Midterm solutions

Thursday, May 19, 2011
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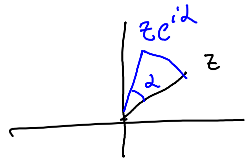
1) Given $z \in \mathbb{C}$ interpret $ze^{i\alpha}$ where $\alpha \in \mathbb{R}$

sol. rotation by α radians



2) Prove $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

Pf: Using Eulers identity $(e^{i\theta})^n = (\cos\theta + i\sin\theta)^n$

$$\begin{array}{ccc} (e^{i\theta})^n & = & (\cos\theta + i\sin\theta)^n \\ \| & & \| \\ e^{in\theta} & = & \cos n\theta + i\sin n\theta \end{array}$$

3) a) what is the definition of a simply connected set and Domain

sol: A domain is an open connected set.
A simply connected set D is a domain such that all simple closed curves contained in D can be contracted to a point.

OR A simply connected set D is a domain such that for every closed curve γ the interior of γ is contained in D

OR A simply connected set D is a domain s.t. D is path connected a if γ, ξ are two paths sharing endpoints $\exists$ a continuous deformation from $\gamma \leftrightarrow \xi$

b) what is the Def: of analytic function

sol: $f(z)$ is analytic in a domain D if $\forall z \in D$ $\exists$ a power series representation for $f(z)$ in $D(z, \varepsilon) \subset D$ for some $\varepsilon > 0$

OR $f(z)$ satisfies the C-R eqns $f = u + iv$

$$\text{analytic} \iff \begin{array}{l} u_x = v_y \\ u_y = -v_x \end{array} \iff \frac{\partial f}{\partial \bar{z}} = 0$$

c) What is the Def of a conformal map.

sol: $f: D \to \mathbb{C}$ is conformal if $f'(z) \neq 0$ in D

and it preserves oriented angles between curves contained in D

d) what is a generalized circle

sol: Associating $\mathbb{C} \cup \{\infty\}$ with the Riemann sphere a generalized circle is a circle on the Riemann sphere i.e a circle or line in the complex plane

If $\alpha, \beta, p \in \mathbb{C}$, then $\alpha z\bar{z} - p\bar{z} - \bar{p}z + \beta = 0$ is the equation for a generalized circle

4) Evaluate $\int_0^{4+2i} \bar{z}\, dz$ along:

a) $z = t^2 + it$ $\quad t \in [0, 2]$

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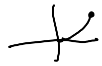
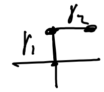


sol: $\displaystyle\int_0^2 (t^2 - it)(2t+i)\,dt = \int_0^2 (2t^3+t) - it^2\,dt$

$\displaystyle = \int_0^2 \frac{1}{2}t^4 + \frac{t^2}{2} - \frac{it^3}{3}\Big|_0^1 = 8 + 2 - \frac{8}{3}i = 10 - \frac{8}{3}i$

b) $z = 0$ to $z = 2i$, $z = 2i$ to $4 + 2i$



Sol: $\gamma_1 = it \quad t \in [0,2]$
$\gamma_2 = 2i + t \quad t \in [0,4]$ $\Rightarrow \displaystyle\int_0^2 -it \cdot i\,dt + \int_0^4 (t - 2i)\,dt$

$\displaystyle = \int_0^2 t\,dt + \int_0^4 t - 2i\,dt = \frac{t^2}{2}\Big|_0^2 + \frac{t^2}{2} - 2it\Big|_0^4$

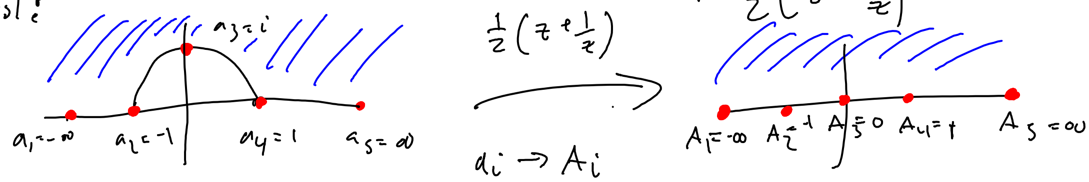
$= 2 + 8 - 8i = 10 - 8i$

c) is $\bar{z}$ analytic?

Sol: No does not satisfy C-R, i.e. $\frac{\partial}{\partial \bar{z}} \bar{z} = 1 \neq 0$

5) a) Find the image of the upper half plane with the semi circle removed under the map $\frac{1}{2}\left(z + \frac{1}{z}\right)$

sol:



$\displaystyle X + iY = \frac{1}{2}\left(\frac{z^2+1}{z}\right) = \frac{1}{2|z|}\left((z^2+1)\bar{z}\right) = \frac{1}{2|z|}\left(z|z|^2 + \bar{z}\right)$

$\displaystyle = \frac{1}{2(x^2+y^2)}\left((x+iy)(x^2+y^2) + x - iy\right)$

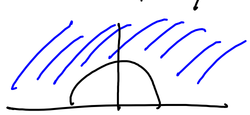
$\displaystyle = \frac{1}{2(x^2+y^2)}\left(x(x^2+y^2) + x\right) + \frac{i}{2(x^2+y^2)}\left(y(x^2+y^2) - y\right)$

$\displaystyle = \frac{x}{2(x^2+y^2)}\left(x^2+y^2+1\right) + \frac{iy}{x^2+y^2}\left(x^2+y^2-1\right)$

b) Find a solution to

$$\begin{cases} \Delta u = 0 \quad \text{in } \Omega \\ \text{B.C.} \begin{cases} u = \frac{1}{2}\left(x + \frac{1}{x}\right) & |x| \geq 1 \quad y = 0 \\ u = \cos\theta & x^2+y^2 = 1 \quad y > 0 \end{cases} \end{cases}$$

where $\Omega =$



$= \{z : \operatorname{Im}(z) > 0\} \setminus \{z : |z| < 1\}$

sol: let $\zeta = \frac{1}{2}\left(z + \frac{1}{z}\right)$, when $z = x \Rightarrow \zeta = \frac{1}{2}\left(x + \frac{1}{x}\right)$

$= X + iY$ where $z = e^{i\theta} \Rightarrow \xi = \frac{1}{2}\left(e^{i\theta} - e^{-i\theta}\right) = \cos\theta$

then $\begin{cases} \Delta u = 0 \\ \text{B.C.} \end{cases} \Rightarrow \begin{cases} \Delta_\xi U = 0 \\ \text{B.C.} \{ U = X \end{cases}$

$\Rightarrow U(X, Y) = X + cY \quad c \in \mathbb{C}$

and $U(X(x,y), Y(x,y)) = u(x, y) = \frac{x}{2(x^2+y^2)}(x^2+y^2+1) + \frac{y}{2(x^2+y^2)}(x^2+y^2-1)$