From Hypergroups to Anyonic Twines

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Outline

Feudal hypergroups

Feudal fusion categories

Fermionic Moore-Read twines

Fractional quantum Hall wavefunctions
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Definition

An *(absolutely regular) hypergroup* is a set \( L \) with an operation \( L \times L \to 2^L \) satisfying

\[
\forall x, y, z : \bigcup_{u \in xy} uz = \bigcup_{v \in yz} xv,
\]

\[
\exists 1 \forall x : 1x = x1 = x,
\]

\[
\forall x \exists \bar{x} \forall y : 1 \in xy \iff 1 \in yx \iff y = \bar{x}.
\]
Example

- Groups are hypergroups.
- For a group $A$, the *Tambara-Yamagami* hypergroup $A \sqcup \{m\}$ has the following multiplication: for $a, b \in A$,
  \[ a \ast b = ab, \quad a \ast m = m \ast a = m, \quad m \ast m = A. \]

For $A = \mathbb{Z}_2$, this is the *Ising* hypergroup $\{1, \psi, \sigma\}$.
- The *fermionic Moore-Read* hypergroup $\{1, \alpha, \psi, \alpha', \sigma, \sigma'\}$ has the following commutative multiplication:
  \[ \{1, \alpha, \psi, \alpha'\} \cong \{0, 1, 2, 3\} = \mathbb{Z}_4 \]

\[
\begin{align*}
\psi \sigma &= \sigma & \psi \sigma' &= \sigma' & \sigma \sigma' &= \{1, \psi\} \\
\alpha \sigma &= \sigma' & \alpha \sigma' &= \sigma & \sigma \sigma &= \{\alpha, \alpha'\} \\
\alpha' \sigma &= \sigma' & \alpha' \sigma' &= \sigma & \sigma' \sigma' &= \{\alpha, \alpha'\}
\end{align*}
\]
Definition (Gelaki and Nikshych)

The \textit{adjoint subhypergroup} \(L_{\text{ad}}\) of a hypergroup \(L\) is the subhypergroup generated by all \(x\bar{x}\) for \(x \in L\).

\(L\) is \textit{nilpotent} if

\[
L \supseteq L_{\text{ad}} \supseteq (L_{\text{ad}})_{\text{ad}} \supseteq \cdots
\]

terminates at \(\{1\}\).

Example

- Any group \(G\) is a nilpotent hypergroup, adjoint subhypergroup = \(\{1\}\).
- Tambara-Yamagami \(A \sqcup \{m\}\) is nilpotent, adjoint subhypergroup = \(A\).
- Fermionic Moore-Read \(L = \{1, \alpha, \psi, \alpha', \sigma, \sigma'\}\) is nilpotent, adjoint subhypergroup = \(\{1, \psi\}\).
**Definition**
A hypergroup element $x \in L$ is a *simple current* if $x \bar{x} = 1$. The simple current group acts on $L$; # orbits = *simple current index*.

**Example**
- Groups = hypergroups of simple current index 1.
- Tambara-Yamagami $A \sqcup \{m\}$: if $A \neq \{1\}$, simple currents = $A$, simple current index = 2.
- Fermionic Moore-Read $\{1, \alpha, \psi, \alpha', \sigma, \sigma'\}$: simple currents = $\{1, \alpha, \psi, \alpha'\}$, simple current index = 2.
Definition
A feudal hypergroup $L = S \sqcup M = \text{serfs} \sqcup \text{lords}$:

$$
\text{serf} \cdot \text{lord} = \text{lord}, \quad \text{serf} \cdot \text{serf} = \text{serf}, \\
\text{lord} \cdot \text{serf} = \text{lord}, \quad \text{lord} \cdot \text{lord} \subseteq \text{serfs}.
$$

Proposition
$Feudal \iff \text{either nilpotent with simple current index 2, or a } \mathbb{Z}_2\text{-graded group}.$

Theorem
$Feudal \text{ hypergroups} \iff \text{group homomorphisms with cokernel } \mathbb{Z}_2.$
Construction: Given any group homomorphism $S \xrightarrow{u} G$ with $\mathbb{Z}_2 \cong G/\text{im}(u)$, let $M = G \setminus \text{im}(u)$. Then $L = S \sqcup M$ is a feudal hypergroup: for $a, b \in S$ and $m, l \in M$,

$$a \ast m = u(a)m, \quad a \ast b = ab,$$

$$m \ast a = mu(a), \quad m \ast l = u^{-1}(ml).$$

Note: adjoint subhypergroup $=$ ker$(u)$.

Example

- $G$ a group, $S$ an index 2 subgroup: $S \hookrightarrow G$.
- Tambara-Yamagami: $A \xrightarrow{0} \mathbb{Z}_2$.
- Fermionic Moore-Read: $\mathbb{Z}_4 \xrightarrow{2} \mathbb{Z}_4$. 
Definition
A fusion rule is a set $L$ with $N^{xy}_r \in \mathbb{N}$ for each $r, x, y \in L$, satisfying

$$\forall x, y, z:\quad \sum_{u \in xy} N^{xy}_u N^{uz}_r = \sum_{v \in yz} N^{xv}_r N^{yz}_v < \infty$$

$$\exists 1 \forall x, y:\quad N^1_x = N^1_y = \delta_{x,y}$$

$$\forall x \exists \bar{x} \forall y:\quad N^{xy}_1 = N^{yx}_1 = \delta_{y,\bar{x}}$$

Underlying hypergroup: $xy = \{r \in L \mid N^{xy}_r > 0\}$

Proposition
Feudal fusion rules $\longleftrightarrow$ feudal hypergroups with finite adjoint subhypergroup $\longleftrightarrow$ group homomorphisms with cokernel $\mathbb{Z}_2$ and kernel finite.
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Definition

A fusion category over a field $\mathbb{F}$ is a rigid strongly semisimple monoidal $\mathbb{F}$-linear category with simple monoidal unit.

Underlying fusion rule: $L$ a representative set of simples, for $x, y, r \in L$,

$$N_{r}^{xy} = \dim_{\mathbb{F}} \text{mor}(r, x \Box y)$$

where $\Box$ is the monoidal bifunctor.

Theorem (Ocneanu rigidity)

On a given finite fusion rule (up to isomorphism), only finitely many fusion categories (up to equivalence).
Example (known)

- On a group $G$, fusion categories $\longleftrightarrow H^3(G, \mathbb{F}^\times) / \text{aut } G$.
- On a Tambara-Yamagami fusion rule $A \sqcup \{m\}$, fusion categories $\longleftrightarrow$ pairs $(\chi, \tau)$ such that $\chi: A \times A \to \mathbb{F}^\times$ is a nondegenerate symmetric bicharacter

\[
\chi(b, a) = \chi(a, b) \\
\chi(ab, c) = \chi(a, c)\chi(b, c)
\]

and $\tau \in \mathbb{F}$ satisfies $|A|\tau^2 = 1$, where $\chi$ is up to $\text{aut } A$.
- On the fermionic Moore-Read fusion rule, four fusion categories (Bonderson).
Tool for characterizing feudal fusion categories: For $L = S \subseteq M$ a feudal fusion rule, $\mathbb{F}$ a field, let $B = \mathbb{F}^M$ be the ring of functions $M \to \mathbb{F}$. Then $B$ is an “involuntary ambidextrous $S$-algebra”: for $\mu \in B$ and $a, b \in S$,

\begin{align*}
\alpha \mu^b \in B : & \quad a \mu^b(m) = \mu(\bar{a}m\bar{b}) \\
\bar{\mu} \in B : & \quad \bar{\mu}(m) = \mu(\bar{m})
\end{align*}
Theorem

Let \( L = S \sqcup M \) be a feudal fusion rule, and \( B = \mathbb{F}^M \).

On \( L \), fusion categories \( \leftrightarrow \) triples \((\chi, \nu, \tau)\) such that \( \chi, \nu : S \times S \to B^\times \) and \( \tau \in B^\times \) satisfy

\[
\bar{\chi}(b, a) = \bar{a} \chi(a, b)^\bar{b} \cdot \frac{\bar{a}_\tau \bar{b} \cdot \tau}{\bar{a}_\tau \bar{b}}
\]

\[
\nu(a, b) \frac{\chi(ab, c)}{\nu(a, b)c} = \chi(a, c) \cdot a \chi(b, c)
\]

\( \chi|_{A \times A} \) is nondegenerate

and \( |A|^{\tau \bar{\tau}} \equiv 1 \), where \( A \subseteq S \) is the adjoint subhypergroup, but \((\chi, \nu, \tau)\) is up to equivalence...
...where $(\chi, \nu, \tau) \sim (\tilde{\chi}, \tilde{\nu}, \tilde{\tau})$ if there exist

$$\theta: S \times S \to F^\times \quad \phi: S \to B^\times \quad \varsigma \in B^\times$$

such that for $a, b \in S$,

$$\frac{\tilde{\chi}(a, b)}{\chi(a, b)} = \frac{\phi(a) \cdot \frac{a}{\phi(b)}^b}{\phi(a)^b \cdot \frac{b}{\phi(b)}^b} \cdot \frac{a \varsigma^b \cdot \varsigma}{a \varsigma \varsigma^b}$$

$$\frac{\tilde{\nu}(a, b)}{\nu(a, b)} = \frac{\phi(a) \cdot \frac{a}{\phi(b)}^b}{\phi(ab) \theta(a, b)}$$

$$\frac{\tilde{\tau}}{\tau} = \frac{\tilde{\varsigma}}{\varsigma}$$

This theorem generalizes the original classifications of Tambara-Yamagami and fermionic Moore-Read fusion categories.
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Braided fusion categories model quasiparticle motion in fractional quantum Hall liquids.

**Problem:** fermionic Moore-Read fusion categories aren’t braided (Bonderson). Possible braiding substitute:

**Definition**

A *twine* or *pure braiding* on a monoidal category is a natural automorphism of the monoidal bifunctor, satisfying coherence diagrams.

**Theorem (Bruguières)**

*Twines yield object-colored pure braid representations.*
A twine on a monoidal category makes the identity functor monoidal.

**Definition**
Two twines on the same monoidal category are *equivalent* if they are isomorphic as monoidal functors.

**Observation**
*On a fusion category on a group $G$, twines $\longleftrightarrow H^2(G, \mathbb{F}^\times)$.*

**Theorem**
*On a fermionic Moore-Read fusion category,* all twines $\sim$ the trivial twine.
Definition

Two twines on the same fusion category are *homothetically equivalent* if their pure braid actions on splitting spaces coincide up to homotheties.

Theorem

*Up to homothetic equivalence, there is a unique nontrivial twine on any fermionic Moore-Read fusion category, acting on triples over* $\{1, \alpha, \psi, \alpha', \sigma, \sigma'\}$ *as*

\[
\begin{align*}
\xi^{1,\sigma} &= 1 & \xi^{1,\sigma'} &= 1 & \xi^{\sigma,\sigma'} &= -i & \xi^{\sigma',\sigma} &= -i \\
\xi^{\psi,\sigma} &= -1 & \xi^{\psi,\sigma'} &= -1 & \xi^{\sigma,\sigma'} &= i & \xi^{\sigma',\sigma} &= i \\
\xi^{\alpha,\sigma'} &= 1 & \xi^{\alpha',\sigma} &= -1 & \xi^{\sigma,\sigma} &= i & \xi^{\sigma',\sigma'} &= -i \\
\xi^{\alpha',\sigma'} &= -1 & \xi^{\alpha',\sigma} &= 1 & \xi^{\sigma,\sigma'} &= -i & \xi^{\sigma',\sigma'} &= i
\end{align*}
\]
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Entwined fusion categories may describe quasiparticles in fractional quantum Hall liquids. The underlying electron wavefunction is a Gaussian, which we ignore, times a translation invariant antisymmetric polynomial over \( \mathbb{C} \).

\[
\text{(antisymmetric poly)} = \prod_{i<j}(z_i - z_j) \cdot \text{(symmetric poly)}
\]

**Conjecture (Haldane)**

*If a homogeneous symmetric polynomial is translation invariant, its squeezing poset has a maximum.*

**Theorem**

*The conjecture fails at arity 4 and degree 14.*
Squeezing poset of a symmetric polynomial:

symmetrized monomial \[\sum_{\sigma \in \text{Sym}(n)} z_{\sigma(1)}^{l_1} \cdots z_{\sigma(n)}^{l_n}\] \[\longleftrightarrow\] multiset over \(\mathbb{N}\)

\[2(z_1^6 z_2^6 z_3 + z_2^6 z_3^6 z_1 + z_3^6 z_1^6 z_2)\] \[\longleftrightarrow\] \([6, 6, 1]\)
Theorem

Ring of translation invariant symmetric \( n \)-variate polynomials
\( \cong \) full polynomial ring in \( n - 1 \) variables.

\[(z_1 - z_{\text{avg}})^k + \cdots + (z_n - z_{\text{avg}})^k, \quad 2 \leq k \leq n\]
The Pfaffian wavefunction

\[ \text{Pf} \left( \frac{1}{z_i - z_j} \right) \prod_{i<j} (z_i - z_j)^2 \]

is believed to produce the fermionic Moore-Read fusion rule for \( \nu = \frac{5}{2} \) fractional quantum Hall; it somehow contains the categorical structures we’ve considered.
Thanks!

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