Real Analysis qual study guide

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1. Measure Theory

Exercise 1.1. *If $A \subset \mathbb{R}$ and $\epsilon > 0$ show $\exists$ open sets $O \subset \mathbb{R}$ such that $m^*(O) \leq m^*(A) + \epsilon$.*

Proof: Let $\{I_n\}$ be a countable cover for A, then $A \subset \bigcup_{n=1} I_n$. Since $m^*(O) \leq m^*(A) + \epsilon$. This implies that

$$m^*(O) - \epsilon \leq m^*(A) \text{ where } m^*(A) = \inf_{A \subset \bigcup I_n} \left\{ \sum_{n=1}^{\infty} l(I_n) \right\}$$

If $l(I_k) = \infty$ for some k then there is nothing to show, so suppose $(a_n, b_n) = I_n$ then $l(I_n) < \infty, \forall n$. Let $O_n = (a_n + 2^{-n}\epsilon, b_n)$ then we have

$$\begin{aligned} l(O_n) &= b_n - a_n - 2^{-n}\epsilon \leq l(I_n) \\ \Rightarrow \quad \sum l(O_n) &= \sum b_n - a_n - \sum 2^{-n}\epsilon = \sum b_n - a_n - \epsilon \\ \Rightarrow \quad & m^*(\bigcup_n O_n) - \epsilon \leq m^*(A) \end{aligned}$$

So let $O = \bigcup_n O_n$, then $m^*(O) - \epsilon \leq m^*(A) \therefore \exists O \subset \mathbb{R}$ st $m^*(O) \leq m^*(A) + \epsilon$ $\square$

Exercise 1.2. *If $A, B \subset \mathbb{R}, m^*(A) = 0$, then $m^*(A \cup B) = m^*(B)$*

Proof: $m^*(A \cup B) \leq m^*(A) + m^*(B)$, and $m^*(B) \leq m^*(A \cup B)$, hence we have

$$\begin{aligned} m^*(B) \leq m^*(A \cup B) &\leq m^*(A) + m^*(B) = m^*(B) \\ \therefore m^*(A \cup B) &= m^*(B) \ \square \end{aligned}$$

Exercise 1.3. *Prove $E \in \mathbb{M}$ iff $\forall \epsilon > 0, \exists O \subset \mathbb{R}$ open, such that $E \subset O$ and $m^*(O \backslash E) < \epsilon$*

Proof: $(\Rightarrow)$ $O \backslash E = E^c \cap O$ implies that $m^*(O \backslash E) = m^*(E^c \cap O)$, but we have

$$m^*(O) = m^*(E^c \cap O) + m^*(E \cap O)$$

So suppose $m^*(E) < \infty \Rightarrow m^*(E^c \cap O) = m^*(O) - m^*(E \cap O)$. Let I_n be a countable cover for E, so $I_n = (a_n, b_n)$. Let $O_n = (a_n, b_n + 2^{-n}\epsilon)$ and let $O = \bigcup O_n$. Then

$$m^*(O) = \sum l(O_n) = \sum 2^{-n}\epsilon + b_n - a_n = \epsilon + \sum b_n - a_n, \text{ and } m^*(E \cap O) = m^*(E)$$

since $E \subset O$. So we have

$$\begin{aligned} m^*(E \cap O) = m^*(E) \leq \sum l(I_n) &= \sum b_n - a_n \\ \Rightarrow \quad m^*(E \cap O) &\leq \sum l(O_n) - \sum l(I_n) \\ &= \epsilon + \sum b_n - a_n - \sum b_n - a_n = \epsilon \end{aligned}$$

$\therefore \exists O \subset \mathbb{R}$ open, st $E \subset O$ and $m^*(O \backslash E) \leq \epsilon$

$(\Leftarrow)$ Conversely, suppose $\forall \epsilon > 0, \exists O \subset \mathbb{R}$, such that $E \subset O$ and $m^*(O \backslash E) < \epsilon$ and that $O \in \mathbb{M}$. Then

$$m^*(O) = m^*(E^c \cap O) + m^*(E \cap O), \text{ but } m^*(E^c \cap O) = m^*(O \backslash E) < \epsilon$$

This implies that

$$m^*(O) = m^*(E \cap O) + \epsilon \quad \Rightarrow \quad m^*(O) = m^*(E) + \epsilon \quad \therefore E \in \mathbb{M} \ \square$$

Exercise 1.4. *Prove $E \in \mathbb{M}$ iff $\forall \epsilon > 0$ $\exists F \subset \mathbb{R}$ closed, such that $F \subset E$ and $m^*(E \backslash F) < \epsilon$*

Proof: ($\Rightarrow$) $E \backslash F = F^c \cap E$ this implies that $m^*(E\backslash F) = m^*(F^c \cap E)$, but we have

$$m^*(F) = m^*(F^c \cap E) + m^*(E \cap F)$$

So suppose $m^*(E) < \infty \Rightarrow m^*(F^c \cap E) = m^*(F) - m^*(F \cap E)$. Let I_n be a countable cover for E, where $I_n = (a_n, b_n)$. Let $F_n = [a_n, b_n - 2^{-n}\epsilon]$ and let $F = \bigcup F_n$. Then we have

$$m^*(F) = \sum l(F_n) = \sum b_n - a_n - 2^{-n}\epsilon = \sum b_n - a_n - \epsilon,$$

and $m^*(E \cap F) = m^*(F)$, since $F \subset E$. So

$$\begin{aligned} m^*(E \cap F) \;=\; m^*(F) \quad &\leq \quad \sum l(I_n) \;=\; \sum b_n - a_n \\ \Rightarrow \quad m^*(E \cap F) \leq \sum l(I_n) - \sum l(F_n) \quad &= \quad \sum b_n - a_n - \sum b_n - a_n + \epsilon \;=\; \epsilon \end{aligned}$$

$\therefore \exists F \subset \mathbb{R}$ Closed, st $F \subset E$ and $m^*(E \backslash F) \leq \epsilon$

($\Leftarrow$) Conversely, suppose $\forall \epsilon > 0, \exists F \subset \mathbb{R}$, such that $F \subset E$ and $m^*(E\backslash F) < \epsilon$ and that $F \in \mathbb{M}$. Then

$$m^*(E) = m^*(F^c \cap E) + m^*(E \cap F),$$

but $m^*(F^c \cap E) = m^*(E \backslash F) < \epsilon$. This implies that

$$m^*(E) \leq m^*(F \cap E) + \epsilon \quad \Rightarrow \quad m^*(E) \leq m^*(F) + \epsilon \quad \therefore E \in \mathbb{M} \; \square$$

Vitali Let E be a set of finite outer measure and $\mathfrak{F}$ a collection of intervals that cover E in the sence of Vitali. Then, given $\epsilon > 0$ there is a finite disjoint collection $\{I_N\}$ of intervals in $\mathfrak{F}$ such that

$$\mu^*\left(E \backslash \bigcup_{n=1}^{N} I_n\right) < \epsilon$$

Exercise 1.5. *Does there exists a Lebesgue measurable subset A of $\mathbb{R}$ such that for every interval (a, b) we have $\mu(A \cap (a, b)) = (b - a)/2$?*

Proof: First suppose that there is such a mesurable set A such that $0 \neq \mu(A \cap (a, b)) = \alpha \leq (b-a)/2$. Then there exsits an open set $\mathcal{O}$ such that $A \subset \mathcal{O}$ and $\mu(\mathcal{O}\backslash A) < \epsilon$, so let $\epsilon = \alpha/2$. Now $\mathcal{O}$ is open, so there are disjoint intervals (x_k, y_k) such that $\mathcal{O}$ is a countable union of these intervals. So

$$\mathcal{O} \cap (a, b) = \bigcup_{k=1}^{\infty} [(x_k, y_k) \cap (a, b)] = \bigcup_{l} (c_{k_l}, d_{k_l}).$$

Hence $\mu(\mathcal{O} \cap (a, b)) = \sum_l d_{k_l} - c_{k_l}$, and we have

$$A \cap \mathcal{O} \cap (a, b) = A \cap (a, b) = \bigcup_{l} [A \cap (c_{k_l}, d_{k_l})]$$

Now

$$\alpha = \mu(A \cap (a, b)) = \frac{1}{2} \sum_{l} (d_{k_l} - c_{k_l})$$

but

$$\begin{aligned} \sum_{l} (d_{k_l} - c_{k_l}) &= \mu(\mathcal{O} \cap (a, b)) \\ &= \mu((\mathcal{O}\backslash A) \cap (a, b)) + \mu(A \cap (a, b)) \\ &\leq \mu(\mathcal{O}\backslash A) + \frac{1}{2} \sum_{l} (d_{k_l} - c_{k_l}) \\ &< \epsilon + \frac{1}{2} \sum_{l} (d_{k_l} - c_{k_l}) \end{aligned}$$

But this implies that

$$\alpha/2 = \epsilon \leq \frac{1}{2} \sum_{l} (d_{k_l} - c_{k_l}) \geq \alpha$$

So $\mu(A) = 0$. which implies that $\mu(A^c) = \infty$. Now if there were to exsits such a set A we have $\mu(A^c) = 0$, and so

$$b - a = \mu((a,b)) = \mu(A \cap (a,b)) + \mu(A^c \cap (a,b)) = \mu(A^c \cap (a,b)) = \frac{1}{2}(b-a)$$

So there cannot exist such a set $\square$.

Exercise 1.6. *Assume that $E \subset [0,1]$ is measurable and for any $(a,b) \subset [0,1]$ we have*

$$\mu(E \cap [a,b]) \geq \frac{1}{2}(b-a)$$

Show that $\mu(E) = 1$.

Proof: By the previous problem, using the same proof, we know that $\mu(E^c) = 0$. So the result is shown.

Exercise 1.7. *Let $E_1, \ldots, E_n$ be measurable subsets of $[0,1]$. Suppose almost every $x \in [0,1]$ belongs to at least k of these subsets. Prove that atleast one of the $E_1, \ldots, E_n$ has measure of at least k/n.*

Proof: Suppose not, then for each i we have $\mu(E_i) < k/n$. Define a function $f(x)$ as follows.

$$f(x) = \sum_{i=1}^{n} \chi_{E_i}$$

where χ_{E_i} denotes the characteristic function of E_i. Now since all most all $x \in [0,1]$ are in at least k of the E_i we have $f(x) \geq k$ almost everywhere in $[0,1]$. Now

$$k = \int_{[0,1]} k \ dx \leq \int_{[0,1]} f(x) \ dx = \sum_{i=1}^{n} \int_{[0,1]} \chi_{E_i} \ dx = \sum_{i=1}^{n} \mu E_i$$

But this implies that

$$\sum_{i=1}^{n} \mu E_i < \sum_{i=1}^{n} \frac{k}{n} = k$$

Which is a contradiction, hence at least one E_i has $\mu(E_i) \geq \frac{k}{n}$ $\square$.

Exercise 1.8. *Consider a measure space $(\mathcal{X}, \mathcal{A}, \mu)$ and a sequences of measurable sets E_n, $n \in \mathbb{N}$, such that*

$$\sum_{n=1}^{\infty} \mu(E_n) < \infty$$

Show that almost every $x \in \mathcal{X}$ is an element of at most finitely many $E_n' s$.

Proof: It suffices to show that $\mu(x : x \in \cap E_{n_k}) = 0$. So consider the following

$$\lim_{m \to \infty} \mu\left(x : x \in \bigcap_{k=1}^{m} E_{n_k}\right)$$

If we have shown the above limit is zero, then we're done. To see this look at the following sum,

$$\sum_{N=1}^{\infty} \mu\left(x : x \in \bigcap_{k=1}^{N} E_{n_k}\right) < \sum_{n=1}^{\infty} \mu(E_n) < \infty$$

and hence

$$\lim_{m \to \infty} \mu\left(x : x \in \bigcap_{k=1}^{m} E_{n_k}\right) = 0$$

Therefore almost every $x \in \mathcal{X}$ is an element of at most finitely many $E_n' s$ $\square$.

Exercise 1.9. *Consider a measure space $(\mathcal{X}, \mathcal{A}, \mu)$ with $\mu(\mathcal{X}) < \infty$, and a sequences $f_n : \mathcal{X} \to \mathbb{R}$ of measurable functions such that $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in \mathcal{X}$. Show that for every $\epsilon > 0$ there exists a set E of measure $\mu(E) \leq \epsilon$ such that f_n converges uniformly to f outside the set E.*

Proof: This is Ergoroff's theorem. See below.

Theorem (Egoroff's) If f_n is a sequence of measurable functions that converge to a real-valued function f a.e. on a measurable set E of finite measure, then given $\eta > 0$, there is a subset A of E with $\mu(A) < \eta$ such that f_n converges to f uniformly on $E \backslash A$

Proof: Let $\eta > 0$, then for each n, there exists a set $A_n \subset E$ with $\mu A_n < \eta 2^{-n}$, and there is an N_n such that for all $x \notin A_n$ and $k \geq A_n$ we have $|f_k(x) - f(x)| < 1/n$. Let $A = \cup A_n$, then by construction $A \subset E$ and $\mu A < \eta$. Choose n_0 such that $1/n_0 < \eta$. Now if $x \notin A$ and $k \geq N_{n_0}$ then $|f_k(x) - f(x)| < 1/n_0 < \eta$. Therefore f_n converges uniformly on $E \backslash A$.

Exercise 1.10. *Let g be an absolutely continuous monotone function on $[0, 1]$. Prove that if $E \subset [0, 1]$ is a set of Lebesgue measure zero, then the set $g(E) = \{g(x) : x \in E\} \subset \mathbb{R}$ is also a set of Lebesgue measure zero.*

Proof: Let $E \subset [0, 1]$ with zero measure, then for any epsilon $\epsilon > 0$, there exists an open cover $\mathcal{O}$ for E, such that $\mu(\mathcal{O} \backslash E) < \epsilon$. Now $\mathcal{O}$ being open in $[0, 1]$ implies that $\mathcal{O} = \cup(a_n, b_n)$, where (a_n, b_n) are disjoint. Now by absolutely continuity of $g(x)$ we have

$$\forall \eta > 0 \; \exists \delta \; s.t. \; \sum_{n=1}^{\infty} \mu(I_n) < \delta \quad \to \quad \sum_{n=1}^{\infty} |g(I_n \cap [0, 1])| < \eta$$

Now $g(E) \subset \cup |g(I_n \cap [0, 1])|$ which implies that $\mu(g(E)) < \eta$, so given an η there exists a $\delta > 0$ such that the above hold, then let $\delta = \epsilon$. Since η is arbitrary we have $\mu(g(E)) = 0$ □

Remark: The above problem (1.10) is commonly refered to as Lusin's N condition.

Exercise 1.11. *Suppose f is Lipschitz continuous in $[0, 1]$. Show that*

(a) $\mu(f(E)) = 0$ if $\mu(E) = 0$.

(b) If E is measurable, then $f(E)$ is also measurable.

Proof: For part (a) if f is Lipschitz continuous then it is absolutely continuous, and so if $\mu(E) = 0$, then $\mu(f(E)) = 0$ (see above proof).

For part (b) Let E be a measurable set and let $\epsilon > 0$. Now there exists an open set $\mathcal{O}$ such that $\mu(\mathcal{O} \backslash E) < \epsilon$, where $\mathcal{O}$ is a disjoint union of intervals $I_n = (a_n, b_n)$. Now since f is absolutely continiuous, it can be approximated by simple functions, namely χ_{I_n}. Choose these functions such that

$$\left| f - \sum_{n=1}^{\infty} c_n \chi_{I_n} \right| < \epsilon$$

Now $\mu(\chi_{I_n}) = b_n - a_n > 0$, so it is measurable. Let $\alpha \in \mathbb{R}$, then the $f(E)$ is measurable if $\{x : f(x) \leq \alpha\}$ is a measurable set for any $\alpha \in \mathbb{R}$. but we have now

$$\{x : f(x) \leq \alpha\} \subset \{x : \chi_{I_n} + \epsilon \leq \alpha\}$$

We know simple functions are measurable, and our choice of simple functions approximates $f(x)$, therefore f is measurable □.

Theorem (Lusin's) Let f be a measurable real-valued function on an interval $[a, b]$. Then given $\delta > 0$, there is a continuous function ϕ on $[a, b]$ such that $\mu\{x : f(x) \neq \phi(x)\} < \delta$

Proof: Let $f(x)$ be measurable on $[a,b]$ and let $\delta > 0$. For each n, there is a continuous function h_n on $[a,b]$ such that

$$\mu\{x : |h_n(x) - f(x)| \geq \delta 2^{-n-2}\} < \delta 2^{-n-2}$$

Denote these sets as E_n. Then by construction we have

$$|h_n(x) - f(x)| < \delta 2^{-n-2}, \text{ for } x \in [a,b]\backslash E_n$$

Let $E = \cup E_n$, then $\mu E < \delta/4$ and $\{h_n\}$ is a sequence of continuous, thus measurable, functions that converges to f on $[a,b]\backslash E$. By Egoroff's theorem, there is a subset $A \subset [a,b]\backslash E$ such that $\mu A < \delta/4$ and h_n converges uniformly to f on $[a,b]\backslash(E\cup A)$. Thus f is continuous on $[a,b]\backslash(E\cup A)$ with $\mu(E\cup A) < \delta/2$. Now there is an open set O such that $(E \cup A) \subset O$ and $\mu(O\backslash(E \cup A)) < \delta/2$. Then we have f is continuous on $[a,b]\backslash O$, which is closed. Hence there exists a ϕ that is continuous on $(-\infty,\infty)$ such that $f = \phi$ on $[a,b]\backslash O$, where $\mu\{x : f(x) \neq \phi(x)\} \leq \mu(O) < \delta$

Exercise 1.12. *Prove the following statement. Supoose that F is a sub-σ-algebra of the Borel σ-algebra on the real line. If $f(x)$ and $g(x)$ are F-measurable and if*

$$\int_A f\ dx = \int_A g\ dx, \quad \forall A \in F$$

Then $f(x) = g(x)$ almost everywhere.

Proof: Let μ denote the Lebesgue measure on the Borel sets. Now since both f and g are F-measurable, for any $n \geq 1$, the sets

$$A_n = \{x : f(x) - g(x) \geq 1/n\}, \quad B_n = \{x : g(x) - f(x) \geq 1/n\}$$

are both measurable and contained in F. Now we also have

$$A = \{x : f(x) - g(x) > 0\} = \bigcap_{n=1}^{\infty} A_n, \quad B = \{x : g(x) - f(x) > 0\} = \bigcap_{n=1}^{\infty} B_n$$

contained in F since F is a σ-algebra. Now using the convenetion that $\infty - \infty = 0$, we have

$$\int_A f - g\ dx = 0$$

If $\mu(A) > 0$ then as $f - g > 0$ implies by that $\int_A f - g\ dx > 0$, which is a contradiction. Hence we have $\mu(A) = 0$. By the same argument also have

$$\int_B g - f\ dx = 0 \quad \to \quad \mu(B) = 0.$$

Now $A \cap B = \emptyset$ and $A \cup B$ is the set of points where $f(x) \neq g(x)$, hence $f = g$ almost everywhere $\square$.

Exercise 1.13. *Let $E \subset \mathbb{R}$. Let $E^2 = \{e^2 : e \in E\}$*

(a) Show that if $\mu^*(E) = 0$, then $\mu^*(E^2) = 0$
(b) Suppose $\mu^*(E) < \infty$, it it true that $\mu^*(E^2) < \infty$

Proof: For part (a) consider the intervales $I_n = [n, n+1]$ for in $\mathbb{Z}$. Now consider the function $f(x) = x^2$. If $p_n = \cup(a_k, b_k)$ is an open subset of I_n such that for $\delta < 0$

$$\mu(p_n) < \delta \quad \Rightarrow \quad \sum_{k=1}^{n} |f(b_k) - f(a_k)| = \sum_{k=1}^{N} |b_k^2 - a_k^2| \leq (2|n| + 1)\delta$$

Hence $f(x)$ is absolutely continuous on I_n. Now a function is absolutely continuous on an interval I if and only if the following are satisfies:

f is continuous on I

f is of bounded variation on I
f satisfies Lusin's (N) condition, or for every subset E of I such that $\mu(E) = 0$, $\mu(f(E)) = 0$.

Remark: The above condition for absolute continuity is the Banach-Zarecki Theorem.

Now define $E_n = E \cap I_n$, then $E_n \subset I_n$ and hence by Lusin's (N) condition $\mu(f(E_n)) = 0$. Now the set $f(E_n)$ is given by

$$f(E_n) = \{e^2 : e \in E \cap I_n\}$$

Now

$$E^2 = \bigcup_{n\in\mathbb{Z}} \{e^2 : e \in E \cap I_n\} = \bigcup_{n\in\mathbb{Z}} f(E_n)$$

and so

$$\mu^*(E^2) \leq \sum_{n\in\mathbb{Z}} \mu^*(E_n) = \sum_{n\in\mathbb{Z}} \mu(E_n) = 0$$

For part (b), the statement is not always true. For each $n \in \mathbb{N}$, let $E_n = [n, n + n^{-3/2})$, then for each $\mu(E_n) = n^{-3/2}$. Now if $E = \cup E_n$, then

$$\mu(E) = \sum_{n=1}^{\infty} \mu(E_n) = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

Now $E_n^2 = [n^2, n^2 + 2n^{-1/2} + n^{-3})$, and so $\mu(E_n^2) = 2n^{-1/2} + n^{-3} \geq n^{-1/2}$. Also $E^2 = \cup E_n^2$, and the sets E_n^2 are mutually disjoint. Hence

$$\mu(E^2) = \sum_{n=1}^{\infty} \mu(E_n^2) \leq \sum_{n=1}^{\infty} \frac{1}{n^{1/2}} = \infty$$

Exercise 1.14. *Suppose a measure μ is defined on a σ-algebra $\mathcal{M}$ of subset of $\mathcal{X}$, and μ^* is the corresponding outer measure. Suppose $A, B \subset \mathcal{X}$. Then $A \sim B$ if $\mu^*(A\Delta B) = 0$. Prove that $\sim$ is an equivalence relation.*

Proof: For symmetry we have, by definition, $A\Delta B = (A \cup B)\backslash(A \cap B) = (B \cup A)\backslash(B \cap A) = B\Delta A$, and so if $\mu^*(A\Delta B) = 0$, then $\mu^*(B\Delta A) = 0$. Hence $A \sim B$ if and only if $B \sim A$.

For reflexivity, we have $(A\Delta A) = A\backslash A = \emptyset$, hence $A \sim A$.

For transitivity, let $A, B, C \subset \mathcal{X}$. First notice, by element chasing, $A\Delta C \subset (A\Delta B) \cup (B\Delta C)$, and so we have

$$0 \leq \mu^*(A\Delta B) = \mu^*((A\Delta B) \cup (B\Delta C)) \leq \mu^*(A\Delta B) + \mu^*(B\Delta C)$$

Now if $A \sim B$ and $B \sim C$, then $\mu^*(A\Delta B) = \mu^*(B\Delta C) = 0$, and so $\mu^*(A\Delta B) = 0$, hence $A \sim C$. Therefore $\sim$ is an equivalence relation on $\mathcal{X}$ $\square$.

Exercise 1.15. *Let $(\mathcal{X}, \mathcal{M}, \mu)$ be a measure space.*

(a) Suppose $\mu(\mathcal{X}) < \infty$. If f and f_n are measurable functions with $f_n \to f$ almost everywhere, prove that there exists sets $H, E_k \in \mathcal{M}$ such that $\mathcal{X} = H \cup \bigcup_{k=1}^{\infty} E_k$, where $\mu(H) = 0$ and $f_n \to f$ uniformly on each E_k

(b) Is the result of (a) still true if $(\mathcal{X}, \mathcal{M}, \mu)$ is σ-finte?

Proof: For part (a), since $\mu(\mathcal{X}) < \infty$ and $f_n \to f$ almost everywhere, by Egoroff's theorem, for any

$k \in \mathbb{N}$, there is $H_k \in \mathcal{M}$ such that $\mu(H_k) < 1/k$ and $f_n \to f$ uniformly on $E_k = H_k^c$. Now define $H = \cap_{k=1}^{\infty} H_k$, then $H \subset H_k$, and so $0 \le \mu(H) \le 1/k$ for all k, hence $\mu(H) = 0$. Now

$$\bigcup_{k=1}^{\infty} E_k = \bigcup_{k=1}^{\infty} H_k^c = \left(\bigcap_{k=1}^{\infty} H_k\right)^c = H^c$$

and so

$$\mathcal{X} = H \cup \left(\bigcup_{k=1}^{\infty} E_k\right)$$

where f_k converges uniformly to f on any E_k $\square$.

For part (b), the statement is true. Since $\mathcal{X}$ is σ-finite, we can write $\mathcal{X}$ as a disjoint union of finite sets, i.e.

$$\mathcal{X} = \bigcup_{n=1}^{\infty} \mathcal{X}_n \text{ where } \mu(\mathcal{X}_n) < \infty \quad \forall n \quad \mathcal{X}_i \cap \mathcal{X}_j = \emptyset \text{ for } i \neq j$$

Now for each $\mathcal{X}_n$ apply part (a). Then we have

$$\mathcal{X}_n = H_n \cup \bigcup_{k=1}^{\infty} E_{k,n} \text{ with } \mu(H_n) = 0$$

Let $H = \cup_{n=1}^{\infty} H_n$, then $\mu(H) = \sum_{n=1}^{\infty} \mu(H_n) = 0$. So we have

$$\begin{aligned}
\mathcal{X} = \bigcup_{n=1}^{\infty} \mathcal{X}_n &= \bigcup_{n=1}^{\infty} \left(H_n \cup \left(\bigcup_{k=1}^{\infty} E_{k,n}\right)\right) \\
&= \left(\bigcup_{n=1}^{\infty} H_n\right) \cup \left(\bigcup_{n,k=1}^{\infty} E_k\right) \\
&= H \cup \left(\bigcup_{n,k=1}^{\infty} E_{k,n}\right)
\end{aligned}$$

Now H has measure zero and $\{E_{k,n}\}_{n,k=1}^{\infty}$ is a countable collection of open sets for which $f_n \to f$ uniformly $\square$.

Exercise 1.16. *Suppose f_n is a sequence of measurable functions on $[0,1]$. For $x \in [0,1]$ define $h(x) = \#\{n : f_n(x) = 0\}$ (the number of indicies n for which $f_n(x) = 0$. Assuming that $h < \infty$ everywhere, prove that the function h is measurable.*

Proof: First consider the measure space $([0,1], \sigma[0,1], \mu)$, where μ is the Lebesgue measure. Since f_n is measurable for all n we know that the set $\{x : f_n(x) = \alpha\}$ is measurable, for $\alpha \in \mathbb{R}$. In particular, the set $\{x : f_n(x) = 0\}$ is measurable. Now we have

$$\bigcup_{n=1}^{\infty} \{x : f_n(x) = 0\}$$

is measurable with respect to μ, since it is the countable union of measurable sets. Now consider the measure space $(\mathbb{N}, \sigma(\mathbb{N}), \nu)$ where ν is the counting measure. Now we know that

$$h(x) = \{n : f_n(x) = 0\} < \infty$$

So consider the following:

$$\begin{aligned} \{x : h(x) = \alpha\} &= \left\{ x : \# \left| \bigcup_n f_n(x) = 0 \right| = \alpha \right\} \\ &= \left\{ x : \sum_{n=1}^{\infty} \nu\{n : f_n(x) = 0\} = \alpha \right\} \\ &\subset \left\{ x : \sum_{n=1}^{\infty} \nu\{n : f_n(x) = 0\} < \infty \right\} \\ &\subset [0,1] \end{aligned}$$

Hence the function $h(x)$ is measurable $\square$.

2. Lebesgue Integration

Exercise 2.1. *Consider the Lebesgue measure space $(\mathbb{R}, \mathcal{M}, \mu)$. Let f be an extended real-valued $\mathcal{M}$-measurable function on $\mathbb{R}$. For $x \in \mathbb{R}$ and $r > 0$ let $B_r(x) = \{y \in \mathbb{R} : |y - x| < r\}$. With $r > 0$ fixed, define a function g on $\mathbb{R}$ by setting*

$$g(x) = \int_{B_r(x)} f(y)\mu(dy) \quad \textit{for} \quad x \in \mathbb{R}$$

(a) Suppose f is locally μ-integrable on $\mathbb{R}$. Show that g is a real-valued continuous function on $\mathbb{R}$.

(b) Show that if f is μ-integrable on $\mathbb{R}$ then g is uniformly continuous on $\mathbb{R}$.

Proof: If we show part (b), then part (a) follows by the same argument. Let $x \in \mathbb{R}$. Now if f is integrable on $\mathbb{R}^2$ so is $|f|$. Hence if $\epsilon > 0$, there is $\delta > 0$ such that if $\mu(A) < \delta$, then we have

$$\int_A |f| \; dy \le \frac{\epsilon}{2}.$$

Now as $B(x, r)$ and $B(y, r)$ are open balls with area πr^2 with centers offset by $|y - x|$, we have that

$$\mu(B(x, r)\backslash B(y, r)) = \mu(B(y, r)\backslash B(x, r)) \to 0 \text{ as } y \to x$$

Hence given $\delta > 0$, there is an $\eta > 0$ such that if $|y - x| < \eta$, then

$$\mu(B(x, r)\backslash B(y, r)) = \mu(B(y, r)\backslash B(x, r)) < \delta$$

So for $|y - x| < \eta$, we have

$$|g(x) - g(y)| \le \int_{B(y,r)\backslash B(x,r)} |f| \; d\mu + \int_{B(x,r)\backslash B(y,r)} |f| \; d\mu < \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon.$$

That is, $g(x)$ is uniformly continuous on $\mathbb{R}^2$ $\square$.

Theorem (Jensen's Inequality) If ϕ is a convex function on $\mathbb{R}$ and f an integrable function on $[0, 1]$.

$$\int \phi(f(t)) \; dt \ge \phi\left(\int f(t) \; dt\right).$$

Proof: Let

$$\alpha = \int f(t) \; dt, \qquad y = m(x - \alpha) + \phi(a)$$

Then y is the equation of a supporting line at α. Now we have

$$\phi(f(t)) \ge m(f(t) - \alpha) + \phi(\alpha) \qquad \Rightarrow \qquad \int \phi(f(t)) \; dt \ge \phi(\alpha) \; dt \; \square$$

Theorem (Bounded Convergence) Let f_n be a sequence of measurable functions defined on a set E of finite measure, and suppose that there is a real number M such that $|f_n| \leq M$ for all N and all x. If $f(x) = \lim f_n(x)$ pointwise in E, then

$$\int_E f = \lim \int_E f_n.$$

Proof: Let $\epsilon > 0$, thn there is an N and a measurable set $A \subset E$ with $\mu A < \frac{\epsilon}{4M}$ such that for all $n \geq N$ and $x \in E \backslash A$ we have $|f_n(x) - f(x)| < \frac{\epsilon}{2\mu(E)}$. Now,

$$\begin{aligned}
\left| \int_E f_n - \int_E f \right| &= \left| \int_E f_n - f \right| \\
&\leq \int_E |f_n - f| \\
&= \int_{E\backslash A} |f_n - f| + \int_A |f_n - f| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

Therefore we have $\int_E f_n \to \int_E f$ □.

Exercise 2.2. *Suppose f_n is a sequence of measurable functions such that f_n converges to f almost everywhere. If for each $\epsilon > 0$, there is a C such that*

$$\int_{|f_n|>C} |f_n| \; dx < \epsilon.$$

Show that f is integrable on $[0, 2]$

Proof: First the interval $[0, 2]$, is not important. The result can be shown for any finite interval. Fix $\epsilon > 0$, now if f is to be integrable, then so is $|f|$. Let C be such in the hypothesis, by Fatou's lemma we have

$$\begin{aligned}
\int_0^2 |f| \; dx &\leq \liminf \int_0^2 |f_n| \; dx \\
&= \liminf \left(\int_{[0,2]\cap\{|f_n|>C\}} |f_n| \; dx + \int_{[0,2]\cap\{|f_n|\leq C\}} |f_n| \; dx \right) \\
&\leq \epsilon + C\mu(0, 2)
\end{aligned}$$

Therefore $\int_0^2 |f| \; dx$ is bounded and hence f is integrable □.

Theorem (Fatou's Lemma) If f_n is a sequence of nonnegative measurable functions and $f_n(x) \to f(x)$ almost everywhere on a set E, then

$$\int_E f \leq \liminf \int_E f_n.$$

Proof: Since the integral over a set of measure zero is zero, (WLOG) we can assume that the converges is everywhere. Let h be a bounded measurable fuction which is not greater that f and which vanishes outside a set $A \subset E$ of finite measure. Define a function h_n, by

$$h_n(x) = \min\{h(x), f_n(x)\}.$$

Then h_n is bounded by the bound for h and vanishes outside A. Now $h_n \to h$ pointwise in A, hence we have by the bounded convergence theorem

$$\int_E h = \int_A h = \lim \int_A h_n \leq \liminf \int_E f_n.$$

Taking supremum over h gives us the result $\square$.

Theorem (Monotone Convergence) Let f_n be an increasing sequence of nonnegative measurable functions, and let $f = \lim f$ a.e. Then

$$\int f = \lim \int f_n.$$

Proof: By Fatou's lemma we have

$$\int f \leq \liminf \int_E f_n.$$

Now for each n, since f is monotone, we have $f_n \leq f$, and so

$$\int_n f \leq \int_E f \quad \Rightarrow \quad \limsup \int_n f \leq \int_E f \quad \Rightarrow \quad \int f \lim \int f_n \ \square.$$

Remark: Let the positive part of f be denoted by $f^+(x) = \max\{f(x), 0\}$, and the negative part be denoted by $f^-(x) = \max\{-f(x), 0\}$. If f is measurable then so are f^+ and f^-. Futhermore $f = f^+ - f^-$ and $|f| = f^+ + f^-$.

Exercise 2.3. *Let f be a real-valued continuous function on $[0, \infty)$ such that the improper Riemann integral $\int_0^\infty f(x)\ dx$ converges. Is f Lebesgue integrable on $[0, \infty)$?*

Proof: f does not have to be Lebesgue integrable. Let $n \geq 0$ and define a function f_n as follows

$$f_n(x) = \begin{cases} \frac{4}{n+1}x & x \in [2n, 2n + \frac{1}{2}] \\ \frac{-4}{n+1}x & x \in [2n + \frac{1}{2}, 2n + \frac{3}{2}] \\ \frac{4}{n+1}x & x \in [2n + \frac{3}{2}, 2n + 2] \end{cases}$$

Now f_n is continuous on $[0, \infty)$ and when considering Riemann integration, we have

$$\int_0^{2n+1} f_n(x)\ dx = \frac{1}{n+1} \text{ and } \int_0^{2n+2} f_n(x)\ dx = 0 \quad \Rightarrow \quad \int_0^\infty f_n\ dx = 0$$

for each fixed n. Now define

$$f(x) = \sum_{n=0}^\infty f_n(x)$$

Then since f_n has disjoint support for any $N \in \mathbb{N}$ and $2N < y < 2N + 2$, we have

$$\int_0^y f(x)\ dx = \int_{2N}^y f(x)\ dx,$$

and so the Riemann integral of $f(x)$ converges to 0 on $[0, \infty)$. Now if a measurable function f is Lebesgue integrablem then so is $|f|$. But,

$$\int_0^\infty |f|\ dx = 2\sum_{n=1}^\infty \frac{1}{n+1} = \infty.$$

Therefore f is Riemann integrable but not Lebesgue integrable $\square$.

Exercise 2.4. *Consider the real valued function $f(x, t)$, where $x \in \mathbb{R}^n$ and $t \in I = (a, b)$. Suppose the following hold.*

(1) $f(x, \cdot)$ is integrable over I for all $x \in E$
(2) There exists an integrable function $g(t)$ on I such that $|f(x, t)| \leq g(t)$, $\forall x \in E, t \in I$.
(3) For some $x_0 \in E$ then function $f(\cdot, t)$ is continuous on I

Then the function $F(x) = \int_I f(x, t)\ dt$ is continuous at x_0

Proof: Let x_n be any sequence in E such that $x_n \to x_0$. Define a sequence of functions as $f_n(t) =$

$f(x_n, t)$. Then by hypothesis we have $f_n(t) \leq g(t)$, for $t \in I$ almost everywhere. Let $f(t) = f(x_0, t)$, now since $f(x,t)$ is continuous at x_0, we have $f_n \to f$. So by the Lebesgue Dominated Convergence theorem we have

$$\lim_{n\to\infty} \int_I |f_n(t) - f(t)| \, dt = 0$$

Hence we have

$$|F(x_n) - F(x_0)| = \left| \int_I f_n(t) - f(t) \, dt \right| \leq \int_I |f_n(t) - f(t)| \, dt \to 0$$

Or $F(x)$ is continuous at x_0 □

Theorem (Lebesgue Dominated Convergence) Let g be integrable over E and let f_n be a sequence of measurable functions such that $|f_n| \leq g$ on E and for almost all $x \in E$ we have $f(x) = \lim f_n(x)$. Then

$$\int_E f = \lim \int_E f_n.$$

Proof: Assuming the hypothesis, the function $g - f_n$ is nonnegative, so by Fatou's lemma we have

$$\int_E (g - f) \leq \liminf \int_E (g - f_n)$$

Now since $|f| \leq g$, f is integrable and we have

$$\int_E g - \int_E f \leq \int_E g - \limsup \int_E f_n$$

Hence we have

$$\int_E f \geq \limsup \int_E f_n$$

Considering $g + f_n$, we have the result

$$\int_E f \leq \liminf \int_E f_n$$

and so the result follows □.

Exercise 2.5. *Show that the Lebesgue Dominated Convergence theorem holds if almost everywhere convergence is replaced by convergence in measure.*

Proof: Suppose that $f_n \to f$ in measure, and there is an integrable function g such that $f_n \leq g$ almost everywhere. Now $|f_n - f|$ is integrable for each n, and $|f_n - f|\chi_{[-k,k]}$ converges to $|f_n - f|$. By the Lebesgue Dominated Convergence theorem we have

$$\int_{-k}^{k} |f_n - f| \to \int_{\mathbb{R}} |f_n - f|$$

Let $\epsilon > 0$, then there exsits an N_0 such that

$$\int_{|x|>N_0} |f_n - f| < \frac{\epsilon}{3}$$

also for each n, given $\epsilon > 0$, there exists $\delta > 0$ such that for any set A with $\mu(A) < \delta$ we have

$$\int_A |f_n - f| < \frac{\epsilon}{3}$$

Let $A = \{|f_n - f| \geq \delta\}$. Then there exists an N_1, such that for all $n \geq N_1$, we have $A = \{|f_n - f| \geq \delta\} < \delta$. Let $N = \max\{N_0, N_1\}$

$$\int_{\mathcal{X}} |f_n - f| = \int_{|x|>N} |f_n - f| + \int_{[-N,N]\cap A} |f_n - f| + \int_{[-N,N]\cap A^c} |f_n - f| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + 2N\delta < \epsilon$$

Let $\delta = \dfrac{\epsilon}{6N}$, therefore we have $\displaystyle\int_{\mathcal{X}} |f_n - f| \to 0$, as $n \to \infty$ □.

Exercise 2.6. *Show that an extended real valued integrable function is finite almost everywhere.*

Proof: Consider the measur space $(\mathcal{X}, \mathcal{M}, \mu)$. Let $E = \{x \in C : |f| = \infty\}$. Now since f is integrable, it is measurable hence the set E is measurable. Now suppose $\mu(E) > 0$, then as $|f| > 0$ on E we have

$$\infty > \int_{\mathcal{X}} |f| \, d\mu \geq \int_E |f| \, d\mu = \infty$$

This contradicts to the integrability of f, thus $\mu(E) = 0$. Therefore f is finite almost everywhere $\square$.

Exercise 2.7. *If f_n is a sequence of measurable functions such that*

$$\sum_{n=1}^{\infty} \int |f_n| < \infty$$

Show that $\sum_{n=1}^{\infty} f_n$ converges almost everywhere to an integrable function f and that

$$\int f = \sum_{n=1}^{\infty} \int f_n < \infty$$

Proof: Define g_N to be the partial sums of $|f_n|$. Then g_N is measurable since each f_n, and hence $|f_n|$ is measurable. Let $g = \lim g_n$, then g is measurable as it is the limit of measurable functions. Now

$$\int f = \int \sum_{n=1}^{\infty} |f_n| = \sum_{n=1}^{\infty} \int |f_n| < \infty$$

So g is integrable, and hence g is finite almost everywhere. Define $f(x)$ as follows

$$f(x) = \begin{cases} \sum_{n=1}^{\infty} \int f_n & \text{if } |g(x)| < \infty \\ 0 & \text{otherwise} \end{cases}$$

Then $g_N \to f$ as $N \to \infty$ almost everywhere. We also have

$$\begin{aligned} \left| \int f \right| &\leq \int |f| \\ &= \int \left| \sum_{n=1}^{\infty} f_n \right| \\ &\leq \int \sum_{n=1}^{\infty} |f_n| \\ &= \int g < \infty \end{aligned}$$

We also have that

$$|g_N| = \left| \sum_{n=1}^{N} f_n \right| \leq \sum_{n=1}^{N} |f_n| \leq \sum_{n=1}^{\infty} |f_n| = g$$

almost everywhere. Now by the Lebesgue Dominated Convergence theorem, we have

$$\int f = \int \lim g_N = \lim \int g_N = \lim \sum_{n=1}^{N} \int f_n = \sum_{n=1}^{\infty} \int f_n \ \square$$

Exercise 2.8. *Let $(\mathcal{X}, \mathcal{M}, \mu)$ be a measure space, and let f_n be a sequences of nonnegative extended real-valued $\mathcal{M}$-measurable functions on $\mathcal{X}$. Suppose $\lim f_n = f$ exists almost everywhere on $\mathcal{X}$ and $f_n \leq f$ almost everywhere. For $n \in \mathbb{N}$, show that*

$$\int_{\mathcal{X}} f \, d\mu = \lim_{n\to\infty} \int_{\mathcal{X}} f_n \, d\mu$$

Proof: First if $\int f \, dx = \infty$, applying Fatou's lemma we have

$$\int_{\mathcal{X}} \lim_{n\to\infty} \inf f_n \, d\mu \leq \lim_{n\to\infty} \inf \int_{\mathcal{X}} f_n \, d\mu \leq \lim_{n\to\infty} \int_{\mathcal{X}} f_n \, d\mu \leq \infty.$$

And so $\lim_{n\to\infty} \int_{\mathcal{X}} f_n \, d\mu = \int f \, dx = \infty$.

Now if $\int f \, dx < \infty$, since $f_n \leq f$ almost everywhere, we have $|f_n| \leq |f|$ almost everywhere, and we have $\lim f_n = f$ exists almost everywhere, we have by the Lebesgue Dominated Convergence theorem

$$\int_{\mathcal{X}} \left| |f_n| - |f| \right| \, d\mu \leq \int_{\mathcal{X}} |f_n - f| \, d\mu = 0 \quad \Rightarrow \quad \lim_{n\to\infty} \int_{\mathcal{X}} f_n \, d\mu = \int_{\mathcal{X}} f \, d\mu \; \square$$

Exercise 2.9. *Let f be a nonnegative Lebesgue measurable function on $[0,1]$. Suppose f is bounded above by 1 and $\int_0^1 f \, dx = 1$. Show that $f = 1$ almost everywhere on $[0,1]$*

Proof: let $1 > \epsilon > 0$ and define the set E as

$$E = \{x \in [0,1] : 0 \leq f \leq 1 - \epsilon\}$$

Now we have

$$\begin{aligned} 1 = \int_0^1 f \, dx &= \int_{E^c} f \, dx + \int_E f \, dx \\ &\leq \int_{E^c} f \, dx + \int_E 1 - \epsilon \, dx \\ &\leq \mu(E^c) + \mu(E) - \epsilon\mu(E) \\ &= 1 - \epsilon\mu(E) \end{aligned}$$

Hence since this holds for any $\epsilon \in (0,1)$, we must have $\mu(E) = 0$. Therefore $f = 1$ almost everywhere on $[0,1]$ $\square$.

Exercise 2.10. *Let f be a real-valued Lebesgue measurable function on $[0, \infty)$ such that:*

(1) f is locally integrable
(2) $\lim_{x\to\infty} f = c$

Show that $\lim_{a\to\infty} \frac{1}{a} \int_0^a f \, dx = c$.

Proof: Let $\epsilon > 0$, then there is an $M > 0$ such that if $x > M$, then $|f(x) - c| < \epsilon$. Let $a > M$, now

$$\begin{aligned}\left|\frac{1}{a}\int_0^a f\ dx - c\right| &= \frac{1}{a}\left|\int_0^a f - c\ dx\right| \\ &\leq \frac{1}{a}\int_0^a |f-c|\ dx \\ &= \frac{1}{a}\int_0^M |f-c|\ dx + \frac{1}{a}\int_M^a |f-c|\ dx \\ &< \frac{1}{a}\int_0^M |f-c|\ dx + \epsilon\frac{1}{a}(a-M) \\ &= \frac{1}{a}\int_0^M |f-c|\ dx + \epsilon(1-\frac{M}{a})\end{aligned}$$

Now since M is fixed, and by the integrability of f, we have

$$\left|\frac{1}{a}\int_0^a f\ dx - c\right| < \epsilon$$

and since this is for any $\epsilon > 0$ and all $a > M$, we have

$$\lim_{a\to\infty}\left|\frac{1}{a}\int_0^a f\ dx - c\right| = 0 \quad \Leftrightarrow \quad \lim_{a\to\infty}\frac{1}{a}\int_0^a f\ dx = c\ \square.$$

Exercise 2.11. *Let f be a real-valued uniformly continuous function on $[0,\infty)$. Show that if f is Lebesgue integrable on $[0,\infty)$, then $\lim_{x\to\infty} f(x) = 0$.*

Proof: First if f is Lebesgue integrable, then so is $|f|$. Now decompose the integral as follows

$$\infty > \int_0^\infty |f(x)|\ dx = \sum_{k=1}^\infty \int_k^{k+1} |f(x)|\ dx,\ \text{denote}\ a_k = \int_k^{k+1} |f(x)|\ dx.$$

Now $a_k > 0$, and since the integral is convergent this implies that $a_k \to 0$ as $k \to \infty$, which inturn implies that a_k is Cauchy. So we have

$$\forall \epsilon > 0,\ \exists N\ s.t.\ \left|\sum_{k=n}^m a_k\right| < \epsilon,\ \ \forall n, m > N \quad \Rightarrow \quad \int_{N+1}^\infty |f(x)|\ dx < \epsilon$$

Since $|f(x)|$ is positive and ϵ is arbitrary this implies that $f(x) \to 0$ as $N \to \infty$ $\square$.

Exercise 2.12. *Let $f \in \mathcal{L}_1(\mathbb{R})$. With $h > 0$ fixed, define a function ϕ_h on $\mathbb{R}$ by setting*

$$\phi_h(x) = \frac{1}{2h}\int_{x-h}^{x+h} f(t)\ \mu(dt), \textit{for } x \in \mathbb{R}$$

(a) Show that ϕ_h is measurable on $\mathbb{R}$.
(b) Show that $\phi_h \in \mathcal{L}_1(\mathbb{R})$ and $\|\phi_h\|_1 \leq \|f\|_1$.

For part (a) since f is integrable, then f is measurable. So the integral of a measurable function is measurable, thus $\phi_h(x)$ is measurable.

For part (b) First apply the change of variable $y = x - t$, then we have

$$\int_{x-h}^{x+h} f(t)\ \mu(dt) = -\int_h^{-h} f(x-y)\ \mu(dy) = \int_{-h}^h f(x-y)\ \mu(dy) = \int_{-\infty}^\infty f(x-y)\chi_{[-h,h]}(y)\ \mu(dy)$$

Where $\chi_{[-h,h]}(y)$ is the charactistic function on $[-h,h]$. So we have

$$\phi_h(x) = \frac{1}{2h} f * \chi_{[-h,h]} \quad \Rightarrow \quad \|\phi_h(x)\|_1 = \frac{1}{2h}\|f\chi_{[-h,h]}\|_1 \leq \frac{1}{2h}\|f\|_1\|\chi_{[-h,h]}\|_1 = \|f\|_1\ \square.$$

Exercise 2.13. *Let f be a Lebesgue integrable function of the real line. Prove that*

$$\lim_{n\to\infty}\int_{\mathbb{R}} f(x)\sin(nx)\ dx = 0.$$

Proof: If f is intergrable, then there exists a sequences of step function ϕ_n such that

$$\forall\epsilon > 0\ \exists N\ s.t. \int |f-\phi_n| < \frac{\epsilon}{2}$$

Now we have

$$\begin{aligned}
\left|\int_{\mathbb{R}} f(x)\sin(nx)\ dx\right| &\leq \int_{\mathbb{R}} |f(x)\sin(nx)|\ dx \\
&\leq \int_{\mathbb{R}} |(f(x)-\phi_n(x))\sin(nx)|\ dx + \int_{\mathbb{R}} |\phi_n(x)\sin(nx)|\ dx \\
&< \frac{\epsilon}{2} + \int_{\mathbb{R}} |\phi_n(x)\sin(nx)|\ dx
\end{aligned}$$

Now ϕ_n being a step function we have it as the sum of simple functions over disjoint interval I_n, where $\bigcup_{n=1}^{\infty} I_n = \mathbb{R}$, i.e.

$$\phi_n = \sum_{k=1}^{\infty} a_{k,n}\chi_{I_{k,n}}$$

and so we have

$$\begin{aligned}
\int_{\mathbb{R}} |\phi_n(x)\sin(nx)|\ dx &= |a_{k,n}| \int_{\mathbb{R}} \left|\chi_{I_{k,n}}\sin(nx)\right|\ dx \\
&= \sum_{k=1}^{\infty} |a_{k,n}| \int_{I_{k,n}} |\sin(nx)|\ dx \to 0 \text{ as } n\to\infty
\end{aligned}$$

Hence for some N large enough and all $n > N$ we have

$$\left|\int_{\mathbb{R}} f(x)\sin(nx)\ dx\right| < \frac{\epsilon}{2} + \int_{\mathbb{R}} |\phi_n(x)\sin(nx)|\ < \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon\ \square$$

3. Convergence

Exercise 3.1. *Consider the Lebesgue measure space $(\mathbb{R},\mathcal{M},\mu)$ on $\mathbb{R}$. Let f be a μ-integrable extended real-valued $\mathcal{M}$-measurable function on $\mathbb{R}$. Show that*

$$\lim_{h\to 0}\int_{\mathbb{R}} |f(x+h)-f(x)|\ \mu(dx) = 0.$$

Proof: First since $f(x)$ is integrable, we have

$$\int_{\mathbb{R}} f(x+h)\ \mu(dx) = \int_{\mathbb{R}} f(x)\ \mu(dx) \quad \forall h\in\mathbb{R}$$

Also since f is integrable, there exists a sequence of continuous function ϕ_n, such that

$$\int |f(x)-\phi_n(x)|\ \mu(dx) < \frac{\epsilon}{3}$$

Now $|\phi_n(x+h)-\phi_n(x)| < \frac{\epsilon}{3}$ if $|h| < \delta$. Let N be large enough, then

$$\int |f-f(x+h)|\ \mu(dx) \leq \int |f-\phi_n(x)|+|\phi_n(x+h)-f(x+h)|+|\phi_n(x+h)-\phi_n(x)|\mu(dx) < \frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3} = \epsilon$$

Therefore we have

$$\lim_{h\to 0}\int_{\mathbb{R}} |f(x+h)-f(x)|\ \mu(dx) = 0\ \square.$$

Exercise 3.2. *Let $(\mathcal{X}, \mathcal{M}, \mu)$ be a measure space. Let f_n and f be an extended real-valued $\mathcal{M}$- measurable fuctions on a set $E \in \mathcal{X}$ such that $\lim_{n\to\infty} f_n = f$ on E. Then for every $\alpha \in \mathbb{R}$ we have*

$$\mu\{E : f > \alpha\} \leq \lim_{n\to\infty} \inf \mu\{E : f_n \geq \alpha\} \text{ and } \mu\{E : f < \alpha\} \leq \lim_{n\to\infty} \inf \mu\{E : f_n \leq \alpha\}$$

Proof: I will only show the first inequality since the proofs are identical. Let $A_\alpha = \{x \in E : f(x) \geq \alpha\}$, $A_{\alpha,n} = \{x \in E : f_n(x) \geq \alpha\}$ and let χ_A denote the characteristic function of A. First we need to show $\chi_{A_n} \to \chi_A$ in measure. Let $\epsilon > 0$, denote the set $F_{\alpha-\epsilon,n}$ by

$$F_{\alpha-\epsilon,n} = \{x \in E : |\chi_{A_{\alpha-\epsilon,n}} - \chi_{A_{\alpha-\epsilon}}| \geq \epsilon\}.$$

Now we want to show that the measure of this set is small. First notice that

$$F^c_{\alpha-\epsilon,n} \supset \{x \in A_\alpha : |f - f_n| < \epsilon\}$$

Let x be in this subset, then this implies two thing. First if $f(x) > \alpha > \alpha - \epsilon$, and $f_n(x) > f(x) - \epsilon > \alpha - \epsilon$. So we must have

$$F_{\alpha-\epsilon,n} \subset \{x \in A_\alpha : |f - f_n| \leq \epsilon\}$$

Now since f_n converges to f almost everywhere in E, it converges in measure, and hence the measure of the set $\mu(F_{\alpha-\epsilon,n}) < \epsilon$. This implies that $\chi_{A_{\alpha,n}}$ converges to χ_{A_α} in measure. Now Fatou's lemma holds for a sequence of functions converging in measure, so we have

$$\int_E \chi_{A_\alpha}\, d\mu \leq \liminf \int_E \chi_{A_{\alpha,n}}\, d\mu \quad \Rightarrow \quad \mu\{E : f > \alpha\} \leq \lim_{n\to\infty} \inf \mu\{E : f_n \geq \alpha\} \ \square$$

Exercise 3.3. *Let $g(x)$ be a real-valued function of bounded variation on an interval $[a, b]$. Suppose that f is a real-valued decreasing function on $[a, b]$. Show that $g(f(x))$ is also of bounded variation. If f is just a bounded continuous function is $g(f(x))$ still of bounded variation.*

Proof: Since g is of bounded variation we have, let $\mathcal{P}$ be all the possible partitions of $[a, b]$

$$V_a^b(g) = \sup_{\{x_i\}\in\mathcal{P}} \sum_{i=1}^{n} |g(x_i) - g(x_{i-1})|$$

Now fix $\epsilon > 0$ and pick an $\{x_i\}$ such that

$$V_a^b(g) < \sum_{i=1}^{N} |g(x_i) - g(x_{i-1})| + \epsilon$$

Now since f is decreasing on we have that $f(x_{i+1}) < f(x_i)$. Now call $y_i = f_{x_i}$, $\{y_i\} \cup \{a, b\}$ then is a partition of $[a, b]$, and so we have

$$\sum_{i=1}^{N} |g(y_i) - g(y_{i-1})| < \sup_{\{x_i\}\in\mathcal{P}} \sum_{i=1}^{n} |g(x_i) - g(x_{i-1})| = V_a^b(g)$$

This can be done for any partition of $[a, b]$.Therefore $g(f(x))$ is also of bounded variation.

For the second part, no. Consider the function

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 1 & x = 0 \end{cases}$$

and let $g(x) = x$. Now $g(x)$ is a function of bounded variation on $[-1, 1]$ and $f(x)$ is a bounded and continuous on $[-1, 1]$, but $g(f(x)) = f(x)$, which not a function of bounded variation on $[-1, 1]$.

Exercise 3.4. *Let $(\mathcal{X}, \mathcal{M}, \mu)$ be a measure space. Let f_n and f be an extended real-valued $\mathcal{M}$- measurable fuctions on a set $E \in \mathcal{X}$ with $\mu(E) < \infty$. Show that f_n converges to 0 in measure on E if and only if $\lim_{n\to\infty} \int_E \frac{|f_n|}{1 + |f_n|}\, d\mu = 0$*

Proof: ($\Rightarrow$) If f_n converges to 0 in measure then we have

$$\mu\{x \in E : |f_n| \geq \epsilon\} < \epsilon.$$

Call this set A_ϵ. Now

$$\int_E \frac{|f_n|}{1+|f_n|}\, d\mu = \int_{A_\epsilon} \frac{|f_n|}{1+|f_n|}\, d\mu + \int_{A_\epsilon^c} \frac{|f_n|}{1+|f_n|}\, d\mu$$

Now the function $\frac{1}{1+x}, x \geq 0$ is monotone, and uniformly continuous on any bounded interval. Now $\mu(A_\epsilon) < \epsilon$, and so there is a δ such that So we have

$$\int_{A_\epsilon} \frac{|f_n|}{1+|f_n|}\, d\mu + \int_{A_\epsilon^c} \frac{|f_n|}{1+|f_n|}\, d\mu < \mu(A_\epsilon) + \mu(E)\epsilon < \epsilon(1+\mu(E))$$

Hence we have

$$\int_E \frac{|f_n|}{1+|f_n|}\, d\mu \to 0 \text{ as } \epsilon \to 0$$

($\Leftarrow$) Now suppose that

$$\int_E \frac{|f_n|}{1+|f_n|}\, d\mu \to 0 \text{ as } \epsilon \to 0$$

and suppose that there exists and ϵ_0 such that $\mu\{x \in E : |f_n| \geq \epsilon_0\} \geq \epsilon_0$. Then we have

$$\begin{aligned} \int_{A_{\epsilon_0}} \frac{|f_n|}{1+|f_n|}\, d\mu + \int_{A_{\epsilon_0}^c} \frac{|f_n|}{1+|f_n|}\, d\mu &\geq \frac{\epsilon_0^2}{1+\epsilon_0^2} + \int_{A_{\epsilon_0}^c} \frac{|f_n|}{1+|f_n|}\, d\mu \\ &\geq \frac{\epsilon_0^2}{1+\epsilon_0^2} \end{aligned}$$

But this implies that

$$\frac{\epsilon_0^2}{1+\epsilon_0^2} \leq \int_{A_{\epsilon_0}} \frac{|f_n|}{1+|f_n|}\, d\mu + \int_{A_{\epsilon_0}^c} \frac{|f_n|}{1+|f_n|} < \epsilon$$

let $\epsilon = \dfrac{\epsilon_0^2}{2(1+\epsilon_0^2)}$, then we have a contradiction $\square$.

Exercise 3.5. *Suppose $\mu(E) < \infty$ and f_n converges to f in measure on E and g_n converges to measure on E. Prove that $f_n g_n$ converges to fg in measure on E.*

Proof: Let $h_n = f_n g_n$ and let $h = fg$. Now h and h_n are measurable since f_n and g_n are. For each $\delta > 0$ define

$$A_n(\delta) = \{x : |h_n(x) - h(x)| \geq \delta\}$$

and let $a_n(\delta) = \mu(A_n(\delta))$. Now because f_n and g_n converge in measure, for any subsequences f_{n_k}, g_{n_k} there are subsequences $f_{n_{k_j}}$ and $g_{n_{k_j}}$, such that both $f_{n_{k_j}}$ and $g_{n_{k_j}}$ converge almost everywhere to f and g respectively. Hence we have $h_{n_{k_j}} = f_{n_{k_j}} h_{n_{k_j}}$, which converges to $h = fg$ almost everywhere on E. Now since $h_{n_{k_j}}$ converges almost everywhere and $\mu(E)$ is finite we have that h_{n_k} converges in measure. Now

$$\lim_{n\to\infty} |h - h_n| \leq \lim_{k\to\infty} \sup_n |h - h_{n_k}| \to 0$$

Hence $\lim_{n\to\infty} a_n(\delta) = \lim_{k\to\infty} a_{n_k}(\delta) = 0$, or h_n converges in measure $\square$.

(Convergence in measure) A sequences f_n of measurable functions is said to converge to f in measure if, given $\epsilon > 0$, there is an N such that for all $n \geq N$ we have

$$\mu\{x : |f(x) - f_n(x)| \geq \epsilon\} \leq \epsilon$$

Remark: Let a_n be a sequence of real numbers. If there is an $a \in \mathbb{R}$, such that for every subsequence a_{n_k}, there is a subsequences for which $a_{n_{k_l}} \to a$, then $a_n \to a$.

Exercise 3.6. *If f_n, $f \in \mathcal{L}_2$ and $f_n \to f$ almost everywhere, then $\|f_n - f\|_2 \to 0$ if and only if $\|f_n\|_2 \to \|f\|_2$.*

Proof: ($\Rightarrow$) Suppose $\|f_n - f\|_2 \to 0$, now

$$\begin{aligned}
\|f_n - f\|_2^2 &= \int f_n^2 - 2ff_n + f^2 \\
&\geq \|f\|_2^2 - 2\int |f_n f| + \|f\|_2^2 \\
\text{Holder's inequality} &\geq \|f\|_2^2 - 2\|f_n\|_2\|f\|_2 + \|f\|_2^2 \\
&= |\|f\|_2 - \|f_n\|_2|^2
\end{aligned}$$

Therefore as $\|f_n - f\|_2^2 \to 0$ we have $\|f_n\|_2 \to \|f\|_2$.

($\Leftarrow$) Now suppose $\|f_n\|_2 \to \|f\|_2$ and $f_n \to f$ almost everywhere. Now for $p \geq 1$, and for finite a,b, we have

$$|a+b|^p \leq 2^p(|a|^p + |b|^p)$$

For each n, let

$$g_n = 4(|f_n|^2 + |f|^2) - |f_n - f|^2.$$

Now $g_n \geq 0$ almost everywhere. Since f_n and f are finite almost everywhere, by Fatou's lemma we have

$$\int \liminf g_n \leq \liminf \int g_n$$

Now since $f_n \to f$ almost everywhere we have $\liminf g_n = 8|f|^2$ almost everywhere. So we have $8\|f\|_2^2 \leq \liminf \int g_n$. Now

$$\begin{aligned}
\liminf \int g_n &= 4\liminf \int |f_n|^2 + 4\liminf \int |f|^2 = \limsup \int |f_n - f|^2 \\
&= 4\liminf \|f_n\|_2^2 + 4\|f\|_2^2 - \limsup \|f_n - f\|_2^2 \\
&= 8\|f_n\|_2^2 - \limsup \|f_n - f\|_2^2
\end{aligned}$$

so we have $0 \leq -\limsup \|f_n - f\|_2^2$, hence $0 \leq \limsup \|f_n - f\|_2^2 \leq 0$. Therefore we have

$$\limsup \|f_n - f\|_2 = \liminf \|f_n - f\|_2 = 0 \quad \Rightarrow \quad \|f_n - f\|_2 \to 0 \ \square.$$

Remark: A sequences of functions f_n converges in measure to f if and only if for every sequences f_{n_k}, there is a subsequence $f_{n_{k_j}}$ that converges almost everywhere to f.

Exercise 3.7. *If $f_n \geq 0$ and $f_n(x) \to f(x)$, in measure then*

$$\int f(x)\ dx \leq \liminf \int f_n(x)\ dx$$

Proof: Let f_{n_k} be any subsequence of f_n. then there exists an $f_{n_{k_j}}$ such that $f_{n_{k_j}}$ converges to f almost everywhere. By Fatou's Lemma we have

$$\int f \leq \liminf \int f_{n_{k_j}} = \lim \int f_{n_k} \leq \liminf \int f_n$$

Exercise 3.8. *Suppose f_n converges to two functions f and g in measure on D. Show that $f = g$ almost everywhere on D*

Proof: Define the set E as $E = \{x \in D : |f_n(x) - f(x)| > 0\}$. Then if $E_n = \{x \in D : |f_n(x) - f(x)| \geq 1/m\}$, we have $E = \lim E_n$. Now if for some n we have $|f_n(x) - f(x)| < \frac{1}{2m}$ and $|f_n(x) - g(x)| < \frac{1}{2m}$, then we have

$$|f - g| \leq |f_n - f| + |f_n - g| < \frac{1}{m}$$

And so

$$\left\{x : |f_n(x) - f(x)| < \frac{1}{2m}\right\} \cap \left\{x : |f_n(x) - g(x)| < \frac{1}{2m}\right\} \quad \subset \quad \left\{x : |f(x) - g(x)| < \frac{1}{2m}\right\}$$

which implies that

$$\left\{x : |f_n(x) - f(x)| \geq \frac{1}{2m}\right\} \cap \left\{x : |f_n(x) - g(x)| \geq \frac{1}{2m}\right\} \quad \supset \quad \left\{x : |f(x) - g(x)| \geq \frac{1}{2m}\right\}$$

This implies that $\mu\left\{x : |f(x) - g(x)| \geq \frac{1}{2m}\right\} < \frac{2}{m}$. Now as $n \to \infty$, we have $\frac{2}{m} \to 0$. Hence $\mu\{x : |f(x) - g(x)| > 0\} = 0$ □.

Exercise 3.9. *Let $f_n \to f$ in $\mathcal{L}_p(\mathcal{X}, \mathcal{M}, \mu)$, with $1 \leq p < \infty$, and let g_n be a seqences of measurable functions such that $|g_n| \leq M < \infty$ for all n, and $g_n \to g$ almost everywhere. Prove that $g_n f_n \to gf$ in $\mathcal{L}_p(\mathcal{X}, \mathcal{M}, \mu)$*

Proof: Since $f_n \to f$ in $\mathcal{L}_p$, since $\mathcal{L}_p$ is complete we have $f \in \mathcal{L}_p$. Also since $|g_n| \leq M$, for all n this implies that $|g| \leq M$. Now

$$\|f_n g_n - g_n f\|_p^p = \int (f_n g_n - g_n f)^p \leq M^p \int |f_n - f|^p \quad \to \quad M^p \|f_n - f\|_p^p$$

So we have $\|f_n g_n - g_n f\|_p \leq M\|f_n - f\|_p$, and so

$$\|f_n g_n - gf\|_p \leq M\|f_n - f\|_p \to 0 \text{ as } n \to \infty$$

Therefore $g_n f_n \to gf$ in $\mathcal{L}_p(\mathcal{X}, \mathcal{M}, \mu)$ □.

Exercise 3.10. *Suppose f is differentiable everywhere on (a, b). Prove that f' is a Borel measurable function on (a, b)*

Proof: f' is Borel measurable if $\{x : f'(x) \leq \alpha\}$ is a Borel set. So

$$\begin{aligned}
f'(x) \leq \alpha \quad &\Leftrightarrow \quad \lim_{n\to\infty} n\left(f\left(x + \frac{1}{n}\right) - f(x)\right) \leq \alpha \\
&\Leftrightarrow \quad \lim_{n\to\infty} \left(f\left(x + \frac{1}{n}\right) - f(x)\right) - \frac{\alpha}{n} \leq 0 \\
\text{for all but finitely many n} \quad &\Leftrightarrow \quad \left(f\left(x + \frac{1}{n}\right) - f(x)\right) - \frac{\alpha}{n} \leq \frac{1}{m} \; \forall m \\
&\Leftrightarrow \quad x \in \liminf \left\{x : f\left(x + \frac{1}{n}\right) - f(x) - \frac{\alpha}{n} \leq \frac{1}{m}\right\} \forall m \\
&\Leftrightarrow \quad x \in \bigcup_{n\geq 1} \bigcap_{k\geq n} \left\{x : f\left(x + \frac{1}{k}\right) - f(x) - \frac{\alpha}{k} \leq \frac{1}{m}\right\} \forall m \\
&\Leftrightarrow \quad x \in \bigcap_{m\geq 1} \bigcup_{n\geq 1} \bigcap_{k\geq n} \left\{x : f\left(x + \frac{1}{k}\right) - f(x) - \frac{\alpha}{k} \leq \frac{1}{m}\right\}
\end{aligned}$$

Now since $f(x)$ is differentiable almost everywhere, it is continuous almost everywhere and so the $f(x)$, and $f(x + 1\backslash k)$ are measurable. Any linear combination of them is measurable, and so the set

$$\left\{x : f\left(x + \frac{1}{k}\right) - f(x) - \frac{\alpha}{k} \leq \frac{1}{m}\right\}$$

is measurable. Now the collection of all such sets form a σ-algebra, and hence the countable union and intersection of these sets are measurable. Therefore $f'(x)$ is measurable □.

Exercise 3.11. *Let $c_{n,i}$ be an array of nonnegative exteneded real numbers for $n, i \in \mathbb{N}$.*

(a) Show that

$$\lim_{n\to\infty} \inf \sum_{i\in\mathbb{N}} c_{n,i} \geq \sum_{i\in\mathbb{N}} \lim_{n\to\infty} \inf c_{n,i}$$

(b) If $c_{n,i}$ is an increasing sequences for each $i \in \mathbb{N}$ then

$$\lim_{n\to\infty} \sum_{i\in\mathbb{N}} c_{n,i} = \sum_{i\in\mathbb{N}} \lim_{n\to\infty} c_{n,i}$$

Proof: For part (a) first let ν denote the counting measure. Now if $\mathcal{M} = \mathcal{P}(\mathbb{N})$, then $(\mathbb{N}, \mathcal{M}, \nu)$ forms a measure space. Now let a_n be sequences with $c_n \in [0, \infty]$. Then the function $a(n) = a_n$ is $\mathcal{M}$-measurable, and so

$$\int_{n\in\mathbb{N}} c \, d\nu = \sum_{n\in\mathbb{N}} c_n$$

Then by Fatou's lemma we have

$$\int_{\mathbb{N}} \lim_{n\to\infty} \inf c_n \leq \lim_{n\to\infty} \inf \int_{\mathbb{N}} c_n \quad \Rightarrow \quad \sum_{i\in\mathbb{N}} \lim_{n\to\infty} \inf c_{n,i} \leq \lim_{n\to\infty} \inf \sum_{i\in\mathbb{N}} c_{n,i} \; \square$$

For part (b) using the same measure space $(\mathbb{N}, \mathcal{M}, \nu)$, we know that $c_n(i) \leq c_n(i+1)$, so by the Monotone convergence theorem we have

$$\int_{\mathcal{N}} \lim_{n\to\infty} c_n = \lim_{n\to\infty} \int_{\mathcal{N}} c_n \quad \Rightarrow \quad \sum_{i\in\mathbb{N}} \lim_{n\to\infty} c_{n,i} \leq \lim_{n\to\infty} \sum_{i\in\mathbb{N}} c_{n,i} \; \square$$

Theorem (Ascoli-Arzela) Let $\mathcal{F}$ be an equicontinuous family of functions from a separable space X to a metric space Y. Let f_n be a sequence in $\mathcal{F}$ such that for each $x \in X$ the closure of the set $\{f_n(x) : 0 \leq n < \infty\}$ is compact. Then there is a subsequence f_{n_k} that converges pointwise to a continuous function f, and the convergence is uniform on each compact subset X.

Exercise 3.12. *Let $\{q_k\}$ be all the rational numbers in $[0, 1]$. Show that*

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \frac{1}{\sqrt{|x - q_k|}} \text{ converges a.e. in } [0, 1]$$

Proof: Fix $\epsilon_0 > 0$, consider the two sets

$$E_1 = \frac{1}{\sqrt{|x - q_k|}} \leq \frac{1}{\epsilon_0} \quad \text{ and } E_2 = \frac{1}{\sqrt{|x - q_k|}} > \frac{1}{\epsilon_0}$$

Now for each fixed $x \in [0, 1] \backslash \mathbb{Q}$ we can enumerate the rationals however we want (Zorn's Lemma). Choose such an ordering so that

$$x \in E_2 \quad \rightarrow \frac{1}{\sqrt{|x - q_k|}} < k^{1-\epsilon_0}$$

That is the closer q_k gets to x, the large the index. Now let ν be the counting measure, then we have

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{1}{k^2} \frac{1}{\sqrt{|x - q_k|}} &= \int_{E_1} \frac{1}{k^2} \frac{1}{\sqrt{|x - q_k|}} \, d\nu + \int_{E_2} \frac{1}{k^2} \frac{1}{\sqrt{|x - q_k|}} \, d\nu \\
&< \int_{E_1} \frac{1}{k^2} \frac{1}{\epsilon_0} + \int_{E_2} \frac{1}{k^{1+\epsilon_0}} \, d\nu \\
&< \int_{\mathbb{N}} \frac{1}{k^2} \frac{1}{\epsilon_0} + \int_{\mathbb{N}} \frac{1}{k^{1+\epsilon_0}} \, d\nu < \infty
\end{aligned}$$

This can be done for all $x \in [0, 1] \backslash \mathbb{Q}$. Therefore the series converges almost everywhere in $[0, 1]$ $\square$.

Exercise 3.13. *Let $(\mathcal{X}, \mathcal{M}, \mu)$ be a finite measure space. Let f_n be an arbitrary sequence of real-valued measurable functions on $\mathcal{X}$. Show that for every $\epsilon > 0$ there exists $E \subset \mathcal{M}$ with $\mu(E) < \epsilon$ and a sequence of positive real numbers a_n such that $a_n f_n \to 0$ for $x \in \mathcal{X} \backslash E$*

Proof: First denote the set $E_m = \{x : m-1 \leq |f_n| < m\}$, then the sets E_m are disjoint and cover $\mathcal{X}$. Now define α as such

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \alpha$$

Since $\mu(\mathcal{X}) < \infty$, if $\epsilon > 0$, there is an M_n such that

$$\frac{\epsilon}{\alpha n^2} > \sum_{m \geq M_n} \mu(E_n) = \mu\{x : |f_n| \geq M_n\}$$

Now choose these M_n such that $M_n > M_{n-1}$ for all n. Define the sets $F_n = \{x : |f_n| \geq M_n\}$, then we have $\mu(F_n) < \frac{\epsilon}{\alpha n^2}$. Now if $E = \cup E_n$, then

$$\mu(E) \leq \sum_{n=1}^{\infty} \mu(E_n) < \frac{\epsilon}{\alpha} \sum_{n=1}^{\infty} \frac{1}{n^2} = \epsilon$$

Let $a_n = 1/M_n^3$, then if $x \in \mathcal{X} \backslash E$, then we have

$$a_n |f_n(x)| < \frac{1}{M_n^3} M_n = \frac{1}{M_n^2} \quad \forall n$$

And so we have

$$\left| \sum_{n=1}^{\infty} a_n f_n(x) \right| \leq \sum_{n=1}^{\infty} |a_n f_n(x)| \leq \sum_{n=1}^{\infty} \frac{1}{M_n^2} \leq \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

Therefore we must have $a_n f_n(x) \to 0$ on $\mathcal{X} \backslash E$ $\square$.

Exercise 3.14. *Prove that the gamma function*

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t}$$

is well defined and continuous for $x > 0$

Proof: Let let $f(t,x) = t^{x+1} e^{-t}$, and $x > 0$ and decompose the integral into two integrals $(0,1]$ and $(1,\infty)$. For the first we have

$$\int_0^1 t^{x-1} e^{-t} \, dt \leq \int_0^1 t^{x-1} \, dt = \frac{t^x}{x} \Big|_0^x < \infty$$

Now $f(t,x)$ is continuous on $(1,\infty)$, and also $t^2 f(t,x) \to 0$ as $t \to \infty$, so there is an M such that M bounds $t^2 f(t,x)$ on $(1,\infty)$. Now

$$\int_1^{\infty} t^{x-1} e^{-t} \, dt = \int_0^1 t^{x+1} e^{-t} t^{-2} \, dt = M \int_0^{\infty} \frac{1}{t^2} \, dt = M$$

And so $\Gamma(x)$ is well defined on $(0,\infty)$.

To show continuity, let x_n, be a cauchy sequence, and define $f_n(t) = f(t, x_n)$. Now by continuity of $f(t,x)$ on $(0,\infty) \times (0,\infty)$, we have that for each x, $f_n \to f$ on $t \in (0,\infty)$. now $f(t,x)$ is bounded on $(1,\infty)$, call this bound $M > 1$. Define a function $g(t)$ by

$$g(t) = \begin{cases} t^{x-1} & 0 < t \leq 1 \\ t^M e^{-t} & 1 < t \leq \infty \end{cases}$$

Now $f_n, f \leq g$ on $(0,\infty)$, so by the Lebesgue Dominated Convergence theorem we have

$$\int_0^{\infty} |f_n - f| \to 0 \text{ as } n \to \infty$$

and so we have

$$|\Gamma(x_n) - \Gamma(x)| = \left|\int_0^\infty f_n(t) - f(t,x)\ dt\right| \leq \int_0^\infty f_n(t) - f(t,x)|\ dt \to 0 \text{ as } n \to \infty$$

This holds for any sequence such that $x_n \to x \in (0,\infty)$, therefore $\Gamma(x)$ is continuous on $(0,\infty)$ $\square$.

4. Lp spaces

Exercise 4.1. *Let $1 \leq p < q < \infty$. Which of the following statements are true and which are false?*

(a) $\mathcal{L}_p(\mathbb{R}) \subset \mathcal{L}_q(\mathbb{R})$
(b) $\mathcal{L}_q(\mathbb{R}) \subset \mathcal{L}_p(\mathbb{R})$
(c) $\mathcal{L}_p([2,5]) \subset \mathcal{L}_q([2,5])$
(d) $\mathcal{L}_q([2,5]) \subset \mathcal{L}_p([2,5])$

Proof: Only part (d) is true. This can easily be shown for any finite interval, let $I = [a,b]$ Let $f \in \mathcal{L}_q(I)$. Then $|f|^p \in \mathcal{L}_{q/p}(I)$. Now by Holder's inequality we have

$$\int_I |f|^p = \leq \||f|^p\|_{q/p} \|1\|_r$$

where r is conjugate to $\frac{q}{p}$. Now

$$\|f\|_p^p \leq \||f|^p\|_{q/p} \|1\|_r = \left(\int_I (|f|^p)^{q/p}\right)^{p/q} \mu(I)^{\frac{q-p}{q}} = \|f\|_q^p \mu(I)^{\frac{q-p}{q}}$$

Hence we have $\|f\|_p \leq \|f\|_q \mu(I)^{\frac{q-p}{qp}}$, therefore $f \in \mathcal{L}_p(I)$ $\square$.

For a counterexample to part (c) consider the function $f(x) = (x-2)^{-1/2}$, and let $p = 1$ and $q = 2$, then $f \in \mathcal{L}_p([2,5])$, but $f \notin \mathcal{L}_q([2,5])$.

For a counterexample to part (b) consider the function $f(x) = (1+x^2)^{-1/2}$, and let $p = 1, q = 2$, then $f \in \mathcal{L}_q(\mathbb{R})$ but $f \notin \mathcal{L}_p(\mathbb{R})$.

For a counterexample to part (a) consider the counterexample to part (c) with the zero extension.

Theorem (Holder Inequality) If p and q are nonnegative extended real numbers such that

$$\frac{1}{p} + \frac{1}{q} = 1$$

and if $f \in \mathcal{L}_p$ and $g \in \mathcal{L}_q$ then $fg \in \mathcal{L}_1$ and

$$\int |fg| \leq \|f\|_p \|g\|_q$$

Proof: Assume $1 < p < \infty$, and suppose that $f,\ g \geq 0$. Let $h = g^{q-1}$, then $g = h^{p-1}$. Now

$$ptf(x)g(x) = ptf(x)h^{p-1} \leq (h(x) + tf(x))^p - h(x)^p$$

so we have

$$pt\int fg \leq \int |h + tf|^p - \int h^p = \|h + tf\|_p^p - \|h\|_p^p$$

and we also have

$$pt\int fg \leq \|h\|_p^p + \|tf\|_p^p - \|h\|_p^p$$

now differentiating bothsides with respect to t at $t = 0$, we have

$$p\int fg \leq p\|f\|_p \|h\|_p^{p-1} = p\|f\|_p \|g\|_q \ \square.$$

Exercise 4.2. *Let $f \in \mathcal{L}_{3/2}([0,5])$. Prove that*

$$\lim_{t\to 0+} \frac{1}{t^{1/3}} \int_0^t f(s)\ ds = 0.$$

Proof: Applying Holders inequality we have

$$\begin{aligned}
\left|\frac{1}{t^{1/3}} \int_0^t f(s)\ ds\right| &\leq \frac{1}{t^{1/3}} \int_0^t |f(s)|\ ds \\
&\leq \frac{1}{t^{1/3}} \left(\int_0^t |f(s)|\ ds\right)^{2/3} \left(\int_0^t\ ds\right)^{1/3} \\
&\leq \frac{1}{t^{1/3}} \|f(s)\|_{3/2} t^{1/3} \\
&\leq \left(\int_0^t |f(s)|^{3/2}\ ds\right)^{2/3} \to 0 \quad \text{as} \quad t \to 0+ \ \square
\end{aligned}$$

Exercise 4.3. *Suppose $f \in C^1[0,1]$, $f(0) = f(1)$, and $f > f'$ everywhere.*
(1) Prove that $f > 0$ everywhere.
(2) Prove that

$$\int_0^1 \frac{f^2}{f - f'}\ d\mu \geq \int_0^1 f\ d\mu$$

Proof: For (1), if $f(x) = \alpha \in \mathbb{R}^+$ then everything holds. So suppose that, there exists an $c \in (a,b) \subset (0,1)$ such that $f'(c) = 0$. (WLOG) suppose that this c is not a saddle point for $f(x)$, also suppose that $f(c) < 0$. Now if there is a $\delta > 0$ such that $f(c) > f(x)$, for all $x \in B(c,\delta)$, then we have $f'(x) > 0$ for $x \in (c-\delta, c)$. This implies that $f'(x) > f(x)$ for $x \in (c-\delta, c)$. If there is a $\delta > 0$ such that $f(c) < f(x)$, for all $x \in B(c,\delta)$, then we have $f'(x) > 0$ for $x \in (c, c+\delta)$, which implies that $f'(x) > f(x)$ for $x \in (c, c+\delta)$. For both cases we have a contradiction. Therefore $f(x) > 0$ for all $x \in (0,1)$. Now if $f(0) = 0$, f cannot be constant since $0 \ngtr 0$. this implies that, for some $\delta > 0$, $f'(x) > 0$ for $x \in [0,\delta)$, which is a contradiction. Therefore $f(x) > 0$ for all $x \in [0,1]$.

For (2) since $f > f'$ we have that $\sqrt{f - f'}$ is well defined on $[0,1]$. So,

$$\begin{aligned}
\left(\int_0^1 f\right)^2 d\mu &= \left(\int_0^1 \frac{f}{\sqrt{f-f'}} \sqrt{f - f'}\ d\mu\right)^2 \\
\text{Hölder's inequality} \quad &\leq \int_0^1 \frac{f^2}{f-f'}\ d\mu \int_0^1 f - f'\ d\mu \\
&\leq \int_0^1 \frac{f^2}{f-f'}\ d\mu \int_0^1 f\ d\mu
\end{aligned}$$

The last line holds since $f - f' > 0$. This implies that:

$$\int_0^1 f\ d\mu \leq \int_0^1 \frac{f^2}{f - f'}\ d\mu \ \square$$

Exercise 4.4. *If $f(x) \in \mathcal{L}_p \cap \mathcal{L}_\infty$ for some $p < \infty$. Show that*

(a) $f(x) \in \mathcal{L}_q$ *for* $q > p$.
(b) $\lim_{q\to\infty} \|f\|_q = \|f\|_\infty$.

Proof: For part (a) Let $0 < p < q < \infty$ and let $f \in \mathcal{L}_p \cap \mathcal{L}_\infty$. Then if $\alpha = \frac{q}{p}$ and if $\beta = \frac{q}{q-p}$, then we have $\frac{1}{\alpha}+\frac{1}{\beta}=1$. Now applying Holder's inequality we have

$$\begin{aligned}
\|f\|_q^q &= \int |f|^q \\
&= \int |f|^{q(\frac{1}{\alpha}+\frac{1}{\beta})} \\
&= \int |f|^p |f|^{q-p} \\
&= \int |f|^p |f|^{q-p} \\
&\leq \||f|^p\|_1 \||f|^{q-p}\|_\infty
\end{aligned}$$

Now since $|f| \leq \|f\|_\infty$ almost everywhere and $q-p>0$ we have $|f|^{q-p} \leq \||f|^{q-p}\|_\infty$ almost everywhere, and so $\||f|^{q-p}\|_\infty < \infty$. Also since f is monotone increasing, we have $\||f|^{q-p}\|_\infty = \|f\|_\infty^{q-p}$. We also have $\||f|^p\|_1 = \|f\|_p^p < \infty$. Therefore $f \in \mathcal{L}_q$ $\square$.

For part (b), first suppose that $\|f\|_\infty = 0$. This implies that $f=0$ almost everywhere and hence $\|f\|_q = 0$ for all q. Hence $\lim_q \|f\|_q \to \|f\|_\infty$ trivially.

Now suppose that $f \in \mathcal{L}_p \cap \mathcal{L}_\infty$ and $\|f\| \neq 0$. From part (a) we have

$$\|f\|_q \leq \left(\|f\|_p^p\right)^{1/q} \left(\|f\|_\infty\right)^{1-\frac{p}{q}}$$

Now let $\epsilon > 0$, then on a set E of nonzero measure, $|f| > \|f\|_\infty - \epsilon$. If $\mu(E)=\infty$, shoose a subset of E with finite measure. Then we have

$$\begin{aligned}
\|f\|_q^q &= \int_E |f|^q d\mu \\
&\geq \int_E (\|f\|_\infty - \epsilon)^q \, d\mu \\
&= \mu(E) |\|f\|_\infty - \epsilon|^q.
\end{aligned}$$

Now this is for all $q>p$. Let q_n be a sequence of numbers greater than p that converges to ∞. Then

$$\begin{aligned}
\lim_{n\to\infty} \mu(E)^{\frac{1}{q_n}} |\|f\| - \epsilon| &\leq \lim_{n\to\infty} \inf \|f\|_{q_n} \\
&\leq \lim_{n\to\infty} \sup \|f\|_{q_n} \\
&\leq \lim_{n\to\infty} \sup \left(\|f\|_p^p\right)^{\frac{1}{q_n}} \left(\|f\|_\infty\right)^{1-\frac{p}{q_n}}
\end{aligned}$$

and so

$$|\|f\|_\infty - \epsilon| \leq \lim_{n\to\infty} \inf \|f\|_{q_n} \leq \lim_{n\to\infty} \sup \|f\|_{q_n} \leq \|f\|_\infty$$

Since this holds for all $\epsilon > 0$ we have $\lim_{n\to\infty} \|f\|_{q_n} = \|f\|_\infty$. Now since this is for any sequence q_n, we have $\lim_{q\to\infty} \|f\|_q = \|f\|_\infty$.

Exercise 4.5. *Suppose that $f \in \mathcal{L}_p([0,1])$ for some $p>2$. Prove that $g(x) = f(x^2) \in L_1([0,1])$*

Proof: $f \in \mathcal{L}_p([0,1])$ implies that $\|f\|_p < \infty$. In particular this implies that $\|g\|_p = \|f(x^2)\|_p < \infty$. Now

$$\begin{aligned}
\int_0^1 |g(x)| \, dx &= \int_0^1 |f(x^2)| \, dx \\
\text{change of variables } (y=x^2) &= \int_0^1 |f(y) \frac{1}{2\sqrt{y}}| \, dy \\
\text{hölder's inequality} &\leq \frac{1}{2}\|f\|_p \left\| \frac{1}{\sqrt{y}} \right\|_{\frac{p}{p-1}}
\end{aligned}$$

Now $f \in \mathcal{L}_p([0,1])$ and since $p > 2$ we have $\left\| \frac{1}{\sqrt{y}} \right\|_{\frac{p}{p-1}} < \infty$, therefore $g(x) \in L_1([0,1])$ $\square$.

Exercise 4.6. *Let $f \in \mathcal{L}_p(\mathcal{X}) \cap \mathcal{L}_q(\mathcal{X})$ with $1 \le p < q < \infty$. Prove that $f \in \mathcal{L}_r(\mathcal{X})$ for all $p \le r \le q$.*

Proof: Let $E_1 = \{x : 0 \le |f(x)| \le 1\}$, and $E_2 = \{x : 1 > |f(x)|\}$, then E_1, E_2 are a Hahn decomposition for $\mathcal{X}$. Now suppose $f \in \mathcal{L}_p \cap \mathcal{L}_q$. Now

$$\begin{aligned} \|f\|_r^r &= \int_{E_1} |f|^r + \int_{E_2} |f|^r \\ &\le \int_{E_1} |f|^p + \int_{E_2} |f|^q \\ &\le \int_{\mathcal{X}} |f|^p + \int_{\mathcal{X}} |f|^q \\ &= \|f\|_p^p + \|f\|_q^q \qquad \therefore f \in \mathcal{L}_r(\mathcal{X}) \end{aligned}$$

Exercise 4.7. *Suppose f and g are real-valued μ-measurable functions on $\mathbb{R}$, such that*
(1) f is μ-integrable.
(2) $g \in C_0(\mathbb{R})$.
For $c > 0$ define $g_c(t) = g(ct)$. Prove that:

(a) $\lim_{c \to \infty} \int_{\mathbb{R}} f g_c \, d\mu = 0$,
(b) $\lim_{c \to 0} \int_{\mathbb{R}} f g_c \, d\mu = g(0) \int_{\mathbb{R}} f \, d\mu$.

Proof: For part (a) define $h_n(x) = f(x) g_n(x)$. Now since $f \in \mathcal{L}_1(\mathbb{R})$ we know that $f(x) < \infty$ a.e., and since $g \in C_0(\mathbb{R})$ we know that

$$g_n(x) \to 0 \text{ as } n \to \infty.$$

For a fixed x such that $f(x) < \infty$ we have

$$h_n(x) \to 0 \text{ as } n \to \infty$$

Hence $h_n \to 0 a.e.$. Also since $g \in C_0(\mathbb{R})$ we have that there is some M such that $|g(x)| < M$. So we have

$$\left| \int_{\mathbb{R}} h_n(x) \, d\mu \right| \le \int_{\mathbb{R}} |f(x) g_n(x)| \, d\mu \le M \int_{\mathbb{R}} |f(x)| \, d\mu < \infty$$

since $f \in \mathcal{L}_1(\mathbb{R})$. Hence by the Lebesgue Dominated Convergence theorem we have

$$\lim_{n \to \infty} \int_{\mathbb{R}} f g_n \, d\mu = \lim_{n \to \infty} \int_{\mathbb{R}} h_n \, d\mu = \int_{\mathbb{R}} \lim_{n \to \infty} h_n \, d\mu = 0$$

Proof: For part (b) we know that for all $n > 0$, $f g_n \in \mathcal{L}_1(\mathbb{R})$. Define $h_n(x) = |f(x) g(xn^{-1})|$, again since $g \in C_0(\mathbb{R})$ we have that there is some M such that $|g(x)| < M$. So

$$\left| \int_{\mathbb{R}} h_n(x) \, d\mu \right| \le \int_{\mathbb{R}} |f(x) g_n(x)| \, d\mu \le M \int_{\mathbb{R}} |f(x)| \, d\mu < \infty$$

Hence by the Lebesgue Dominated Convergence theorem we have

$$\begin{aligned} \lim_{n \to \infty} \int_{\mathbb{R}} f g_{1/n} \, d\mu &= \lim_{n \to \infty} \int_{\mathbb{R}} h_n \, d\mu \\ &= \int_{\mathbb{R}} \lim_{n \to \infty} h_n \, d\mu \\ &= \int_{\mathbb{R}} \lim_{n \to \infty} f g_{1/n} \, d\mu \\ &= g(0) \int_{\mathbb{R}} f \, d\mu \end{aligned}$$

Exercise 4.8. *Let E be a measurable subset of the real line. Prove that $\mathcal{L}_\infty(E)$ is complete.*

Proof: Let f_n be a Cauchy sequence of measurable functions in $\mathcal{L}_\infty$. Then there exists and $k \in \mathbb{N}$ such that if $m, n \geq N_k$, then

$$\|f_n - f_m\|_\infty < \frac{1}{k},\ \forall n, m > \mathbb{N}_k \quad \rightarrow \quad |f_n - f_m| < \frac{1}{k}\ a.e.$$

Now define the sets $E_{n,m,k}$ by

$$E_{n,m,k} = \left\{x \in E : |f_n(x) - f_m(x)| \geq \frac{1}{k}\right\}$$

then for each $n, m > N_k$, the set $E_{n,m,k}$ is empty. Let F be defined by

$$F = \bigcup_{k \geq m,n,N_k} E_{n,m,k}$$

Now F is a countable union of empty sets, and therefore is empty. Now for any $x \in E \backslash F$ we have

$$|f_n(x) - f_m(x)| < \frac{1}{k}$$

and so $f_n(x)$ is a Cauchy sequence in $\mathbb{R}$. Now

$$|f_m(x)| \leq |f_m(x) - f_n(x)| + |f_n(x)| < \frac{1}{k} + |f_n(x)|.$$

Taking $m \rightarrow \infty$, we have

$$|f(x)| \leq \frac{1}{k} + |f_n(x)| < \frac{1}{k} + \|f_n(x)\|_\infty \quad a.e.$$

Hence for each n we have $|f| \leq \frac{1}{k} + \|f_n\|_\infty$ almost everywhere so $f \in \mathcal{L}_\infty$. Therefore $\mathcal{L}_\infty$ is complete $\square$.

Theorem (Riesz-Fischer) The $\mathcal{L}_p(E)$ spaces are complete.

Proof: For $1 \leq p < \infty$, let f_n be a Cauchy sequence on $\mathcal{L}_p$.

$$\forall \epsilon > 0\ \exists N_\epsilon\ s.t.\ \|f_m - f_n\|_p < \epsilon\ \forall n, m > N$$

Now let $n_k = N2^{-k}$, then the subsequence f_{n_k}, satisfies

$$\|f_{n_{k+1}} - f_{n_k}\|_p < \frac{1}{2^k}$$

Define the function f by

$$f(x) = f_{n_1} + \sum_{k=1}^{\infty}(f_{n_{k+1}} - f_{n_k}) \quad \text{for } x \in E$$

Now the partial sums $S_N(f)$ is just

$$S_N(f) = f_{n_1} + \sum_{k=N}^{\infty}(f_{n_{k+1}} - f_{n_k}) = f_{n_N}$$

Define the function $g(x)$ by,

$$f(x) = |f_{n_1}| + \sum_{k=1}^{\infty}|f_{n_{k+1}} - f_{n_k}| \quad \text{for } x \in E$$

Now by Minikowski's inequality we have

$$\|S_N(g)\|_p \leq \|f_{n_1}\|_p + \|\sum_{k=1}^{N-1}|f_{n_{k+1}} - f_{n_k}|\|_p \leq \|f_{n_1}\|_p + \sum_{k=1}^{N-1}\frac{1}{2^k}$$

So the increasing sequences of partial sums $\|S_n(g)\|_p$ is bounded above by $\|f_{n_1}\| + 1$. Hence we have

$$\int_E g^p < \infty \quad \Rightarrow \quad \int_E |f|^p < \infty \quad \Rightarrow \quad \int_E f^p < \infty$$

This implies that the series f_{n_k} converges almost everywhere. Now

$$|f - f_{n_N}| = |S_\infty(f) - S_{N-1}(f)| = \left|\sum_{k=1}^{N} f_{n_{k+1}} - f_{n_k}\right| \leq g$$

Hence by the Lebesgue dominated convergence thoerem we have

$$\lim_{k\to\infty} \|f - f_{n_k}\|_p^p = \int_E \lim_{k\to\infty} (f(x) - f_{n_k})^p = 0$$

Hence f_{n_k} converges to f in $\mathcal{L}_p(E)$. Now f_n is itself Cauchy, hence f_n converges to f is in $\mathcal{L}_p(E)$.

Exercise 4.9. *Let $g(x)$ be measurable and suppose $\int_a^b f(x)g(x)\, dx$ is finite for any $f(x) \in \mathcal{L}_2$. Prove that $g(x) \in \mathcal{L}_2$.*

Proof: If $f = 1$, then $f \in \mathcal{L}_2([a, b])$ so $\int_a^b g dx < \infty$ which implies that $g \in \mathcal{L}_1[a, b]$. Let $F = \int_a^b g dx$, then F is a bounded linear functional from $\mathcal{L}_2([a, b])$ to $\mathbb{R}$. So there exists an M such that

$$\|F(f)\| = \sup_{\|f\|_2=1} \left\{ \int_a^b fg \right\} < M, \qquad f \in \mathcal{L}^2([0, 1])$$

Then by the Reisz Representation Theorem g must be in $\mathcal{L}_2([0, 1])$ $\square$.

Theorem (Riesz Representation) Let F be a bounded linear functional on $\mathcal{L}_p$ for $1 \leq p < \infty$. Then there exists a function $g \in \mathcal{L}_q$ such that

$$F(f) = \int fg.$$

We also have $\|F\| = \|g\|_q$.

Proof: Just considering the finite dimensional case. Let μ be of finite measure. Then every bounded measurable function is in $\mathcal{L}_p(\mu)$. Define a set function ν on the measurable sets by $\nu(E) = F(\chi_E)$. If E is the union of a sequence E_n of disjoint measurable sets, define a sequence $\alpha_n = \operatorname{sgn} F\chi_{E_n}$ and set

$$f = \sum \alpha_n \chi_{E_n}$$

Then F is bounded and we have

$$\sum_{n=1}^{\infty} |\nu(E_n)| = F(f) < \infty, \qquad \sum_{n=1}^{\infty} \nu(E_n) = F(f) = \nu(E)$$

Hence ν is a signed measure, and by construction it is absolutely continuous with respect to μ. By the Radon-Nikodym Theorem, there is a measurable function g such that for each measurable set E we have

$$\nu(E) = \int_E g\, d\mu$$

Since ν is always finite implies that g integrable. Now if ϕ is a simple function, the linearity of F and of the integral imply that

$$F(\phi) = \int \phi g\, d\mu$$

Since the left-hand side is bounded by $\|F\|\|\phi\|_p$ we have $g \in \mathcal{L}^q$. Now let G be the bounded linear functional defined on $\mathcal{L}_p$ by

$$G(f) = \int fg\, d\mu$$

Then $G - F$ is a bounded linear function which vanishes on the subspace of simple functions, which are dense in $\mathcal{L}_p$. Hence we must have $G - F = 0$ in $\mathcal{L}_p$. So for all $f \in \mathcal{L}_p$, we have

$$F(f) = \int fg \; d\mu$$

and by construction $\|F\| = \|G\| = \|g\|_q$ $\square$.

Exercise 4.10. *Let $(\mathcal{X}, \mathcal{M}, \mu)$ be a measure space and let f be an extended real-valued $\mathcal{M}$- measurable function on $\mathcal{X}$ such that*

$$\int_{\mathcal{X}} |f|^p \; d\mu < \infty \text{ for } p \in (0, \infty).$$

Show that $\lim_{\lambda \to \infty} \lambda^p \mu\{x : |f(x)| \geq \lambda\} = 0$

Proof: First define the set $E_\lambda = \{x \in \mathcal{X} : f(x) \geq \lambda\}$. Now notice that $E_\nu \subset E_\lambda$ if $\nu > \lambda$, also because $f \in \mathcal{L}_p$ we have $\mu(E_\lambda) < \lambda^{-1}$ if λ is large enough, in particular $\mu(E_\infty) = 0$. Now

$$\lambda^p \mu\{x : |f(x)| \geq \lambda\} \quad = \quad \lambda^p \int_{E_\lambda} d\mu \leq \int_{E_\lambda} |f|^p d\mu$$

Hence we have

$$\lim_{\lambda \to \infty} \lambda^p \mu\{x : |f(x) \geq \lambda\} \leq \int_{E_\infty} |f|^p d\mu = 0$$

Since $f \in \mathcal{L}_p$, then $|f|^p \in \mathcal{L}_1(\mathcal{X})$, $\mu(E_\infty) = 0$ and the integral of an Lebesgue integrable function over a set of measure zero is zero $\square$.

5. Signed Measures

Remark: If $(\mathcal{X}, \mathcal{M})$ is a measure space, and if μ, ν are two measure defined on $(\mathcal{X}, \mathcal{M})$. μ and ν are said to be mutually singular ($\mu \perp \nu$), if there are disjoint stes A and B, in $\mathcal{M}$ such that $X = A \cup B$ and $\nu(A) = \mu(B) = 0$. A measure ν is said to be absolutely continuous with respect to the measure μ, ($\nu << \mu$), if $\nu(A) = 0$ for each set A for which $\mu(A) = 0$.

Exercise 5.1. *Let μ be a measure and let $\lambda, \lambda_1, \lambda_2$ be signed measure on the measurable space $(\mathcal{X}, \mathcal{A})$. Prove:*

(a) If $\lambda \perp \mu$ and $\lambda << \mu$, then $\lambda = 0$

(b) If $\lambda_1 \perp \mu$ and $\lambda_2 \perp \mu$, then, if we set $\lambda = c_1\lambda_1 + c_2\lambda_2$ with $c_1, c_2 \in \mathbb{R}$ such that λ is a signed measure, thwn we have $\lambda \perp \mu$.

(c) If $\lambda_1 << \mu$ and $\lambda_2 << \mu$, then, if we set $\lambda = c_1\lambda_1 + c_2\lambda_2$ with $c_1, c_2 \in \mathbb{R}$ such that λ is a signed measure, thwn we have $\lambda << \mu$.

Proof: For part (a), if ν is a signed measure such that $\nu \perp \mu$ and $\nu << \mu$. There are disjoint measurable sets A and B such that $X = A \cup B$ and $|\nu|(B) = |\mu|(A) = 0$. Then $|\nu(A)| = 0$ so $|\nu|(X) = |\nu|(A) + |\nu|(B) = 0$. Hence we have $\nu^+ = \nu^- = 0$ i.e. $\nu = 0$.

For part (b), there are disjoint measurable sets A_i and B_i such that $X = A_i \cup B_i$ and $\mu(B_i) = \nu_i(A_i) = 0$, for $i = 1, 2$. Now $X = (A_1 \cap A_2) \cup (B_1 \cup B_2)$ and $(A_1 \cap A_2) \cap (B_1 \cup B_2) = \emptyset$. Now we have

$$(c_1\nu_1 + c_2\nu_2)(A_1 \cap A_2) = \mu(B_1 \cup B_2) = 0 \quad \Rightarrow \quad (c_1\nu_1 + c_2\nu_2) \perp \mu$$

For part (c), suppose $\nu_1 << \mu$ and $\nu_2 << \mu$. If $\mu(E) = 0$, then $\nu_1(E) = \nu_2(E) = 0$. Hence

$$(c_1\nu_1 + c_2\nu_2)(E) = 0 \quad \Rightarrow \quad (c_1\nu_1 + c_2\nu_2) << \mu$$

Exercise 5.2. *Let μ be a positive measure and ν be a finite positive measure on a measurable space $(\mathcal{X}, \mathcal{M})$. Show that if $\nu << \mu$, then for every $\epsilon > 0$ there is a $\delta > 0$, such that for every $E \subset \mathcal{M}$ with $\mu(E) < \delta$, we have $\nu(E) < \epsilon$.*

Proof: Suppose not, Then there is an $\epsilon > 0$ such that for every $\delta > 0$, there is $E_\delta \subset \mathcal{M}$, such that $\mu(E_\delta) < \delta$, and $\nu(E_\delta) \geq \epsilon$. In particular, for every $n \geq 1$, there is an E_n such that $\mu(E_n) < \frac{1}{n^2}$ and $\nu(E_n) \geq \epsilon$. Now we have

$$\sum_{n=1}^{\infty} \mu(E_n) = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

Let $E = \limsup E_n$, then $\mu(E) = 0$. Now since $\nu << \mu$, we have $\nu(E) = 0$. Now

$$\nu(E) = \nu(\limsup E_n) \geq \limsup \nu(E_n) \geq \epsilon$$

But this implies that $\nu(E_n) \geq \epsilon > 0$, and hence $\nu(E) > 0$, which is a contradiction. Therefore given $\epsilon > 0$ there is a $\delta > 0$, such that for every $E \subset \mathcal{M}$ with $\mu(E) < \delta$, we have $\nu(E) < \epsilon$ □.

Theorem (Hahn Decomposition) Let ν be a signed measure on the measurable space $(X, \mathcal{M})$. Then there is a positive set A and a negative set B such that $X = A \cup B$ and $A \cap B = \emptyset$.

Theorem (Jordan Decomposition) Let ν be a signed measure on the measurable space $(X, \mathcal{M})$. Then there are two mutually singular measure ν^+ and ν^- on $(X, \mathcal{M})$ such that $\nu = \nu^+ - \nu^-$. Moreover, there is only one such pair of mutually singular measures.

Exercise 5.3. *Suppose $(\mathcal{X}, \mathcal{M})$ is a measurable space, and Y is the set of all signed measure ν on $\mathcal{M}$ for which $\nu(E) < \infty$, whenevery $E \subset \mathcal{M}$. For ν_1, $\nu_2 \in Y$, define*

$$d(\nu_1, \nu_2) = \sup_{E \in \mathcal{M}} |\nu_1(E) - \nu_2(E)|$$

Show that d is a metric on Y and that Y equipped with d is a complete metric space.

Proof: Since ν_i are choosen such that $\nu_i(E) < \infty$, then for any $\nu_1, \nu_2 \in Y$ and $E \in \mathcal{M}$, we have $|\nu_1(E) - \nu_2(E)| < \infty$. So we have $d : Y \times Y \to [0, \infty)$. Now to show d is a metric on Y we need to show symmetry, positive definiteness and the triangle inequality. Clearly $d(\nu_1, \nu_2) = d(\nu_2, \nu_1)$ by definition of d. For the triangle inequality we have

$$\begin{aligned} d(\mu, \nu) &= \sup_{E \in \mathcal{M}} |\mu(E) - \nu(E)| \\ &\leq \sup_{E \in \mathcal{M}} \{|\nu(E) - \sigma(E)| + |\mu(E) - \sigma(E)|\} \\ &\leq \sup_{E \in \mathcal{M}} \{|\nu(E) - \sigma(E)|\} + \left\{ \sup_{F \in \mathcal{M}} |\mu(F) - \sigma(F)| \right\} \\ &= d(\mu, \sigma) + d(\sigma, \nu) \end{aligned}$$

Now to show definiteness, if $\mu = \nu$, then $|\mu(E) - \nu(E)| = 0$ for any $E \in \mathcal{M}$, and so $d(\mu, \nu) = 0$. On the other hand if $d(\mu, \nu) = 0$, then we have $|\mu(E) - \nu(E)| = 0$. Let (A_1, B_1), (A_2, B_2) be Hahn decompostions of μ, and ν respectively.

Case 1: If $E \subset A_1 \cap A_2$, then $\mu(E) = \mu^+(E)$, and $\nu(Y) = \nu^+(Y)$, hence $|\mu(E) - \nu(E)| = |\mu^+(E) - \nu^+(E)|$. So we have $\mu^+ = \nu^+$ on $A_1 \cap A_2$.

Case 2, 3: If $E \subset A_1 \cap B_2$, then we have $\mu(E) = -\mu^-(E)$ and $\nu(E) = \nu^+(E)$, hence

$$0 = |\mu(E) - \nu(E)| = | - \mu^-(E) - \nu^+(E)| = \mu^-(E) + \nu^+(E)$$

Hence $\mu^- = \nu^+ = 0$ on $E \subset A_1 \cap B_2$. If $E \subset A_2 \cap B_1$, by the same proof we have the result $\mu^+ = \nu^- = 0$ on $E \subset A_2 \cap B_1$

Case 3: If $E \subset B_1 \cap B_2$, then $\mu(E) = -\mu^-(E)$ and $\nu(E) = -\nu^-(E)$. So

$$0 = |\mu(E) - \nu(E)| = | - \mu^-(E) + \nu^-(E)|$$

and so $\mu^- = \nu^- = 0$ on $E \subset B_1 \cap B_2$. So definiteness holds, therefore d is a metric on Y.

Now to show the metric space is complete. Let ν_n be a Cauchy sequence. Then for any $\epsilon > 0$, there is an N such that if $m, n > N$, we have

$$\sup_{E \in \mathcal{M}} |v_n(E) - v_m(E)| < \epsilon$$

If particular, for a fixed set E, we have ν_n is a Cauchy sequence in $\mathbb{R}$. Hence there exists some $\mu(E) \in \mathbb{R}$, such that $\nu_n \to \mu$. By the uniform boundedness pricipal we know that μ is bounded, and hence

$$\nu_n \to \mu \text{ in the metric d } \square$$

Remark: The measure $|\nu|$ is defined from the Jordan decomposition by, $|\nu|(E) = \nu^+ E + \nu^- E$.

Theorem (Radon-Nikodym) let $(\mathcal{X}, \mathcal{M}, \mu)$ be a σ-finite measure space, and let ν be a measure defined on $\mathcal{M}$ which is absolutely continuous with respect to μ. Then there is a nonnegative measurable function f such that for each set E on $\mathcal{M}$ we have

$$\nu(E) = \int_E f \, d\mu$$

The function f is unique in the sense that if g is any measurable function with this property then $g = f$ almost everywhere.

Proof: Only the finite case is considered. Let μ be finite then $\nu - \alpha\mu$ is a signed measure for each rational number α. Let (A_α, B_α) be a Hahn decomposition for $\nu - \alpha\mu$, and take $A_0 = \mathcal{X}$ and $B_0 = \emptyset$. Now $B_\alpha \sim B_\beta = B_\alpha \cap A_\beta$. So we have

$$(\nu - \alpha\mu)(B_\alpha \sim B_\beta) \leq 0 \quad (\nu - \beta\mu)(B_\alpha \sim B_\beta) \geq 0$$

hence we must have $\mu(B_\alpha \sim B_\beta) = 0$. Now there exists a measurable function f such that for each rational α we have $f \geq \alpha$ almost everywhere on A_α and $f \leq \alpha$ almost everywhere on B_α. Since $B_0 = \emptyset$ be an arbitrary set in $\mathcal{M}$, and set

$$E_k = E \cap (B_{(k+1)/N} \sim B_{k/N})$$

Then $E = \cup \bigcup_{k=1}^{\infty} E_k$, and this union is disjoint modulo null sets. Hence we have

$$\nu(E) = \nu(E_\infty) + \sum_{k=0}^{\infty} \nu(E_k).$$

Since $E_k \subset B_{(k+1)/N} \cap A_{k/N}$, we have $\frac{k}{N} \leq f \leq \frac{k+1}{N}$ on E_k, and so

$$\mu(E_k)\frac{k}{N} \leq \int_{E_k} f \, d\mu \leq \frac{k+1}{N}\mu(E_k).$$

Now since $\frac{k}{N}\mu(E_k) \leq \nu(E_k) \leq \frac{k+1}{N}\mu(E_k)$, we have

$$\nu(E_k) - \frac{1}{N}\mu(E_k) \leq \int_{E_k} f \, d\mu \leq \nu(E_k) + \frac{1}{N}\mu(E_k).$$

) On E_∞ we have $f = \infty$ almost everywhere. If $\mu(E_\infty) > 0$, we must have $\nu(E_\infty) > 0$, since $(\nu - \alpha\mu)(E_\infty$ is positive for each α. If $\mu(E_\infty) = 0$, we have $\nu(E_\infty) = 0$. Since $\nu << \mu$, for either case we have

$$\nu(E_\infty) = \int_{E_\infty} f \, d\mu.$$

Hence we have

$$\nu(E) - \frac{1}{N}\mu(E) \leq \int_E f \, d\mu \leq \nu(E) + \frac{1}{N}\mu(E).$$

Since $\mu(E)$ is finite and N arbitrary, we must have $\nu(E) = \displaystyle\int_E f \ d\mu$.

The function $f = \left[\frac{d\nu}{d\mu}\right]$ above is called the Radon-Nikodym derivative of ν with respect to μ.

Exercise 5.4. *Suppose ν and μ are σ-finite measures on a measurable space $(\mathcal{X}, \mathcal{A})$, such that $\nu << \mu$, and $\nu << \mu - \nu$. Prove that*

$$\mu\left(\left\{x \in \mathcal{X} : \frac{d\nu}{d\mu} = 1\right\}\right) = 0.$$

Proof: First notice that if $E \subset \mathcal{X}$, such that $(\mu - \nu)E = 0$, then we have $\mu(E) = \nu(E)$. But we have $\nu << \mu - \nu$, hence if $(\mu - \nu)E = 0$, then $\nu(E) = 0 = \mu(E)$. Conversely if $\mu(E) = \nu(E)$ and $\nu << \mu - \nu$, then $\mu(E) - \nu(E) = 0$, and so $\nu = 0$ thus $\mu(E) = 0$. So if $\nu(E) = \mu(E)$, then $\nu(E) = \mu(E) = 0$. Now let $E = \left\{x \in \mathcal{X} : \frac{d\nu}{d\mu} = 1\right\}$ and consider $\nu(E)$. By the Radon-Nikodym theorem we have

$$\nu(E) = \int_E d\nu = \int_E \frac{d\nu}{d\mu} \ d\mu$$

but $\frac{d\nu}{d\mu} = 1$ on E, and so

$$\nu(E) = \int_E \frac{d\nu}{d\mu} \ d\mu = \int_E d\mu = \mu(E)$$

Hence $\mu(E) = \mu\left(\left\{x \in \mathcal{X} : \frac{d\nu}{d\mu} = 1\right\}\right) = 0$ □

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Exercise 5.5. *Let μ and ν be two measure on the same measurable space, such that μ is σ-finite and ν is absolutely continuous with respect to μ.*

(a) If f is a nonngeative measurable function, show that

$$\int f \ d\nu = \int f \left[\frac{d\nu}{d\mu}\right] \ d\mu$$

(b) If f is a measurable function, prove that f is integrable with respect to ν, if and only if $f\left[\frac{d\nu}{d\mu}\right]$ is integralble with respect to μ, and in this case, part (a) still holds.

Proof: For part (a), let E be a measurable set and let $f = \chi_E$. Suppose that $h = \left[\frac{d\nu}{d\mu}\right]$ exists. Then

$$\int f \ d\nu = \int \chi_E \ d\nu = \nu(E) = \int_E h \ d\mu = \int h\chi_E \ d\mu = \int fh \ d\mu.$$

So the equality holds for charactersitc functions. Let $f = \phi$ be a simple function, then by the above we have

$$\int \phi \ d\nu = \int \phi h \ d\mu.$$

Now let f be a nonnegative measurable function. There there exists a monotone sequence of simple functions ϕ_n such that $0 \leq \phi_n \leq f$ and $\phi_n \to f$ almost everywhere. Applying the Monotone Covergence theorem, we have

$$\int f \ d\nu = \lim_{n\to\infty} \int \phi_n \ d\nu = \lim_{n\to\infty} \int \phi_n h \ d\mu = \int fh \ d\mu \ \square.$$

For part (b), f is ν-integrable if and only if $\int f^+ \ d\nu - \int f^- \ d\nu$ is finite. Now by part (a) we have

$$\int f^+ \ d\nu = \int f^+ h \ d\mu \text{ and } \int f^- \ d\nu = \int f^- h \ d\mu.$$

So we have f is ν-integrable if and only if f is μ-integrable □.

Theorem (Lebesgue Decomposition) Let $(\mathcal{X}, \mathcal{M}, \mu)$ be a σ-finite measure space and ν a σ-finite measure defined on $\mathcal{M}$. Then we can find a measure ν_0, singular with respect to μ and a measure ν_1 absolutely continuous with respect to μ, such that $\nu = \nu_0 + \nu_1$. Furthermore, the measures ν_0 and ν_1 are unique.

6. Topological and Product Measure Spaces

Exercise 6.1. *Let L be a normed space. Then every weakly bounded set X is bounded.*

Proof: Let $\phi : L \to L^{**}$, by $\phi(x)(f) = f(x)$, where $x \in L$, $f \in L^*$. Now X^* is a Banach space and $\phi(X)$ is a family of bounded linear functionals on X^*, and for each $f \in L^*$ we have

$$\sup\{\phi(x)(f) : x \in X\} = \sup\{f(x) : x \in X\} < \infty$$

Then from the uniform boundness principle we have

$$\sup\{\|x\| : x \in X\} = \sup\{\|\phi(x)\| : x \in X\} < \infty$$

Therefore every weakly bounded nonempty set of a normed space is bounded $\square$.

Exercise 6.2. *Suppose that A is a subset in $\mathbb{R}^2$. Define for each $x \in \mathbb{R}^2$, $p(x) = \inf\{|y - x| : y \in A\}$. Show that $B_r = \{x : p(x) \leq r\}$ is a closed set for each nonnegative r. Is the measure of B_0 equal to the outer measure of A?*

Proof: Let $z \in (B_r)$, and let $\epsilon > 0$. Then there is $x \in B_r$ such that $|x - z| < \epsilon$. So we have

$$\begin{aligned} p(z) &= \inf\{|z - y| : y \in A\} \\ &\leq \inf\{|z - x| + |x - y| : y \in A\} \\ &\leq \epsilon + \inf\{|x - y| : y \in A\} \\ &\leq \epsilon + r. \end{aligned}$$

This is for all $\epsilon > 0$, therefore $p(z) \leq r$ which implies $z \in B_r$ thus B_r is closed. Now $B_0 = A \cup \partial A$. First by definition of $p(x)$ we have for any $x \in A$, $p(x) = 0$. Hence $x \in B_0$, Now suppose that $x \in \partial A$, then for any $\epsilon > 0$, there is a $y \in A$ such that $|x - y| < \epsilon$. Therefore we have

$$p(z) = \inf\{|z - y| : y \in A\} = 0 \quad \Rightarrow \quad x \in B_0,$$

and so $\overline{A} \subset B_0$. Now suppose $x \in B_0$. Then $\inf\{|x - y| : y \in A\} = 0$, so for every $\epsilon > 0$ there is a $y \in A$ such that $|x - y| < \epsilon$. So $x \in \overline{A}$, therefore we have $B_0 = \overline{A} = A^\circ \cup \partial A$. Now

$$\mu^*(A) \leq \mu^*(B_0) = \mu^*(A^\circ \cup \partial A) = \mu^*(A^\circ) + \mu^*(\partial A) = \mu^*(A) + \mu^*(\partial A)$$

Since A° is open and A is measurable. Therefore $\mu^*(A) = \mu(B_0)$, if and only if $\mu^*(\partial A) = 0$ $\square$.

Exercise 6.3. *Prove that an algebraic basis in any infinite-dimensional Banach space must be uncountable.*

Proof: Let V be an infinite-dimensional Banach space over $\mathbb{F}$, and suppose $\{x_n\}_{n\in\mathbb{N}}$ is a countable Hamel basis. Then $v \in V$ if any only if there exists $a_i \in \mathbb{F}$ such that

$$v = \sum_{i=1}^{k} a_i x_i$$

for some $x_i \in \{x_n\}$. Now let $\langle x_i \rangle$ denote the span of x_i, then we have

$$V = \bigcup_{k\in\mathbb{N}} \langle \{x_n\}_{n=1}^{k} \rangle$$

But this implies that V is a countable union of proper subspace of finite dimension. Which implies that V would be of first category, since every finite dimensional proper subspace of a normed space is nowhere dense. Which is a contradiction to the Baire Category Theorem. Therefore any basis for an infinite-dimensional Banach space must be uncountable □

Theorem (Hahn-Banach) Let p be a real-valued function defined on the vector space X satisfying $p(x+y) \leq p(x)+p(y)$ and $p(\alpha x) = \alpha p(x)$ for each $\alpha \geq 0$. Suppose that f is a linear functional defined on a subspace S and that $f(s) \leq p(s)$ for all $s \in S$. Then there is a linear function F defined on X such that $F(x) \leq p(x)$ for all x, and $F(s) = f(s)$ for all $s \in S$.

Exercise 6.4. *Let ν be a finite Borel measure on the real line, and set $F(x) = \nu\{(-\infty, x]\}$. Prove that ν is absolutely continuous with respect to the Lebesgue measure μ if and only if F is an absolutely continuous function. In this case show that its Radon-Nikodym derivative is the derivative of F, almost everywhere.*

Proof: ($\Rightarrow$) First suppose that $\nu << \mu$. Let $\mu(E)$, then there exists an open set $\mathcal{O}$, such that $E \subset \mathcal{O}$ and $\mu(\mathcal{O}) < \epsilon$. Now $\mathcal{O}$ being open, there are disjoint intervals (x_k, y_k), such that

$$\mathcal{O} = \bigcup_{k=1} (x_k, y_k), \quad \Rightarrow \quad \mu(\mathcal{O}) = \sum_{k=1} (y_k - x_k) < \epsilon$$

Since $\nu << \mu$, there exists a delta such that if $\mu(\mathcal{O}) < \epsilon$, then $\nu(\mathcal{O}) < \delta$. So we have

$$\sum_{k=1} |F(y_k) - F(x_k)| = \sum_{k=1} \nu(x_k, y_k) < \delta$$

So $F(x)$ is an absolutely continuous function.

($\Leftarrow$) Suppose that $F(x)$ is absolutely continuous. Then we have

$$\forall \epsilon > 0 \ \exists \delta > 0 \ s.t. \sum_{k=1} |y_k - x_k| < \delta \quad \Rightarrow \quad \sum_{k=1} |F(y_k) - F(x_k)| < \epsilon$$

Choose such disjoint intervals (x_k, y_k) and call the union of these intervals $\mathcal{O}$, then we have $\mu(\mathcal{O}) < \epsilon$. Now by definition of $F(x)$, we have

$$\nu(\mathcal{O}) = \sum_{k=1} |F(y_k) - F(x_k)| < \delta$$

and so $\nu << \mu$.

To see that F is Radon-Nikodym derivative, we know that since F is absolutely continuous we have that $F'(t)$ exists almost everywhere so

$$\nu(-\infty, x] = F(x) = \int_{-\infty}^{x} F'(t) \ d\mu(t)$$

We also have that

$$\nu(-\infty, x] = \int_{-\infty}^{x} d\nu = \int_{-\infty}^{x} \left[\frac{d\nu}{d\mu}\right] \ d\mu$$

which implies that

$$\nu(-\infty, x] = F(x) = \int_{-\infty}^{x} F'(t) \ d\mu(t) = \int_{-\infty}^{x} \left[\frac{d\nu}{d\mu}\right] \ d\mu$$

Hence by the Radon-Nikodym theorem we know that $F' = \left[\frac{d\nu}{d\mu}\right]$ almost everywhere.

Theorem (Tonneli's) Suppose $(\mathcal{X} \times \mathcal{Y}, \sigma(\mathcal{A} \times \mathcal{B}), \mu \times \nu)$ is the product space of two σ-finite measure spaces, and $f : \mathcal{X} \times \mathcal{Y} \to [0, \infty]$ and $f(x, y)$ be a nonnegative measurable function in the product measure, then

$$F_1(x) = \int_{\mathcal{Y}} f(x, \cdot) \ d\nu \text{ is } \mathcal{A} \text{ measurable of } x \in \mathcal{X}$$

$$F_2(y) = \int_{\mathcal{X}} f(\cdot, y)\ d\mu \text{ is } \mathcal{B} \text{ measurable of } x \in \mathcal{Y}$$

and

$$\int_{\mathcal{X}\times\mathcal{Y}} f\ d(\mu) = \int_{\mathcal{X}} F_1\ d\mu = \int_{\mathcal{Y}} F_2\ d\nu$$

i.e., the iterated integrals is equal to the the integral in the product space

$$\int_{\mathcal{X}} \left(\int_{\mathcal{Y}} f(x, y)\ d\nu \right)\ d\mu = \int_{\mathcal{Y}} \left(\int_{\mathcal{X}} f(\cdot, y)\ d\mu \right)\ d\nu = \int_{\mathcal{X}\times\mathcal{Y}} f(x, y)\ d(\mu \times \nu).$$

Theorem (Fubini's) Suppose$(\mathcal{X} \times \mathcal{Y}, \sigma(\mathcal{A} \times \mathcal{B}), \mu \times \nu)$ is the product space of two σ-finite measure spaces, and $f : \mathcal{X} \times \mathcal{Y} \to [0, \infty]$ and $f(x, y)$ be an integrable function in the product space, then

$$\int_{\mathcal{X}} \left(\int_{\mathcal{Y}} f(x, y)\ d\nu \right)\ d\mu = \int_{\mathcal{Y}} \left(\int_{\mathcal{X}} f(\cdot, y)\ d\mu \right)\ d\nu = \int_{\mathcal{X}\times\mathcal{Y}} f(x, y)\ d(\mu \times \nu).$$

Theorem (Fubini-Tonelli) Suppose $(\mathcal{X} \times \mathcal{Y}, \sigma(\mathcal{A} \times \mathcal{B}), \mu \times \nu)$ is the product space of two σ-finite measure spaces. Let f be an extended real-valued $\sigma(\mathcal{A} \times \mathcal{B})$ measurable function on $\mathcal{X} \times \mathcal{Y}$. If either

$$\int_{\mathcal{X}} \left(\int_{\mathcal{Y}} |f|\ d\nu \right)\ d\mu < \text{ or } \int_{\mathcal{Y}} \left(\int_{\mathcal{X}} |f|\ d\mu \right)\ d\nu < \infty$$

then f is $\mu \times \nu$-integrable, furthermore the iterated integrals are equal to the product integral.

Exercise 6.5. *Let f be a real valued measurable function on the finite measure space $(\mathcal{X}, \mathcal{M}, \mu)$. Prove that the function $F(x, y) = f(x) - 5f(y) + 4$ is measurable in the product measure space $(\mathcal{X} \times \mathcal{X}, \sigma(\mathcal{M} \times \mathcal{M}), \mu \times \mu)$, and that F is integrable if and only if f is integrable.*

Proof: First since $f(x)$ is measurable, we have both sections $F(x_0, y)$, and $F(x, y_0)$ as measurable for each fixed x_0, y_0. Now for $x \in \mathcal{X}$ we have $F(x, x) = -4(f(x) - 1)$, which is measurable. Having $F(x, y)$ being measurable on each section, and the diagonal is enough for $F(x, y)$ to be measurable in the product space.

Now let f be integrable, hence $|f|$ is integrable, so let $M = \int_{\mathcal{X}} |f(x)|\ dx$, now we have

$$\begin{aligned}
\int_{\mathcal{X}} \int_{\mathcal{X}} |f(x) - 5f(y) + 4|\ dxdy &\leq \int_{\mathcal{X}} M + 5|f(y)|\mu(\mathcal{X}) + 4\mu(\mathcal{X})\ dy \\
&= M\mu(\mathcal{X}) + 5M\mu(\mathcal{X}) + 4\mu(\mathcal{X})^2 \\
&= 4\mu(\mathcal{X})(M + \mu(\mathcal{X})) < \infty \\
\text{by the same computation} &\Rightarrow \int_{\mathcal{X}} \int_{\mathcal{X}} |f(x) - 5f(y) + 4|\ dydx < \infty
\end{aligned}$$

Then by Fubini-Tonelli theorem $F(x, y)$ is integrable. Now suppose that $F(x, y)$ is integrable, then by Fubini's theorem we have that the iterations are equal, but this is true if and only if $f(x)$ is integrable $\square$.

Theorem (Stone-Weierstrass) Let X be a compacct space and A an algebra of continuous real-valued functions on X that separates the points of X and contains the constant functions. Then given any continuous real-valued function f on X and any $\epsilon > 0$ there is a function $g \in A$ such that for all $x \in X$ we have $|g(x) - f(x)| < \epsilon$. In other words, A is a dense subset of $C(X)$.

Theorem (Closed Graph) Let A be a linear transformation on a Banach space X to a Banach space Y. Suppose that A has the property that, whenever x_n is a sequence in X that converges to some point x and Ax_n converges in Y to a point y, then $y = Ax$. Then A is continuous.

Exercise 6.6. *Let* $(\mathcal{X}, \mathcal{A}, \mu)$ *and* $(\mathcal{Y}, \mathcal{B}, \nu)$ *be the measure spaces given by*

- $\mathcal{X} = \mathcal{Y} = [0, 1]$
- $\mathcal{A} = \mathcal{B} = \sigma([0, 1])$
- μ *be the Lebesgue measure on* $\mathbb{R}$, *and* ν *the counting measure.*

Consider the product measure space $(\mathcal{X} \times \mathcal{Y}, \sigma(\mathcal{A} \times \mathcal{B}))$, *and its subset* $E = \{(x, y) \in \mathcal{X} \times \mathcal{Y} : x = y\}$

(1) Show that $E \subset \sigma(\mathcal{A} \times \mathcal{B})$
(2) Show that $\int_{\mathcal{X}} \int_{\mathcal{Y}} \chi_E \; d\nu \; d\mu \neq \int_{\mathcal{Y}} \int_{\mathcal{X}} \chi_E \; d\mu \; d\nu$.
(3) Explain why Tonelli's theorem is not applicable.

Proof: For (1) First notice that the following sets

$$A_k = \left[\frac{k-1}{n}, \frac{k}{n}\right] \times \left[\frac{k-1}{n}, \frac{k}{n}\right]$$

are measurable. Now let define E_n as follows

$$E_n = \bigcup_{k=1}^{n} A_k,$$

Then the sets E_n are measurable as they are countable union of measurable sets. Then the set E is given by

$$E = \bigcap_{n=1}^{\infty} E_n = \{(x, y) \in \mathcal{X} \times \mathcal{Y} : x = y\}$$

is measurable since is a countable intersection of measurable sets.

For (2) by a direct computation we have

$$\int_{\mathcal{X}} \int_{\mathcal{Y}} \chi_E \; d\nu \; d\mu = \int_0^1 \nu(E) d\mu = \int_0^1 \; d\mu = 1$$

and

$$\int_{\mathcal{Y}} \int_{\mathcal{X}} \chi_E \; d\mu \; d\nu = \int_0^1 \mu(E) \; d\nu = \int_0^1 0 \; d\nu = 0$$

Tonelli's theorem is not applicable because the measure space $(\mathcal{Y}, \mathcal{B}, \nu)$ is not σ-finite $\square$.