From Hypergroups to Anyonic Twines

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Outline

Feudal hypergroups

Feudal fusion categories

Fermionic Moore-Read twines

Fractional quantum Hall wavefunctions

Outline

Definition

An *(absolutely regular) hypergroup* is a set L with an operation $L \times L \to 2^L$ satisfying

$$\begin{aligned} \forall x, y, z: &\quad \bigcup_{u \in xy} uz = \bigcup_{v \in yz} xv, \\ \exists 1 \forall x: &\quad 1x = x1 = x, \\ \forall x \exists \bar{x} \forall y: &\quad 1 \in xy \iff 1 \in yx \iff y = \bar{x}. \end{aligned}$$

Example

- Groups are hypergroups.
- For a group A, the *Tambara-Yamagami* hypergroup $A \sqcup \{m\}$ has the following multiplication: for $a, b \in A$,

$$a * b = ab, \qquad a * m = m * a = m, \qquad m * m = A.$$

  For $A = \mathbb{Z}_2$, this is the *Ising* hypergroup $\{1, \psi, \sigma\}$.
- The *fermionic Moore-Read* hypergroup $\{1, \alpha, \psi, \alpha', \sigma, \sigma'\}$ has the following commutative multiplication:

$$\{1, \alpha, \psi, \alpha'\} \cong \{0, 1, 2, 3\} = \mathbb{Z}_4$$

$$\begin{aligned} \psi\sigma &= \sigma & \psi\sigma' &= \sigma' & \sigma\sigma' &= \{1, \psi\} \\ \alpha\sigma &= \sigma' & \alpha\sigma' &= \sigma & \sigma\sigma &= \{\alpha, \alpha'\} \\ \alpha'\sigma &= \sigma' & \alpha'\sigma' &= \sigma & \sigma'\sigma' &= \{\alpha, \alpha'\} \end{aligned}$$

Definition (Gelaki and Nikshych)

The *adjoint subhypergroup* L_{ad} of a hypergroup L is the subhypergroup generated by all $x\bar{x}$ for $x \in L$.
L is *nilpotent* if

$$L \supseteq L_{\mathrm{ad}} \supseteq (L_{\mathrm{ad}})_{\mathrm{ad}} \supseteq \cdots$$

terminates at $\{1\}$.

Example

- Any group G is a nilpotent hypergroup,
  adjoint subhypergroup = $\{1\}$.
- Tambara-Yamagami $A \sqcup \{m\}$ is nilpotent,
  adjoint subhypergroup = A.
- Fermionic Moore-Read $L = \{1, \alpha, \psi, \alpha', \sigma, \sigma'\}$ is nilpotent,
  adjoint subhypergroup = $\{1, \psi\}$.

Definition

A hypergroup element $x \in L$ is a *simple current* if $x\bar{x} = 1$.
The simple current group acts on L;
orbits = *simple current index*.

Example

- Groups = hypergroups of simple current index 1.
- Tambara-Yamagami $A \sqcup \{m\}$: if $A \neq \{1\}$,
  simple currents = A,
  simple current index = 2.
- Fermionic Moore-Read $\{1, \alpha, \psi, \alpha', \sigma, \sigma'\}$:
  simple currents = $\{1, \alpha, \psi, \alpha'\}$,
  simple current index = 2.

Definition

A *feudal* hypergroup $L = S \sqcup M = \text{serfs} \sqcup \text{lords}$:

$$\text{serf} \cdot \text{lord} = \text{lord}, \qquad \text{serf} \cdot \text{serf} = \text{serf},$$
$$\text{lord} \cdot \text{serf} = \text{lord}, \qquad \text{lord} \cdot \text{lord} \subseteq \text{serfs}.$$

Proposition

Feudal $\iff$ *either nilpotent with simple current index* 2*,*
or a $\mathbb{Z}_2$*-graded group.*

Theorem

Feudal hypergroups $\longleftrightarrow$
group homomorphisms with cokernel $\mathbb{Z}_2$*.*

Construction: Given any group homomorphism $S \xrightarrow{u} G$ with $\mathbb{Z}_2 \cong G/\operatorname{im}(u)$, let $M = G \setminus \operatorname{im}(u)$.
Then $L = S \sqcup M$ is a feudal hypergroup:
for $a, b \in S$ and $m, l \in M$,

$$a * m = u(a)m, \qquad a * b = ab,$$
$$m * a = mu(a), \qquad m * l = u^{-1}(ml).$$

Note: adjoint subhypergroup = $\ker(u)$.

Example

- G a group, S an index 2 subgroup: $S \hookrightarrow G$.
- Tambara-Yamagami: $A \xrightarrow{0} \mathbb{Z}_2$.
- Fermionic Moore-Read: $\mathbb{Z}_4 \xrightarrow{2} \mathbb{Z}_4$.

Definition

A *fusion rule* is a set L with $N_r^{xy} \in \mathbb{N}$ for each $r, x, y \in L$, satisfying

$$\forall x, y, z : \quad \sum_{u \in xy} N_u^{xy} N_r^{uz} = \sum_{v \in yz} N_r^{xv} N_v^{yz} < \infty$$

$$\exists 1 \forall x, y : \quad N_y^{1x} = N_y^{x1} = \delta_{x,y}$$

$$\forall x \exists \bar{x} \forall y : \quad N_1^{xy} = N_1^{yx} = \delta_{y,\bar{x}}$$

Underlying hypergroup: $\quad xy = \{ r \in L \mid N_r^{xy} > 0 \}$

Proposition

Feudal fusion rules $\longleftrightarrow$
feudal hypergroups with finite adjoint subhypergroup $\longleftrightarrow$
group homomorphisms with cokernel $\mathbb{Z}_2$ *and kernel finite.*

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Definition

A *fusion category* over a field $\mathbb{F}$ is a rigid strongly semisimple monoidal $\mathbb{F}$-linear category with simple monoidal unit.

Underlying fusion rule: L a representative set of simples, for $x, y, r \in L$,

$$N_r^{xy} = \dim_{\mathbb{F}} \mathrm{mor}(r, x \square y)$$

where $\square$ is the monoidal bifunctor.

Theorem (Ocneanu rigidity)

On a given finite fusion rule (up to isomorphism), only finitely many fusion categories (up to equivalence).

Example (known)

- On a group G, fusion categories $\longleftrightarrow H^3(G, \mathbb{F}^\times)/\text{aut}\, G$.
- On a Tambara-Yamagami fusion rule $A \sqcup \{m\}$, fusion categories $\longleftrightarrow$ pairs (χ, τ) such that $\chi\colon A \times A \to \mathbb{F}^\times$ is a nondegenerate symmetric bicharacter

$$\chi(b, a) = \chi(a, b)$$
$$\chi(ab, c) = \chi(a, c)\chi(b, c)$$

  and $\tau \in \mathbb{F}$ satisfies $|A|\tau^2 = 1$, where χ is up to aut A.
- On the fermionic Moore-Read fusion rule, four fusion categories (Bonderson).

Tool for characterizing feudal fusion categories:
For $L = S \sqcup M$ a feudal fusion rule, $\mathbb{F}$ a field,
let $B = \mathbb{F}^M$ be the ring of functions $M \to \mathbb{F}$.
Then B is an "involutory ambidextrous S-algebra":
for $\mu \in B$ and $a, b \in S$,

$$
\begin{aligned}
{}^a\mu^b \in B : \qquad & {}^a\mu^b(m) = \mu(\bar{a} m \bar{b}) \\
\bar{\mu} \in B : \qquad & \bar{\mu}(m) = \mu(\bar{m})
\end{aligned}
$$

Theorem

Let $L = S \sqcup M$ be a feudal fusion rule, and $B = \mathbb{F}^M$.
On L, fusion categories $\longleftrightarrow$ triples (χ, υ, τ) such that $\chi, \upsilon \colon S \times S \to B^\times$ and $\tau \in B^\times$ satisfy

$$\bar{\chi}(b,a) = {}^{\bar{a}}\chi(a,b)^{\bar{b}} \cdot \frac{{}^{\bar{a}}\tau^{\bar{b}} \cdot \tau}{{}^{\bar{a}}\tau\tau^{\bar{b}}}$$

$$\frac{\upsilon(a,b)}{\upsilon(a,b)^c}\chi(ab,c) = \chi(a,c) \cdot {}^{a}\chi(b,c)$$

$$\chi|_{A\times A} \text{ is nondegenerate}$$

and $|A|\tau\bar{\tau} \equiv 1$, where $A \subseteq S$ is the adjoint subhypergroup,
but (χ, υ, τ) is up to equivalence...

...where $(\chi, \upsilon, \tau) \sim (\tilde{\chi}, \tilde{\upsilon}, \tilde{\tau})$ if there exist

$$\theta \colon S \times S \to \mathbb{F}^\times \qquad \phi \colon S \to B^\times \qquad \varsigma \in B^\times$$

such that for $a, b \in S$,

$$\frac{\tilde{\chi}(a,b)}{\chi(a,b)} = \frac{\phi(a) \cdot {}^a\overline{\phi(b)}^b}{\phi(a)^b \cdot \overline{\phi(b)}^b} \cdot \frac{{}^a\varsigma^b \cdot \varsigma}{{}^a\varsigma\varsigma^b}$$

$$\frac{\tilde{\upsilon}(a,b)}{\upsilon(a,b)} = \frac{\phi(a) \cdot {}^a\phi(b)}{\phi(ab)\theta(a,b)}$$

$$\frac{\tilde{\tau}}{\tau} = \frac{\bar{\varsigma}}{\varsigma}$$

This theorem generalizes the original classifications of Tambara-Yamagami and fermionic Moore-Read fusion categories.

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Braided fusion categories model quasiparticle motion in fractional quantum Hall liquids.
Problem: fermionic Moore-Read fusion categories aren't braided (Bonderson). Possible braiding substitute:

Definition

A *twine* or *pure braiding* on a monoidal category is a natural automorphism of the monoidal bifunctor, satisfying coherence diagrams.

Theorem (Bruguières)

Twines yield object-colored pure braid representations.

A twine on a monoidal category makes the identity functor monoidal.

Definition

Two twines on the same monoidal category are *equivalent* if they are isomorphic as monoidal functors.

Observation

On a fusion category on a group G, twines $\longleftrightarrow H^2(G, \mathbb{F}^\times)$.

Theorem

On a fermionic Moore-Read fusion category,
all twines $\sim$ *the trivial twine.*

Definition

Two twines on the same fusion category are *homothetically equivalent* if their pure braid actions on splitting spaces coincide up to homotheties.

Theorem

Up to homothetic equivalence, there is a unique nontrivial twine on any fermionic Moore-Read fusion category, acting on triples over $\{1, \alpha, \psi, \alpha', \sigma, \sigma'\}$ *as*

$$
\begin{array}{llll}
\xi_{\sigma}^{1,\sigma} = 1 & \xi_{\sigma'}^{1,\sigma'} = 1 & \xi_{1}^{\sigma,\sigma'} = -i & \xi_{1}^{\sigma',\sigma} = -i \\
\xi_{\sigma}^{\psi,\sigma} = -1 & \xi_{\sigma'}^{\psi,\sigma'} = -1 & \xi_{\psi}^{\sigma,\sigma'} = i & \xi_{\psi}^{\sigma',\sigma} = i \\
\xi_{\sigma}^{\alpha,\sigma'} = 1 & \xi_{\sigma'}^{\alpha,\sigma} = -1 & \xi_{\alpha}^{\sigma,\sigma} = i & \xi_{\alpha}^{\sigma',\sigma'} = -i \\
\xi_{\sigma}^{\alpha',\sigma'} = -1 & \xi_{\sigma'}^{\alpha',\sigma} = 1 & \xi_{\alpha'}^{\sigma,\sigma} = -i & \xi_{\alpha'}^{\sigma',\sigma'} = i
\end{array}
$$

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Entwined fusion categories may describe quasiparticles in fractional quantum Hall liquids. The underlying electron wavefunction is a Gaussian, which we ignore, times a translation invariant antisymmetric polynomial over $\mathbb{C}$.

$$\text{(antisymmetric poly)} = \prod_{i<j} (z_i - z_j) \cdot \text{(symmetric poly)}$$

Conjecture (Haldane)

If a homogeneous symmetric polynomial is translation invariant, its squeezing poset has a maximum.

Theorem

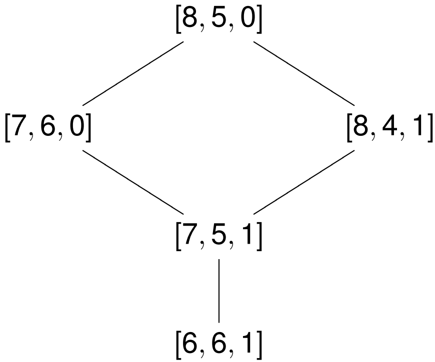
The conjecture fails at arity 4 *and degree* 14.

Squeezing poset of a symmetric polynomial:

$$\text{symmetrized monomial} \quad \longleftrightarrow \quad \text{multiset over } \mathbb{N}$$

$$\sum_{\sigma \in \mathrm{Sym}(n)} z_{\sigma(1)}^{l_1} \cdots z_{\sigma(n)}^{l_n} \quad \longleftrightarrow \quad [l_1, \ldots, l_n]$$

$$2(z_1^6 z_2^6 z_3 + z_2^6 z_3^6 z_1 + z_3^6 z_1^6 z_2) \quad \longleftrightarrow \quad [6, 6, 1]$$



Theorem

Ring of translation invariant symmetric n-variate polynomials
$\cong$ *full polynomial ring in* $n-1$ *variables.*

$$(z_1 - z_{\mathrm{avg}})^k + \cdots + (z_n - z_{\mathrm{avg}})^k, \qquad 2 \le k \le n$$

The Pfaffian wavefunction

$$\mathrm{Pf}\left(\frac{1}{z_i - z_j}\right)\prod_{i<j}(z_i - z_j)^2$$

is believed to produce the fermionic Moore-Read fusion rule for $\nu = 5/2$ fractional quantum Hall; it somehow contains the categorical structures we've considered.

Thanks!

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