$, and in this case, part (a) still holds.

**Proof:** For part (a), let $E$ be a measurable set and let $f = \chi_E$. Suppose that $h = \left[\frac{d\nu}{d\mu}\right]$ exists. Then

$$\int f \, d\nu = \int \chi_E \, d\nu = \nu(E) = \int h \, d\mu = \int h\chi_E \, d\mu = \int f \, h \, d\mu.$$ 

So the equality holds for characteristic functions. Let $f = \phi$ be a simple function, then by the above we have

$$\int \phi \, d\nu = \int \phi h \, d\mu.$$ 

Now let $f$ be a nonnegative measurable function. There there exists a monotone sequence of simple functions $\phi_n$ such that $0 \leq \phi_n \leq f$ and $\phi_n \to f$ almost everywhere. Applying the Monotone Convergence theorem, we have

$$\int f \, d\nu = \lim_{n \to \infty} \int \phi_n \, d\nu = \lim_{n \to \infty} \int \phi_n h \, d\mu = \int f h \, d\mu.$$ 

For part (b), $f$ is $\nu$-integrable if and only if $\int f^+ \, d\nu - \int f^- \, d\nu$ is finite. Now by part (a) we have

$$\int f^+ \, d\nu = \int f^+ h \, d\mu \text{ and } \int f^- \, d\nu = \int f^- h \, d\mu.$$ 

So we have $f$ is $\nu$-integrable if and only if $f$ is $\mu$-integrable □.

**Theorem (Lebesgue Decomposition)** Let $(\mathcal{X}, \mathcal{M}, \mu)$ be a $\sigma$-finite measure space and $\nu$ a $\sigma$-finite...
measure defined on $M$. Then we can find a measure $\nu_0$, singular with respect to $\mu$ and a measure $\nu_1$ absolutely continuous with respect to $\mu$, such that $\nu = \nu_0 + \nu_1$. Furthermore, the measures $\nu_0$ and $\nu_1$ are unique.

6. Topological and Product Measure Spaces

Exercise 6.1. Let $L$ be a normed space. Then every weakly bounded set $X$ is bounded.

Proof: Let $\phi : L \to L^{**}$, by $\phi(x)(f) = f(x)$, where $x \in L$, $f \in L^*$. Now $X^*$ is a Banach space and $\phi(X)$ is a family of bounded linear functionals on $X^*$, and for each $f \in L^*$ we have
\[
\sup \{ \phi(x)(f) : x \in X \} = \sup \{ f(x) : x \in X \} < \infty
\]
Then from the uniform boundness principle we have
\[
\sup \{ \|x\| : x \in X \} = \sup \{ \|\phi(x)\| : x \in X \} < \infty
\]
Therefore every weakly bounded nonempty set of a normed space is bounded \(\square\).

Exercise 6.2. Suppose that $A$ is a subset in $\mathbb{R}^2$. Define for each $x \in \mathbb{R}^2$, $p(x) = \inf \{|y - x| : y \in A\}$. Show that $B_r = \{x : p(x) \leq r\}$ is a closed set for each nonnegative $r$. Is the measure of $B_0$ equal to the outer measure of $A$?

Proof: Let $z \in (B_r)$, and let $\epsilon > 0$. Then there is $x \in B_r$ such that $|x - z| < \epsilon$. So we have
\[
p(z) = \inf \{|z - y| : y \in A\} \\
\leq \inf \{|z - x| + |x - y| : y \in A\} \\
\leq \epsilon + \inf \{|x - y| : y \in A\} \\
\leq \epsilon + r.
\]
This is for all $\epsilon > 0$, therefore $p(z) \leq r$ which implies $z \in B_r$ thus $B_r$ is closed. Now $B_0 = A \cup \partial A$. First by definition of $p(x)$ we have for any $x \in A$, $p(x) = 0$. Hence $x \in B_0$. Now suppose that $x \in \partial A$, then for any $\epsilon > 0$, there is a $y \in A$ such that $|x - y| < \epsilon$. Therefore we have
\[
p(z) = \inf \{|z - y| : y \in A\} = 0 \quad \Rightarrow \quad x \in B_0,
\]
and so $A \subset B_0$. Now suppose $x \in B_0$. Then $\inf \{|x - y| : y \in A\} = 0$, so for every $\epsilon > 0$ there is a $y \in A$ such that $|x - y| < \epsilon$. So $x \in A$, therefore we have $B_0 = A = A^o \cup \partial A$. Now
\[
\mu^*(A) \leq \mu^*(B_0) = \mu^*(A^o \cup \partial A) = \mu^*(A^o) + \mu^*(\partial A) = \mu^*(A) + \mu^*(\partial A)
\]
Since $A^o$ is open and $A$ is measurable. Therefore $\mu^*(A) = \mu(B_0)$, if and only if $\mu^*(\partial A) = 0 \square$.

Exercise 6.3. Prove that an algebraic basis in any infinite-dimensional Banach space must be uncountable.

Proof: Let $V$ be an infinite-dimensional Banach space over $\mathbb{F}$, and suppose $\{x_n\}_{n \in \mathbb{N}}$ is a countable Hamel basis. Then $v \in V$ if any only if there exists $a_i \in \mathbb{F}$ such that
\[
v = \sum_{i=1}^{k} a_i x_i
\]
for some $x_i \in \{x_n\}$. Now let $\langle x_i \rangle$ denote the span of $x_i$, then we have
\[
V = \bigcup_{k \in \mathbb{N}} \langle \{x_n\}_{n=1}^{k} \rangle
\]
But this implies that $V$ is a countable union of proper subspace of finite dimension. Which implies that $V$ would be of first category, since every finite dimensional proper subspace of a normed space is nowhere dense. Which is a contradiction to the Baire Category Theorem. Therefore any basis for an
infinite-dimensional Banach space must be uncountable \(\Box\)

**Theorem (Hahn-Banach)** Let \( p \) be a real-valued function defined on the vector space \( X \) satisfying \( p(x + y) \leq p(x) + p(y) \) and \( p(\alpha x) = \alpha p(x) \) for each \( \alpha \geq 0 \). Suppose that \( f \) is a linear functional defined on a subspace \( S \) and that \( f(s) \leq p(s) \) for all \( s \in S \). Then there is a linear function \( F \) defined on \( X \) such that \( F(x) \leq p(x) \) for all \( x \), and \( F(s) = f(s) \) for all \( s \in S \).

**Exercise 6.4.** Let \( \nu \) be a finite Borel measure on the real line, and set \( F(x) = \nu\{(−∞, x]\}. \) Prove that \( \nu \) is absolutely continuous with respect to the Lebesgue measure \( \mu \) if and only if \( F \) is an absolutely continuous function. In this case show that its Radon-Nikodym derivative is the derivative of \( F \), almost everywhere.

**Proof:** \((\Rightarrow)\) First suppose that \( \nu \ll \mu \). Let \( \mu(E) \), then there exists an open set \( \mathcal{O} \), such that \( E \subset \mathcal{O} \) and \( \mu(\mathcal{O}) < \epsilon \). Now \( \mathcal{O} \) being open, there are disjoint intervals \((x_k, y_k)\), such that
\[
\mathcal{O} = \bigcup_{k=1}^{\infty} (x_k, y_k), \quad \Rightarrow \quad \mu(\mathcal{O}) = \sum_{k=1}^{\infty} (y_k - x_k) < \epsilon
\]
Since \( \nu \ll \mu \), there exists a delta such that if \( \mu(\mathcal{O}) < \epsilon \), then \( \nu(\mathcal{O}) < \delta \). So we have
\[
\sum_{k=1}^{\infty} |F(y_k) - F(x_k)| = \sum_{k=1}^{\infty} \nu(x_k, y_k) < \delta
\]
So \( F(x) \) is an absolutely continuous function.

\((\Leftarrow)\) Suppose that \( F(x) \) is absolutely continuous. Then we have
\[
\forall \epsilon > 0 \exists \delta > 0 \text{ s.t. } \sum_{k=1}^{\infty} |y_k - x_k| < \delta \quad \Rightarrow \quad \sum_{k=1}^{\infty} |F(y_k) - F(x_k)| < \epsilon
\]
Choose such disjoint intervals \((x_k, y_k)\) and call the union of these intervals \( \mathcal{O} \), then we have \( \mu(\mathcal{O}) < \epsilon \). Now by definition of \( F(x) \), we have
\[
\nu(\mathcal{O}) = \sum_{k=1}^{\infty} |F(y_k) - F(x_k)| < \delta
\]
and so \( \nu \ll \mu \).

To see that \( F \) is Radon-Nikodym derivative, we know that since \( F \) is absolutely continuous we have that \( F'(t) \) exists almost everywhere so
\[
\nu(−∞, x] = F(x) = \int_{−∞}^{x} F'(t) \, d\mu(t)
\]
We also have that
\[
\nu(−∞, x] = \int_{−∞}^{x} d\nu = \int_{−∞}^{x} \left[ \frac{d\nu}{d\mu} \right] d\mu
\]
which implies that
\[
\nu(−∞, x] = F(x) = \int_{−∞}^{x} F'(t) \, d\mu(t) = \int_{−∞}^{x} \left[ \frac{d\nu}{d\mu} \right] d\mu
\]
Hence by the Radon-Nikodym theorem we know that \( F' = \left[ \frac{d\nu}{d\mu} \right] \) almost everywhere.

**Theorem (Tonneli’s)** Suppose \((\mathcal{X} \times \mathcal{Y}, \sigma(\mathcal{A} \times \mathcal{B}), \mu \times \nu)\) is the product space of two \( \sigma \)-finite measure spaces, and \( f : \mathcal{X} \times \mathcal{Y} \to [0, \infty] \) and \( f(x, y) \) be a nonnegative measurable function in the product measure, then
\[
F_1(x) = \int_{\mathcal{Y}} f(x, \cdot) \, d\nu \text{ is } \mathcal{A} \text{ measurable of } x \in \mathcal{X}
\]
\[ F_2(y) = \int_X f(\cdot, y) \, d\mu \] is \( \mathcal{B} \) measurable of \( x \in \mathcal{Y} \)

and
\[
\int_{X \times Y} f \, d(\mu \nu) = \int_X F_1 \, d\mu = \int_Y F_2 \, d\nu
\]
i.e., the iterated integrals is equal to the the integral in the product space

Suppose (Theorem (Fubini’s)) Suppose (X \times \mathcal{Y}, \sigma(A \times \mathcal{B}), \mu \times \nu) is the product space of two \( \sigma \)-finite measure spaces, and \( f : X \times \mathcal{Y} \rightarrow [0, \infty] \) and \( f(x, y) \) be an integrable function in the product space, then
\[
\int_X \left( \int_Y f(x, y) \, d\nu \right) \, d\mu = \int_Y \left( \int_X f(\cdot, y) \, d\mu \right) \, d\nu = \int_{X \times Y} f(x, y) \, d(\mu \times \nu).
\]

Theorem (Fubini-Tonelli) Suppose (X \times \mathcal{Y}, \sigma(A \times \mathcal{B}), \mu \times \nu) is the product space of two \( \sigma \)-finite measure spaces. Let \( f \) be an extended real-valued \( \sigma(A \times \mathcal{B}) \) measurable function on \( X \times \mathcal{Y} \). If either
\[
\int_X \left( \int_Y |f| \, d\nu \right) \, d\mu < \infty \text{ or } \int_Y \left( \int_X |f| \, d\mu \right) \, d\nu < \infty
\]
then \( f \) is \( \mu \times \nu \)-integrable, furthermore the iterated integrals are equal to the product integral.

Exercise 6.5. Let \( f \) be a real valued measurable function on the finite measure space \((X, \mathcal{M}, \mu)\). Prove that the function \( F(x, y) = f(x) - 5f(y) + 4 \) is measurable in the product measure space \((X \times X, \sigma(\mathcal{M} \times \mathcal{M}), \mu \times \mu)\), and that \( F \) is integrable if and only if \( f \) is integrable.

Proof: First since \( f(x) \) is measurable, we have both sections \( F(x_0, y) \), and \( F(x, y_0) \) as measurable for each fixed \( x_0, y_0 \). Now for \( x \in X \) we have \( F(x, x) = -4(f(x) + 1) \), which is measurable. Having \( F(x, y) \) being measurable on each section, and the diagonal is enough for \( F(x, y) \) to be measurable in the product space.

Now let \( f \) be integrable, hence \( |f| \) is integrable, so let \( M = \int_X |f(x)| \, dx \), now we have
\[
\int_X \int_X |f(x) - 5f(y) + 4| \, dx \, dy \leq \int_X M + 5|f(y)||\mu(X) + 4\mu(X) \, dy \\
= M\mu(X) + 5M\mu(X) + 4\mu(X)^2 \\
= 4\mu(X)(M + \mu(X)) < \infty
\]
by the same computation \( \Rightarrow \int_X \int_X |f(x) - 5f(y) + 4| \, dy \, dx < \infty \)

Then by Fubini-Tonelli theorem \( F(x, y) \) is integrable. Now suppose that \( F(x, y) \) is integrable, then by Fubini’s theorem we have that the iterations are equal, but this is true if and only if \( f(x) \) is integrable.

Theorem (Stone-Weierstrass) Let \( X \) be a compact space and \( A \) an algebra of continuous real-valued functions on \( X \) that separates the points of \( X \) and contains the constant functions. Then given any continuous real-valued function \( f \) on \( X \) and any \( \epsilon > 0 \) there is a function \( g \in A \) such that for all \( x \in X \) we have \( |g(x) - f(x)| < \epsilon \). In other words, \( A \) is a dense subset of \( C(X) \).

Theorem (Closed Graph) Let \( A \) be a linear transformation on a Banach space \( X \) to a Banach space \( Y \). Suppose that \( A \) has the property that, whenever \( x_n \) is a sequence in \( X \) that converges to some point \( x \) and \( Ax_n \) converges in \( Y \) to a point \( y \), then \( y = Ax \). Then \( A \) is continuous.
Exercise 6.6. Let \((\mathcal{X}, \mathcal{A}, \mu)\) and \((\mathcal{Y}, \mathcal{B}, \nu)\) be the measure spaces given by

- \(\mathcal{X} = \mathcal{Y} = [0, 1]\)
- \(\mathcal{A} = \mathcal{B} = \sigma([0,1])\)
- \(\mu\) be the Lebesgue measure on \(\mathbb{R}\), and \(\nu\) the counting measure.

Consider the product measure space \((\mathcal{X} \times \mathcal{Y}, \sigma(\mathcal{A} \times \mathcal{B}))\), and its subset \(E = \{(x,y) \in \mathcal{X} \times \mathcal{Y} : x = y\}\)

1. Show that \(E \subset \sigma(\mathcal{A} \times \mathcal{B})\)
2. Show that \(\int_\mathcal{X} \int_\mathcal{Y} \chi_E \, d\nu \, d\mu \not= \int_\mathcal{Y} \int_\mathcal{X} \chi_E \, d\mu \, d\nu\).
3. Explain why Tonelli’s theorem is not applicable.

Proof: For (1) First notice that the following sets
\[
A_k = \left[\frac{k-1}{n}, \frac{k}{n}\right] \times \left[\frac{k-1}{n}, \frac{k}{n}\right]
\]
are measurable. Now let define \(E_n\) as follows
\[
E_n = \bigcup_{k=1}^{n} A_k
\]
Then the sets \(E_n\) are measurable as they are countable union of measurable sets. Then the set \(E\) is given by
\[
E = \bigcap_{n=1}^{\infty} E_n = \{(x,y) \in \mathcal{X} \times \mathcal{Y} : x = y\}
\]
is measurable since it is a countable intersection of measurable sets.

For (2) by a direct computation we have
\[
\int_\mathcal{X} \int_\mathcal{Y} \chi_E \, d\nu \, d\mu = \int_0^1 \nu(E) \, d\mu = \int_0^1 d\mu = 1
\]
and
\[
\int_\mathcal{Y} \int_\mathcal{X} \chi_E \, d\mu \, d\nu = \int_0^1 \mu(E) \, d\nu = \int_0^1 0 \, d\nu = 0
\]
Tonelli’s theorem is not applicable because the measure space \((\mathcal{Y}, \mathcal{B}, \nu)\) is not \(\sigma\)-finite □.