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From Hypergroups to Anyonic Twines

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Jesse Liptrap

Committee in Charge:

Professor Daryl Cooper, Chair

Professor Jon McCammond

Professor David Morrison

Professor Michael Freedman

Professor Zhenghan Wang

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The Dissertation of
Jesse Liptrap is approved:

Professor Jon McCammond

Professor David Morrison

Professor Michael Freedman

Professor Zhenghan Wang

Professor Daryl Cooper, Committee Chairperson

May 2010

From Hypergroups to Anyonic Twines

To my family.

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Curriculum Vitæ
Jesse Liptrap

Education

June 2010	Ph.D. in Mathematics, UCSB (expected)
June 2006	B.S. in Mathematics, Caltech

Professional employment

2006–2010	Teaching Assistant, Mathematics Department, UCSB.
Winter 2008	Research Assistant, Mathematics Department, UCSB, supported by NSF grant of Professor Daryl Cooper.
Summer 2005	Undergraduate Research Fellow, Caltech. Studied complexes of curves, supervised by Jason Manning.
Summer 2004	Undergraduate Research Fellow, Caltech. Studied networks of relations, supervised by Professor Jehoshua Bruck and Matthew Cook.

Publications

2008	*On Translation Invariant Symmetric Polynomials and Haldane's Conjecture*, Topology and Physics: Proceedings of the Nankai International Conference in Memory of Xiao-Song Lin, Nankai Tracts in Mathematics 19:279-287, World Scientific, `arXiv:1002.3166`.

Preprints

2010	*Generalized Tambara-Yamagami categories*, submitted to *Journal of Algebra*, `arXiv:1004.0364`.

Talks

2009	*Generalized Tambara-Yamagami Categories*, AMS Special Session on Fusion Categories and Applications.

Awards

2007	Wilder award (for first year UCSB math grad students)

Fields of Study

2008–2010	Studies in Quantum Topology with Zhenghan Wang

Abstract

From Hypergroups to Anyonic Twines

by

Jesse Liptrap

We study the construction of hypergroups from groups and of fusion rules from hypergroups. We categorify Yamagami's linear algebraicization of fusion categories, to a 2-categorical equivalence. We classify nilpotent (in the sense of Gelaki and Nikshych) fusion rules of simple current index 2, and characterize the associated fusion categories, which include Tambara-Yamagami and fermionic Moore-Read. We compute all twines (pure braidings) of the latter. Entwined fusion categories may describe fractional quantum Hall quasiparticle motion in the absence of braiding, such as for fermionic Moore-Read, while the underlying electron wavefunction is determined by a translation invariant antisymmetric n-variate polynomial. We show that the ring of translation invariant symmetric n-variate polynomials is isomorphic to the full $(n-1)$-variate polynomial ring, and disprove a conjecture of Haldane regarding their structure.

Contents

Chapter 1

Introduction

Fusion categories can be viewed as quantum analogues of finite groups, or as mild generalizations of the category of finite-dimensional vector spaces. *Braided fusion categories* govern quasiparticle motion in the fractional quantum Hall effect, driving the main mechanism of topological quantum computation (see Chapter 8 or Wang [29]). *Twines* or *pure braidings*, inventions of Bruguieres [6], may be a useful alternative to braidings in some fermionic fractional quantum Hall regimes, but have not previously been studied in this context. Along the way to entwined fusion categories, we ascend a hierarchy of structures, viewing each as a foundation for the next:

- Absolutely regular hypergroups (Chapter 2)
- Fusion rules (Chapter 3)
- Fusion categories (Chapters 4, 5, 6)
- Entwined fusion categories (Chapter 7)

Hypergroups generalize groups by allowing multivalued multiplication. Though studied since the 1930s with various applications, they have not previously interacted with the other structures in this thesis. In Chapter 2 we introduce *absolutely regular hypergroups* (ARHs) as a natural weakening of fusion rules. To each ARH we assign a *simple current index* so that groups are precisely ARHs of simple current index 1. Following Gelaki and Nikshych [13] we assume *nilpotence*, not to be confused with the standard notion for groups. We classify, functorially, nilpotent ARHs of simple current index 2, in terms of group homomorphisms (Corollary 2.2.8), thereby generalizing the *Tambara-Yamagami* fusion rules [28] and the *fermionic Moore-Read* fusion rule [3] (Definition 2.1.2). Generalizing further, we construct ARHs from sequences of group homomorphisms (Theorem 2.3.1). Leaping from sequences to lattices, we conjecturally classify ARHs of nilpotence class two.

Fusion rules provide the combinatorial laws of fusing and splitting of fractional quantum Hall quasiparticles. They can be viewed as nondeterministic groups, obtained from ARHs by refining associativity from setwise to multisetwise. Nilpotent fusion rules of simple current index 2 are mere ARHs. We classify fusion rules with underlying ARHs given by Theorem 2.3.1, in terms of group 2-cocycles over $\mathbb{Z}_+$ (Theorem 3.3.2), and compute some simple examples.

Fusion categories appear in conformal field theory, operator algebras, representation theory, and quantum topology. They underly models of fractional quantum Hall quasiparticles, the proposed raw material for topological quantum computation. The theory of fusion categories is developed in Etingof, Nikshych, and Ostrik [12]. Every fusion category has a fusion rule up to isomorphism. Thus the problem of classifying fusion categories splits into two difficult subproblems: understand fusion rules, and given a fusion rule, understand the associated fusion categories, of which there are only finitely many up to equivalence over an algebraically closed field of characteristic 0 (Ocneanu rigidity [12]).

As a tool for constructing fusion categories from fusion rules, Yamagami [31] introduced linear algebraic gadgets which we call *fusion systems*, and proved them to be merely an alternate formulation of fusion categories. Fusion systems are also used directly in fractional quantum Hall physics (Appendix E of Kitaev [19]). Fusion systems are obtained from fusion categories by considering only the splitting spaces from one simple object to the monoidal product of two simple objects, corresponding in the fractional quantum Hall effect to one quasiparticle splitting into two. (It is equivalent to use fusion spaces.) We observe that fusion systems are objects in a 2-category, which we prove equivalent to a suitably defined 2-category of fusion categories (Theorem 5.1.4).

To compute all fusion systems on a given fusion rule, one picks bases for the splitting spaces, yielding collections of numbers (6j symbols) which we call *6j fusion systems*. When the fusion rule is a group, it is well-known that 6j fusion systems correspond to third group cohomology. Theorem 5.1.4 yields interpretations of second and first group cohomology as well, corresponding to 1- and 2-isomorphisms of fusion categories, respectively (Theorem 6.1.2). Generalizing Tambara and Yamagamis' classification using nondegenerate symmetric bilinear forms on groups [28], we give an elementary algebraic characterization of 6j fusion systems on nilpotent fusion rules of simple current index 2, and of $H^3(G, \mathbb{F}^\times)$ for G a group of even order and $\mathbb{F}$ a field (Corollaries 6.2.7, 6.2.6). For other generalizations of Tambara-Yamagami fusion categories, see Siehler [26] and Etingof, Nikshych, and Ostrik [11].

A *braiding* on a monoidal category yields representations of object-colored braids; a *twine* is similar but limited to pure braids. Some fusion categories such as fermionic Moore-Read are significant in fractional quantum Hall physics but lack braidings (Bonderson [3]). Twines may help describe quasiparticle motion in such cases. We observe that a twine on a monoidal category is merely a monoidal functor whose underlying functor is the identity and which satisfies a certain dodecagon axiom (Definition 7.2). Isomorphism of monoidal functors then gives a categorical definition of equivalence of twines, according to which twines on fusion categories on groups correspond to second group cohomology (Proposition 7.2.10). We compute all twines on fermionic Moore-Read fusion categories (Lemma 7.3.7). They are all trivial under our categorical notion of equivalence, but not all trivial up to homotheties on splitting trees. So it remains unclear whether twines can capture enough of the topological behavior of fermionic Moore-Read quasiparticles or those of other fractional quantum Hall theories. Table 7.1 summarizes various structures on fermionic Moore-Read fusion categories.

Chapter 8 sketches the role of braided or entwined fusion categories in fractional quantum Hall physics and topological quantum computation. Such categories may describe the exotic emergent behavior of quasiparticles in the fractional quantum Hall effect, while the underlying electron wavefunction is characterized by a translation invariant (anti)symmetric complex polynomial. In Section 8.1 we describe the totality of such polynomials. In Section 8.2, we find a counterexample to Haldane's conjecture [15] that every homogeneous translation invariant symmetric polynomial satisfies a certain physically convenient property (Proposition 8.2.7). More precisely, each symmetric polynomial p is associated with a finite poset $B(p)$; Haldane conjectured that if p is homogeneous and translation invariant, then $B(p)$ has a maximum. We prove the conjecture for polynomials of at most three variables, construct a minimal counterexample, and discuss whether a weakened version of the conjecture holds.

Chapter 2

Hypergroups

Consider modding out a group by a non-normal subgroup. The quotient should be some sort of algebraic structure with a single operation, but of course it is not a group because multiplication is not well-defined. For this and other applications, F. Marty in 1934 introduced the following definition [22].

First a bit of notation. Given a set H and a binary operation $*\colon H \times H \to 2^H$, and given $X, Y \subseteq H$, define $X * Y = \bigcup_{x \in X, y \in Y} x * y$. We identify each element x of H with the singleton $\{x\}$.

Definition 2.0.1. A *hypergroup* is a set H equipped with a binary operation $H \times H \to 2^H$ such that:

- $(xy)z = x(yz)$ for all $x, y, z \in H$.
- For all $x, y \in H$ there exist $a, b \in H$ such that $ax, xb \supseteq y$.

Example 2.0.2.

1. Every group is a hypergroup. A hypergroup is a group iff it has single-valued multiplication.

2. Let G be a group and S be an arbitrary subgroup. The set $H = G/S$ of left cosets is a hypergroup as follows. For $C, D, E \in H$, put $E \in C * D$ iff $E \ni cd$ for some $c \in C$, $d \in D$.

3. Let G be a group and S, T be arbitrary subgroups. The set $H = S \backslash G/T$ of double cosets is a hypergroup, defined as in the case of left cosets.

4. Let $\mathcal{C}$ be a fusion category (Definition 4.2.4). The set H of isomorphism classes of simple objects of $\mathcal{C}$ is a hypergroup as follows. For simple objects x, y, z of isomorphism types $[x], [y], [z]$, put $[z] \in [x] * [y]$ iff $\mathrm{mor}(z, x\Box y) \neq \{0\}$, where $\Box$ is the monoidal bifunctor on $\mathcal{C}$.

Hypergroups have been studied for their own sake and for applications to other areas of mathematics (Corsini and Leoreanu [7]). Our interest in hypergroups lies in Example 2.0.2(4), for applications to quantum topology and fractional quantum Hall physics. In these areas, fusion categories are of fundamental importance. To construct them, one often starts with a fusion rule (Chapter 3), commonly regarded as the lowest level of structure. We propose that hypergroups form a natural "basement level" below fusion rules.

Remark. Some authors, especially in harmonic analysis, define a hypergroup to be a sort of generalized fusion rule whose fusion multiplicities are probabilities.

A surprising amount of elementary group theory generalizes to hypergroups, including the first and second isomorphism theorems and the Jordan-Hölder theorem (Dresher and Ore [8], Ore and Eaton [10]). But hypergroups and most of the special families thereof which appear in the literature are too general for our goal of studying fusion rules. For instance, we want an *absolute unit*, an element e satisfying $xe = ex = x$ for all x. This does not exist in Example 2.0.2(2) unless S is normal. So we restrict attention to a special class of hypergroups which does not appear to have been defined in the literature.

2.1 Absolutely regular hypergroups

For the rest of this thesis, if we speak of hypergroups, we mean the following notion, some of whose basic theory is developed in this section.

Definition 2.1.1. An *absolutely regular hypergroup* (ARH) is a set H equipped with a binary operation $H \times H \to 2^H$ such that

$$\forall x, y, z\colon (xy)z = x(yz),$$

$$\exists 1 \forall x\colon 1x = x1 = x,$$

$$\forall x \exists \bar{x} \forall y\colon 1 \in xy \iff 1 \in yx \iff y = \bar{x}.$$

Example 2.1.2.

- Groups are precisely ARHs with single-valued multiplication. Note $\bar{a} = a^{-1}$ for any group element a.

- Given a group A and $m \notin A$, the *Tambara-Yamagami* ARH has elements $A \cup \{m\}$ multiplying as follows: for $a, b \in A$,

$$a * b = ab, \qquad a * m = m * a = m, \qquad m * m = A.$$

- The *fermionic Moore-Read* ARH has six elements $\{1, \alpha, \psi, \alpha', \sigma, \sigma'\}$ with the following commutative multiplication: $\{1, \alpha, \psi, \alpha'\} \cong \{1, i, -1, -i\} = \mathbb{Z}_4$, and

$$\psi\sigma = \sigma \qquad \psi\sigma' = \sigma'$$
$$\alpha\sigma = \alpha'\sigma = \sigma' \qquad \alpha\sigma' = \alpha'\sigma' = \sigma$$
$$\sigma\sigma' = \{1, \psi\} \qquad \sigma\sigma = \sigma'\sigma' = \{\alpha, \alpha'\}$$

Lemma 2.1.3. *Let L be an ARH. Then*

- *$xy \neq \emptyset$ for $x, y \in L$.*
- *$1 \in L$ is unique.*
- *If $x \in L$, then $\bar{x}$ is uniquely determined by x, and $\bar{\bar{x}} = x$.*

Definition 2.1.4. A *hypermagma* is a set H equipped with a binary operation $H \times H \to 2^H$.

Definition 2.1.5. Let L and L' be hypermagmas. A map $f\colon L \to L'$ is a *homomorphism* if $f(xy) \subseteq f(x)f(y)$ for $x, y \in L$.

Definition 2.1.6. Let G be a group and L an ARH. A *G-grading* on L is a surjective homomorphism $L \to G$.

Remark. Many authors do not require gradings to be surjective.

Definition 2.1.7. Let L be an ARH and $S \subseteq L$.

- $S \ni 1$ is a *sub-ARH* if $z \in S$ whenever $\bar{z} \in S$ or $z \in xy$ for some $x, y \in S$.
- A *left coset* of S in L is a subset xS for some $x \in L$. The set ${}^{L}/_{S}$ of all left cosets of S in L is a hypermagma with operation $*_S$ defined as follows: for $X, Y, Z \in {}^{L}/_{S}$, put $Z \in X *_S Y$ iff $z \in xy$ for some $x \in X,\ y \in Y,\ z \in Z$.
- The *index* of S is the cardinality of ${}^{L}/_{S}$.

Definition 2.1.8. The *adjoint sub-ARH* L_{ad} of an ARH L is the smallest sub-ARH containing $x\bar{x}$ for all $x \in L$. For $n \in \mathbb{N}$, we say L is *nil-n* if $L^{(n)} = \{1\}$, where $L^{(0)} = L$ and $L^{(m+1)} = (L^{(m)})_{\text{ad}}$. The smallest such n is the *nilpotence class* of L. We say L is *nilpotent* if it has a nilpotence class.

Remark. This meaning of "nilpotent" conflicts with its usual meaning for groups: every group is nilpotent as an ARH. We use "nilpotent" and "adjoint" for consistency with Gelaki and Nikshych [13].

Theorem 2.1.9 (Dresher and Ore [8], Gelaki and Nikshych [13])**.** *Let L be an ARH. Then L_{ad} is the intersection of all sub-ARHs A such that ${}^{L}/_{A}$ is a group. The hypermagma ${}^{L}/_{L_{\text{ad}}}$ is a group, called the* universal grading group*, partitioning L. Every grading of L factors uniquely through the quotient projection $L \to {}^{L}/_{L_{\text{ad}}}$, called the* universal grading.

Definition 2.1.10. An ARH element a is a *simple current* if $a\bar{a} = 1$. The *simple current index* of an ARH is the index of the set of simple currents.

Lemma 2.1.11. *Let L be an ARH with simple currents S. Then*

(i) $a \in S$ *iff* $\bar{a}a = 1$ *iff* az *and* za *are singletons for all* $z \in L$.

(ii) S *is the largest sub-ARH of* L *which is a group.*

(iii) ${}^{L}/_{S}$ *partitions* L.

(iv) L *is a group iff its simple current index is* 1.

2.2 Feudal hypergroups

Groups are the simplest ARHs. In this section we classify the "most group-like" non-group ARHs, which we call *properly feudal*.

Definition 2.2.1. Let $\mathbb{Z}_2 = \{\pm 1\}$ be a group of order 2. An ARH is *feudal* if it is equipped with a $\mathbb{Z}_2$-grading γ such that $\gamma^{-1}(1)$ is a group. We call elements of $\gamma^{-1}(1)$ *serfs* and elements of $\gamma^{-1}(-1)$ *lords*. An ARH is *properly feudal* if it is nilpotent with simple current index 2.

Example 2.2.2. We revisit Example 2.1.2, not for the last time.

- A $\mathbb{Z}_2$-graded group is improperly feudal. Its adjoint sub-ARH is trivial; it is its own simple current group and universal grading group.

- A Tambara-Yamagami ARH $A \cup \{m\}$ is feudal with serfs A and lord m. It is properly feudal iff $|A| > 1$ iff A is the simple current group. The adjoint sub-ARH is A; the universal grading group is $\{A, m\} \cong \mathbb{Z}_2$. A properly feudal ARH with a lone lord is Tambara-Yamagami.

- The fermionic Moore-Read ARH is feudal with serfs $\{1, \alpha, \psi, \alpha'\}$ and lords $\{\sigma, \sigma'\}$. It is properly feudal, with simple currents $\{1, \alpha, \psi, \alpha'\}$, adjoint sub-ARH $\{1, \psi\}$, and universal grading group $\{\{1, \psi\}, \sigma, \{\alpha, \alpha'\}, \sigma'\} \cong \mathbb{Z}_4$.

Lemma 2.2.3. *Let L be a feudal ARH with serfs S. Then*

(i) *S acts on lords by multiplication, transitively on the left and on the right.*

(ii) *$L_{\text{ad}} \trianglelefteq S$ is the left stabilizer and right stabilizer of any lord.*

(iii) *Two lords m, l multiply to a coset of L_{ad} in S, namely*

$$ml = \{a \in S \mid \bar{m}a = l\} = \{a \in S \mid m = a\bar{l}\}. \tag{2.1}$$

Proof. Let M be the lords and $m, l \in M$. By Lemma 2.1.11(i), S acts on M on the left and on the right by multiplication; by feudality, ml is a subset of S. For $a \in S$,

$$\begin{aligned} a \in ml &\iff 1 \in \bar{a}ml \iff \bar{m}a = l \\ &\iff 1 \in ml\bar{a} \iff m = a\bar{l}, \end{aligned}$$

proving equation (2.1). Then $ml \neq \emptyset$ implies (i). Since $\{ml' \mid l' \in M\}$ and $\{m'l \mid m' \in M\}$ each partition S,

$$mb = m \iff mbl = ml \iff bl = l$$

for $b \in S$, i.e., the right stabilizer of m and the left stabilizer of l coincide for arbitrary $m, l \in M$. Let $A \trianglelefteq S$ be this common stabilizer. Equation (2.1) implies $m\bar{m} = A$ for all $m \in M$. Thus $A = L_{\text{ad}}$, proving (ii). The orbit-stabilizer theorem of elementary group theory then completes (iii). $\square$

Proposition 2.2.4. *A properly feudal ARH is uniquely feudal. A feudal ARH is properly feudal or a* $\mathbb{Z}_2$*-graded group.*

Proof. Let L be properly feudal with simple currents S. By Lemma 2.1.11, $M = L \setminus S \neq \emptyset$ and $am, ma \in M$ whenever $a \in S$ and $m \in M$. Thus ${}^{L}\!/_{S} = \{S, M\}$ with $S *_S S = S$ and $S *_S M = M *_S S = M$.

To show $M *_S M = S$, we first need $L_{\mathrm{ad}} \subseteq S$. Pick $m \in M$. Since $M = mS$ and $ma\overline{ma} = m\bar{m}$ for $a \in S$, we see L_{ad} is the smallest sub-ARH of L containing $m\bar{m}$. If $m \in L_{\mathrm{ad}}$, then $m\bar{m} \subseteq (L_{\mathrm{ad}})_{\mathrm{ad}}$ implies $(L_{\mathrm{ad}})_{\mathrm{ad}} = L_{\mathrm{ad}}$, contradicting nilpotence. Therefore $L_{\mathrm{ad}} \subseteq S$. Now pick any $m, l \in M$. Then $l = \bar{m}a$ for some $a \in S$, whence $ml = m\bar{m}a \subseteq S$. Thus $M *_S M = S$, and $L \to {}^{L}\!/_{S} \cong \mathbb{Z}_2$ is a feudal grading.

Suppose $\gamma\colon L \to \mathbb{Z}_2$ is a different feudal grading. Then $S' = \gamma^{-1}(1) \subset S$. Picking $a \in S \setminus S'$ and $m \in M$, we have $am \in M \cap S'$, a contradiction. Thus L is uniquely feudal. Finally, a non-group feudal ARH is properly feudal by Lemma 2.2.3. □

Definition 2.2.5. Let $\mathcal{H}$ be the following category. An object of $\mathcal{H}$ is a homomorphism $S \xrightarrow{u} G$ of arbitrary groups S, G such that $|\operatorname{coker} u| = 2$, with the innocuous technical conditions $S \cap (G \setminus \operatorname{im} u) = \emptyset$ and $\operatorname{im} u = S/\ker u$. A morphism from $S \xrightarrow{u} G$ to $\tilde{S} \xrightarrow{\tilde{u}} \tilde{G}$ in $\mathcal{H}$ is a pair of homomorphisms (h_0, h_1) making the square

$$\begin{array}{ccc} S & \xrightarrow{u} & G \\ {\scriptstyle h_0}\downarrow & & \downarrow{\scriptstyle h_1} \\ \tilde{S} & \xrightarrow[\tilde{u}]{} & \tilde{G} \end{array} \tag{2.2}$$

commute, with $h_1(G \setminus \operatorname{im} u) \subseteq \tilde{G} \setminus \operatorname{im} \tilde{u}$.

Let $\mathcal{L}$ be the category of feudal ARHs and graded homomorphisms. Let $\Phi\colon \mathcal{H} \to \mathcal{L}$ be the following functor. For $H = (S \xrightarrow{u} G) \in \operatorname{obj} \mathcal{H}$, let ΦH be the feudal ARH with serfs S and lords $M = G \setminus \operatorname{im} u$ multiplying as follows: for $a, b \in S$ and $m, l \in M$,

$$a * b = ab, \qquad a * m = u(a)m, \qquad m * a = mu(a), \qquad m * l = u^{-1}(ml).$$

For $\tilde{H} \in \operatorname{obj} \mathcal{H}$ and $(h_0, h_1) \in \operatorname{mor}_{\mathcal{H}}(H, \tilde{H})$, let $\Phi(h_0, h_1)$ agree with h_0 on S and with h_1 on M.

Inversely, let $\Gamma\colon \mathcal{L} \to \mathcal{H}$ be the following functor. Given $L \in \operatorname{obj} \mathcal{L}$, let $\Gamma L = (S \xrightarrow{u} G)$ with S the serfs (or the simple currents unless L is a group), G the universal grading group, and u the restriction to S of the universal grading. For $\tilde{L} \in \operatorname{obj} \mathcal{L}$ and $t \in \operatorname{mor}_{\mathcal{L}}(L, \tilde{L})$, let $\Gamma t = (h_0, h_1)$ where t agrees with h_0 and induces h_1.

Example 2.2.6.

(i) Let G be a $\mathbb{Z}_2$-graded group with serfs S. Then ΓG is inclusion $S \to G$.

(ii) Let $L = A \cup \{m\}$ be Tambara-Yamagami. Then ΓL is isomorphic to the trivial homomorphism $A \to \mathbb{Z}_2$.

(iii) Let L be the fermionic Moore-Read ARH. Then ΓL is isomorphic to the nontrivial nonidentity homomorphism $\mathbb{Z}_4 \to \mathbb{Z}_4$.

Theorem 2.2.7. *The category of feudal ARHs and graded homomorphisms is isomorphic to the category $\mathcal{H}$ of Definition 2.2.5, via the functors therein.*

Corollary 2.2.8. *Up to isomorphism, properly feudal ARHs are in 1-1 correspondence with noninjective group homomorphisms with cokernels of order* 2.

Proof of Theorem 2.2.7. First we check $\Phi\colon \mathcal{H} \to \mathcal{L}$ is a functor. For $H = (S \xrightarrow{u} G) \in \operatorname{obj} \mathcal{H}$, we check $\Phi H \in \operatorname{obj} \mathcal{L}$. Let $A = \ker u$ and $M = G \setminus \operatorname{im} u$. For $a, b, c \in S$ and

$m, l, r \in M$,

$$(a * b) * c = abc = a * (b * c),$$

$$(a * m) * b = u(a)mu(b) = a * (m * b),$$

$$(a * b) * m = u(ab)m = a * (b * m),$$

$$(m * a) * b = m * (a * b),$$

$$(m * a) * l = u^{-1}(mu(a)l) = m * (a * l),$$

$$(a * m) * l = u^{-1}(u(a)ml) = au^{-1}(ml) = a * (m * l),$$

$$(m * l) * a = m * (l * a),$$

$$(m * l) * r = mlr = m * (l * r).$$

Thus $*$ is associative. Therefore $\Phi H \in \operatorname{obj} \mathcal{L}$. Now let $\tilde{H} = (\tilde{S} \xrightarrow{\tilde{u}} \tilde{G}) \in \operatorname{obj} \mathcal{H}$ and $h = (h_0, h_1) \in \operatorname{mor}_{\mathcal{H}}(H, \tilde{H})$. For $a, b \in S$ and $m, l \in M$,

$$\Phi h(a * b) = h_0(ab) = \Phi h(a) * \Phi h(b),$$

$$\Phi h(a * m) = h_1(u(a)m) = \tilde{u}(h_0(a))h_1(m) = \Phi h(a) * \Phi h(m),$$

$$\Phi h(m * a) = \Phi h(m) * \Phi h(a).$$

Since $|\operatorname{coker} u| = 2$, there exists $b \in u^{-1}(ml)$. Let $\tilde{A} = \ker \tilde{u}$. Since square (2.2) commutes, $h_0(A) \subseteq \tilde{A}$. Then

$$\Phi h(m * l) = h_0(bA) \subseteq h_0(b)\tilde{A} = \tilde{u}^{-1}(\tilde{u}(h_0(b))) = \tilde{u}^{-1}(h_1(ml)) = \Phi h(m) * \Phi h(l)$$

Thus Φh is a homomorphism. Therefore $\Phi \colon \mathcal{H} \to \mathcal{L}$ is a functor.

Now we check $\Gamma \colon \mathcal{L} \to \mathcal{H}$ is a functor. Suppose $L, \tilde{L} \in \operatorname{obj} \mathcal{L}$, with serfs $S, \tilde{S}$, lords $M, \tilde{M}$, adjoint sub-ARHs $A, \tilde{A}$, and restrictions $u, \tilde{u}$ to serfs of the universal gradings, respectively. Then $\Gamma L = (S \xrightarrow{u} G)$ and $\Gamma \tilde{L} = (\tilde{S} \xrightarrow{\tilde{u}} \tilde{G})$ are in $\operatorname{obj} \mathcal{H}$. Suppose $t \in \operatorname{mor}_{\mathcal{L}}(L, \tilde{L})$. Since t is graded, it restricts to a homomorphism $h_0 \colon S \to \tilde{S}$. By Lemma 2.2.3(ii), $h_0(A) \subseteq \tilde{A}$. Let $h_{0.5}$ be the induced homomorphism ${}^{S}\!/_{A} \to {}^{\tilde{S}}\!/_{\tilde{A}}$. Recalling $G = ({}^{S}\!/_{A}) \cup M$ and $\tilde{G} = ({}^{\tilde{S}}\!/_{\tilde{A}}) \cup \tilde{M}$, let $h_1 \colon G \to \tilde{G}$ agree with $h_{0.5}$ on ${}^{S}\!/_{A}$ and with t on M. Then $\Gamma t = (h_0, h_1)$ is defined and square (2.2) commutes. To check h_1 is a homomorphism, let $a, b \in S$ and $m, l \in M$ and $c \in m * l$. By Lemma 2.2.3,

$$h_1((aA)(bA)) = h_{0.5}((aA)(bA)) = h_1(aA)h_1(bA),$$

$$h_1((aA)m) = t(a * m) = t(a) * t(m) = h_1(aA)h_1(m),$$

$$h_1(m(aA)) = h_1(m)h_1(aA),$$

$$h_1(ml) = h_{0.5}(cA) = t(c)\tilde{A} = t(m) * t(l) = h_1(m)h_1(l).$$

Therefore Γ is a functor. It is easy to see Φ and Γ are mutually inverse. $\square$

2.3 Hypergroups from lattices of groups

In this section we seek the most general way of constructing hypergroups from groups and group homomorphisms, dreaming of classifying nilpotent ARHs (Definition 2.1.8). The journey starts at Definition 2.2.5, where we constructed an ARH from a pair of groups connected by a noninjective homomorphism with cokernel of order 2. It is natural to wonder whether the condition on the cokernel is necessary, and how the construction might generalize to three or more groups.

Let

$$G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} \cdots \tag{2.3}$$

be any sequence of groups connected by homomorphisms. The sequence need not terminate. Let $M_1 = G_1$, and for all $i > 1$, let $M_i = G_i \setminus \operatorname{im} f_{i-1}$. Then the disjoint union $H = \bigsqcup_i M_i$ is a hypermagma as follows. For any $i \leq j$, let $f_{i,j} \colon G_i \to G_j$ be the composition of zero or more maps in the sequence. Given $x \in M_i$, $y \in M_j$, $z \in M_k$, let $l = \sup\{i, j, k\}$, and put $z \in x * y$ iff

$$f_{i,l}(z) = f_{j,l}(x) f_{k,l}(y). \tag{2.4}$$

Theorem 2.3.1. *Let H be the hypermagma constructed from sequence* (2.3) *by equation* (2.4)*. The following are equivalent:*

- *H is a hypergroup.*
- $\operatorname{im} f_i \supseteq \ker f_{i+1}$ *for all i.*
- *For each $x \in M_i$ and $y \in M_j$, there exists k such that $x * y \subseteq M_k$.*

Proof. The second and third conditions are easily seen to be equivalent. For the first, one checks associativity $(x * y) * z = x * (y * z)$ for arbitrary $x \in M_i$, $y \in M_j$, $z \in M_k$ by cases according to the relative ordering of i, j, k. □

Theorem 2.3.1 subsumes the previous section's construction of feudal hypergroups (Φ of Definition 2.2.5), but can probably still be generalized. Consider a commutative diagram D formed by assigning a group G_i to each element i of a poset P and a homomorphism $f_{i,j} \colon G_i \to G_j$ to each relation $i \leq j$ in P. More concisely, D is a functor from P to **Grp**. If P is a join-semilattice (any two elements have a least upper bound), the construction of the hypermagma H still makes sense. If P is bounded below (has a minimal element), H is absolutely regular. The only potentially missing ingredient is associativity.

Conjecture 2.3.2. *Let H be the hypermagma constructed by equation (2.4) from a join-semilattice D of groups and group homomorphisms. Then H is a hypergroup iff D is a lattice bounded below such that $\operatorname{im} f_{i,j} \supseteq \ker f_{j,k}$ for any chain*

$$G_i \xrightarrow{f_{i,j}} G_j \xrightarrow{f_{j,k}} G_k$$

in D. Moreover, H determines D if we require the nonidentity homomorphisms in D to be noninjective and nonsurjective.

The correspondence $D \mapsto H$ should be functorial: homomorphisms of the resulting ARHs should be classified by poset homomorphisms and group homomorphisms in the natural way.

Recall that a hypergroup is nil-0 iff it is the trivial hypergroup $\{1\}$, and nil-1 iff it is a group. It is easy to see that the hypergroups of Conjecture 2.3.2 are nil-2.

Conjecture 2.3.3. *Conjecture 2.3.2 classifies nil-2 ARHs.*

From a lattice of nil-2 ARHs and hypergroup homomorphisms satisfying appropriate conditions, could we similarly construct a nil-3 ARH? Continuing recursively, would all nilpotent ARHs be thus classified?

Chapter 3

Fusion rules

In the hierarchy of structures considered in this thesis, *fusion rules* are the next level up from hypergroups. A fusion rule is essentially a hypergroup with a notion of *multiplicity* attached to the hyperproduct: for any elements a, b, c, there is a natural number N_c^{ab}, the multiplicity of c in ab. Fusion rules are the "skeletons" of fusion categories, whereas fusion systems (Chapter 5) are the "flesh" (Yamagami [31]).

Unlike hypergroups, fusion rules have a fairly restricted range of applications in the literature. Their principal use, overlapping with their auxiliary role in fusion category theory, is to describe the combinatorial laws of fusing (and splitting) of particle types in particular physical theories, especially in the areas of conformal field theory and fractional quantum Hall physics.

We now motivate the definition of a fusion rule using naive physical ideas. Consider a physical theory with some set L of particle types. Any two particles can be fused together, resulting in a quantum superposition of particles, i.e., a single particle whose type may be undetermined until measurement. For any particle types a, b, c, there is a *multiplicity* $N_c^{ab} \in \mathbb{N}$, closely related to the probability of observing a particle of type c after fusing a particle of type a with one of type b (the precise relationship involves quantum dimensions). The multiplicities must satisfy an *associativity* constraint for fusion of three or more particles to be well-defined. The *vacuum* is considered a special particle type; fusing with it has no effect. Finally, each particle type has a *dual* particle type: fusion can only result in the vacuum if the fused particles have dual types.

Fusion rules are usually defined as families of multiplicities N_c^{ab} satisfying three axioms which naturally generalize the three axioms of a group. But hypergroups and fusion rules are usually not studied in the same context. To emphasize their close relationship, it is convenient to define a fusion rule using multisets rather than raw multiplicities.

Remark. Many authors refer to a single equation of the form $ab = \sum_{c \in L} N_c^{ab} c$, rather than the totality of such equations for a given theory, as a fusion rule.

3.1 Multiset formalism

Definition 3.1.1. A *multiset* over a set L is a function $X \in \mathbb{N}^L$, assigning a *multiplicity* N_x^X to each $x \in L$. A multiset is *finite* if it has finite support.

Definition 3.1.2. A *multimagma* is a set L equipped with an operation $*\colon L \times L \to \mathbb{N}^L$. If $X, Y \in \mathbb{N}^L$ are finite, define $X * Y \in \mathbb{N}^L$ as follows: for $z \in L$,

$$N_z^{X*Y} = \sum_{x,y \in L} N_x^X N_y^Y N_z^{x*y}$$

We say L is *locally finite* if $x * y$ is finite for $x, y \in L$.

Convention. We identify each subset of a set L with the multiset given by its indicator function, and each element $x \in L$ with the singleton $\{x\}$ given by the Kronecker delta $y \mapsto \delta_{x,y}$. The symbol $*$ is often suppressed.

Definition 3.1.3. A *fusion rule* is a locally finite multimagma satisfying

$$\forall x, y, z\colon (xy)z = x(yz)$$

$$\exists 1 \forall x\colon 1x = x1 = x$$

$$\forall x \exists \bar{x} \forall y\colon N_1^{xy} = N_1^{yx} = \delta_{y,\bar{x}}$$

Example 3.1.4. The hypergroups of Example 2.1.2 are fusion rules except in the case of Tambara-Yamagami hypergroups, where the adjoint subrule A must be finite in order to satisfy local finiteness.

Example 3.1.5. For any finite group A and $m \notin A$ and $k \in \mathbb{N}$, the *near-group* fusion rule $\mathrm{Neargrp}(A, k)$ has elements $A \cup \{m\}$ fusing as follows: for $a, b \in A$,

$$a * b = ab, \qquad a * m = m * a = m, \qquad m * m = A \cup km,$$

the last equation meaning $N_m^{m*m} = k$ and $N_a^{m*m} = 1$ for $a \in A$. Every fusion rule all but one of whose elements are simple currents (Definition 2.1.10) has this form. Fusion categories on such fusion rules were studied by Siehler [26].

Definition 3.1.6. Define the *underlying set* $\lfloor X \rfloor$ of a multiset X by $x \in \lfloor X \rfloor$ iff $N_x^X > 0$. Given a multimagma L, let $\lfloor L \rfloor$ be the multimagma with the same elements with operation $(x, y) \mapsto \lfloor xy \rfloor$. We say L is *multiplicity-free* if $L = \lfloor L \rfloor$.

Observation 3.1.7. *If L is a fusion rule, then $\lfloor L \rfloor$ is an ARH. In particular, every multiplicity-free fusion rule is an ARH.*

In light of Observation 3.1.7 we will freely use previously defined notions of hypergroups, such as simple currents, when discussing a fusion rule, with the understanding that they refer to the underlying hypergroup.

Remark. One might naively hope every locally finite ARH is a fusion rule. Scott Morrison showed me the following counterexample: the second fusion rule of Peters [24]. Arising from subfactor theory, it has four elements and is not nilpotent (Definition 2.1.8). Examples 3.3.4 and 3.3.7 are counterexamples of nilpotence class 2.

Lemma 3.1.8. *Let L be a fusion rule with simple currents S. Then*

(i) *$a \in S$ iff az and za are singletons for all $z \in L$.*

(ii) *If $a \in S$, then $N_a^{xy} \leq 1$ for all $x, y \in L$.*

3.2 Feudal fusion rules

In this section we observe that the notion of a fusion rule offers nothing new for feudal hypergroups (Definition 2.2.1).

Proposition 3.2.1. *Feudal fusion rules are precisely locally finite feudal hypergroups. In other words, every feudal fusion rule is multiplicity-free, and a feudal hypergroup is a fusion rule iff its adjoint subhypergroup is finite.*

Proof. Let L be a feudal fusion rule. Let $x, y \in L$, and let a be any serf. Then $N_y^{ax}, N_y^{xa}, N_a^{xy} \leq 1$ by Lemma 3.1.8. Thus L is multiplicity-free. Conversely, let L be a feudal hypergroup. By Lemma 2.2.3, L, is locally finite iff its adjoint subhypergroup is finite. The only thing to check is associativity of multiplicities, i.e., that the eight equations of sets used to check associativity in the proof of Theorem 2.2.7 remain valid when read as equations of multisets. This holds easily: by Lemma 2.2.3, the first seven equations remain honest sets, while both sides of the last equation become the multiset $g \mapsto |A|\delta_{g,mlr}$. □

Corollary 3.2.2. *Up to isomorphism, properly feudal ARHs are in 1-1 correspondence with group homomorphisms whose cokernels have order* 2 *and whose kernels are nontrivial and finite.*

Proof. Immediate from Corollary 2.2.8 and Proposition 3.2.1. □

3.3 Fusion rules from lattices of groups

In this section we discuss fusion rules with underlying hypergroups constructed from lattices of groups as in Section 2.3. We classify fusion rule structures on some of these hypergroups in terms of group 2-cocycles (Theorem 3.3.2). Computing the simplest examples of this classification, we encounter hypergroups upgradable to fusion rules in finitely many different ways (Examples 3.3.4, 3.3.7).

Definition 3.3.1. Let G be a group and M a monoid. A 2-*cocycle* on G over M is a map $\mu\colon G \times G \to M$ such that $\mu(a,b)\mu(ab,c) = \mu(a,bc)\mu(b,c)$ for all $a, b, c \in G$. For

$T \subseteq G$, we say μ is *T-normalized* if $\mu(a,b) = 1$ whenever $a \in T$ or $b \in T$. *Normalized* means $\{1\}$-normalized.

Theorem 3.3.2. *Let $S \xrightarrow{f} G$ be a homomorphism of groups, nonsurjective, with nontrivial finite kernel. Let H be the associated hypergroup of Definition 2.3.1. Then fusion rules with underlying hypergroup H are in 1-1 correspondence with* $\mathrm{im}(f)$*-normalized* 2*-cocycles μ on G over $\mathbb{Z}_+$ such that $\mu(m,l) = |\ker f|$ whenever $m, l \notin \mathrm{im}\, f \ni ml$.*

Proof. Let $M = G \setminus \mathrm{im}\, f$, and recall $H = S \sqcup M$. Given such a μ, promote H to a fusion rule by introducing multiplicities as follows. By Lemma 3.1.8 all multiplicities are 0 or 1, and hence determined by H, except $N_{ml}^{m,l}$ for $m, l, ml \in M$; in this case let $N_{ml}^{m,l} = \mu(m,l)$. Going in the other direction, given a fusion rule with underlying hypergroup H, let μ be the $\mathrm{im}(f)$-normalized function $G \times G \to \mathbb{Z}_+$ such that $\mu(m,l)$ is the multiset cardinality of $m * l$ for $m, l \in M$. A quick check of 2^3 cases, refining the associativity check in the proof of Theorem 2.3.1, shows that associativity of multiplicities is equivalent to the stated conditions on μ. □

Example 3.3.3. In Theorem 3.3.2, let $G = \mathbb{Z}_2$. Since f is trivial, there is exactly one choice of μ, producing a multiplicity-free fusion rule H, agreeing with Proposition 3.2.1.

Example 3.3.4. In Theorem 3.3.2, let $G = \mathbb{Z}_3 = \{1, \omega, \bar{\omega}\}$. Since f is trivial, the conditions on μ are precisely that it is normalized and

$$\mu(\omega, \bar{\omega}) = \mu(\bar{\omega}, \omega) = |S|, \qquad \mu(\omega, \omega)\mu(\bar{\omega}, \bar{\omega}) = |S|.$$

Thus fusion rule structures on $H = S \sqcup \{\omega, \bar{\omega}\}$ are in 1-1 correspondence with factorizations of $|S|$ into ordered pairs of positive integers. Since swapping ω and $\bar{\omega}$ is a fusion rule automorphism, there are $\lceil d/2 \rceil$ isomorphism classes of fusion rules with underlying hypergroup isomorphic to H, where d is the number of divisors of $|S|$.

Lemma 3.3.5. *Let G be a cyclic group of order n, and let K be an n-divisible abelian group. Then the cohomology $H^i(G, K)$ is trivial for all positive even i.*

Proof. Example III.1.2 of Brown [5]. □

Lemma 3.3.6. *Let K be a 4-divisible abelian group, and $\mathbb{Z}_4 = \{1, -1, i, -i\}$. A normalized map $\mu\colon \mathbb{Z}_4 \times \mathbb{Z}_4 \to K$ is a 2-cocycle iff it is a 1-coboundary (Definition 6.1.1) iff μ is symmetric and*

$$\mu(-1,i)\mu(i,-i) = \mu(i,i)\mu(-1,-1) \qquad \mu(-1,i)\mu(-1,-i) = \mu(-1,-1)$$

$$\mu(-1,-i)\mu(i,-i) = \mu(-i,-i)\mu(-1,-1) \qquad \mu(i,i)\mu(-1,-i) = \mu(-1,i)\mu(-i,-i)$$

Proof. By Lemma 3.3.5, μ is a 2-cocycle iff it is a 1-coboundary. It is straightforward to work out the equations of coboundaryhood. □

Example 3.3.7. In Theorem 3.3.2, let $G = \mathbb{Z}_4 = \{1, -1, i, -i\}$. If $\operatorname{im} f = \mathbb{Z}_2 = \{\pm 1\}$, as in the fermionic Moore-Read hypergroup (Example 2.1.2), then $H = S \sqcup \{i, -i\}$ is a multiplicity-free fusion rule, the only fusion rule with underlying hypergroup H. If $\operatorname{im} f = \{1\}$, the only constraints on μ are that it is symmetric, normalized, and

$$\mu(-1,i) = \mu(i,i) \qquad \mu(-1,-i) = \mu(-i,-i) \qquad \mu(i,i)\mu(-i,-i) = |S|$$

Just as in Example 3.3.4, fusion rule structures on $H = S \sqcup \{-1, i, -i\}$ are in 1-1 correspondence with factorizations of $|S|$ into ordered pairs of positive integers. This example follows without difficulty from Lemma 3.3.6.

Now let D be a lattice of groups as in Section 2.3. We wish to understand how the associated hypergroup H can be promoted to a fusion rule. For simplicity assume the lattice P underlying D is bounded above, and let G be the group at the top. Perhaps, as in Theorem 3.3.2, fusion rule structures on H are classified by 2-cocycles $G \times G \to \mathbb{Z}_+$ satisfying appropriate constraints involving the images in G of the G_i and the kernel sizes of the $f_{i,j}$.

Chapter 4

Fusion categories

In this chapter we define the strict 2-category (Proposition 4.3.3) of fusion categories (Definition 4.2.4). But we do not pause to study their properties. Rather, in the next two chapters we turn to the equivalent structures of fusion systems and 6j fusion systems, which are more amenable to computation. As preparation, we discuss equivalence of strict 2-categories in Section 4.3, and labeling of fusion categories in Section 4.4.

4.1 Monoidal categories

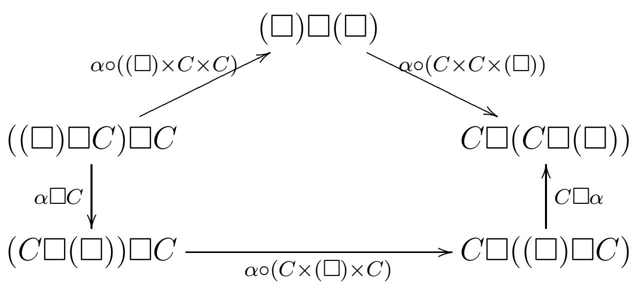
Monoidal categories are ubiquitous. We assume familiarity with horizontal and vertical composition of natural transformations. Leinster [20] and Mac Lane [21] explain the material in this section in greater detail.

Definition 4.1.1. For categories C, D, let $\mathrm{Fun}(C, D)$ be the functor category, whose objects are functors $C \to D$ and whose morphisms are natural transformations under vertical composition.

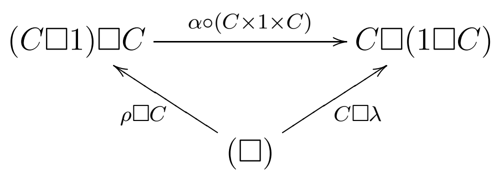
Definition 4.1.2. A *bifunctor* on a category C is a functor $\Box \colon C \times C \to C$, usually written as an infix. We often write $\Box$ as $(\Box)$ when it appears in a formula not as an infix. With C denoting its identity functor, we could then write $(\Box) = C \Box C$.

Definition 4.1.3. A *monoidal category* is a sextuple $\mathcal{C} = (C, \Box, 1, \alpha, \lambda, \rho)$, where C is a category; $\Box$ is a bifunctor on C; $1 \in \operatorname{obj} C$; α is a natural isomorphism $(\Box)\Box C \Rightarrow C\Box(\Box)$ of trifunctors on C; and λ and ρ are natural isomorphisms $C \Rightarrow 1\Box C$ and $C \Rightarrow C\Box 1$, respectively, of functors on C; satisfying

Pentagon axiom: This diagram in $\mathrm{Fun}(C^4, C)$ commutes:



Triangle axiom: This diagram in $\mathrm{Fun}(C^2, C)$ commutes:



The pentagon and triangle axioms were engineered to produce the following coherence theorem, categorifying the fact that a monoid M yields well-behaved multiplication maps $M^n \to M$ for all $n \in \mathbb{N}$.

Theorem 4.1.4 (monoidal coherence)**.** *Let $\mathcal{C} = (C, \Box, 1, \alpha, \lambda, \rho)$ be a monoidal category, let $n \in \mathbb{N}$, and let T, S be functors $C^n \to C$ built from $\Box, 1$. Then there is a unique natural isomorphism $T \cong S$ built from α, λ, ρ.*

Proof. See Mac Lane [21]. This theorem follows from the simple connectivity of the 2-skeleton of the associahedron K_n. □

Definition 4.1.5. Let $\mathcal{C} = (C, \Box, 1, \alpha, \lambda, \rho)$ and $\tilde{\mathcal{C}} = (\tilde{C}, \tilde{\Box}, \tilde{1}, \tilde{\alpha}, \tilde{\lambda}, \tilde{\rho})$ be monoidal categories. An *oplax monoidal functor* $\mathcal{T} \colon \mathcal{C} \to \tilde{\mathcal{C}}$ is a triple (T_1, T_2, T_0), where $T = T_1 \colon C \to \tilde{C}$ is an ordinary functor, T_2 is a natural transformation $T \circ (\Box) \Rightarrow T \tilde{\Box} T$ of functors $C \times C \to \tilde{C}$, and $T_0 \colon T1 \to \tilde{1}$, such that this diagram in $\mathrm{Fun}(C^3, \tilde{C})$ commutes:

$$\begin{array}{ccc}
(T\tilde{\Box}T)\tilde{\Box}T & \xrightarrow{\tilde{\alpha}\circ(T\Box T\Box T)} & T\tilde{\Box}(T\tilde{\Box}T) \\
\uparrow{\scriptstyle T_2\tilde{\Box}T} & & \uparrow{\scriptstyle T\tilde{\Box}T_2} \\
(T\circ(\Box))\tilde{\Box}T & & T\tilde{\Box}(T\circ(\Box)) \\
\uparrow{\scriptstyle T_2\circ((\Box)\times C)} & & \uparrow{\scriptstyle T_2\circ(C\times(\Box))} \\
T\circ((C\Box C)\Box C) & \xrightarrow[T\circ\alpha]{} & T\circ(C\Box(C\Box C))
\end{array}$$

and these two diagrams in $\mathrm{Fun}(C, \tilde{C})$ commute:

$$\begin{array}{ccc}
T & \xrightarrow{\tilde{\lambda}\circ T} & \tilde{1}\tilde{\Box}T \\
\downarrow{\scriptstyle T\circ\lambda} & & \uparrow{\scriptstyle T_0\tilde{\Box}T} \\
T\circ(1\Box C) & \xrightarrow[T_2\circ(1\times C)]{} & T1\tilde{\Box}T
\end{array}
\qquad
\begin{array}{ccc}
T & \xrightarrow{\tilde{\rho}\circ T} & T\tilde{\Box}\tilde{1} \\
\downarrow{\scriptstyle T\circ\rho} & & \uparrow{\scriptstyle T\tilde{\Box}T_0} \\
T\circ(C\Box 1) & \xrightarrow[T_2\circ(C\times 1)]{} & T\tilde{\Box}T1
\end{array}$$

A *monoidal functor* is an oplax monoidal functor (T, T_2, T_0) such that T_2 and T_0 are invertible. A *colax monoidal functor* is like an oplax monoidal functor, but with the directions of T_2 and T_0 reversed.

Remark. In this thesis we use oplax monoidal functors rather than colax monoidal functors. The choice seems arbitrary, but the former go a bit better with braid actions on morphism spaces: oplax monoidal functors relate splitting spaces, on which braids act covariantly, while colax monoidal functors relate fusion spaces, on which braids act contravariantly.

Definition 4.1.6. Let $\mathcal{C}, \tilde{\mathcal{C}}$ be monoidal categories with bifunctors $\Box, \tilde{\Box}$ and units $1, \tilde{1}$, respectively. Let $\mathcal{S} = (S, S_2, S_0)$ and $\mathcal{T} = (T, T_2, T_0)$ be oplax monoidal functors $\mathcal{C} \to \tilde{\mathcal{C}}$. A *monoidal natural transformation* $\eta\colon \mathcal{S} \Rightarrow \mathcal{T}$ is a natural transformation $\eta\colon S \Rightarrow T$ such that these diagrams, in $\mathrm{Fun}(C \times C, \tilde{C})$ and $\tilde{C}$ respectively, commute:

$$\begin{array}{ccc} S\tilde{\Box}S & \xrightarrow{\eta\tilde{\Box}\eta} & T\tilde{\Box}T \\ {\scriptstyle S_2}\uparrow & & \uparrow{\scriptstyle T_2} \\ S \circ \Box & \xrightarrow[\eta\circ\Box]{} & T \circ \Box \end{array} \qquad \begin{array}{ccccc} & & \tilde{1} & & \\ & {\scriptstyle S_0}\nearrow & & \nwarrow{\scriptstyle T_0} & \\ S1 & & \xrightarrow[\eta 1]{} & & T1 \end{array}$$

The following definition generalizes the phenomenon of dual vector spaces.

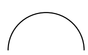
Definition 4.1.7. Let x be an object in a monoidal category $\mathcal{C}$ with bifunctor $\Box$ and unit 1. A *right dual* to x is an object x^* and morphisms $b\colon 1 \to x\Box x^*$ and $d\colon x^*\Box x \to 1$, called *birth* and *death* respectively, such that

$$x^* \xrightarrow{\rho} x^*1 \xrightarrow{x^*\Box b} x^*(xx^*) \xrightarrow{\alpha^{-1}} (x^*x)x^* \xrightarrow{d\Box x^*} 1x^* \xrightarrow{\lambda^{-1}} x^* = \mathrm{id}_{x^*}$$

$$x \xrightarrow{\lambda} 1x \xrightarrow{b\Box x} (xx^*)x \xrightarrow{\alpha} x(x^*x) \xrightarrow{x\Box d} x1 \xrightarrow{\rho^{-1}} x = \mathrm{id}_x$$

A *left dual* to x is an object ${}^*x$ and morphisms $b\colon 1 \to {}^*x\Box x$ and $d\colon x\Box{}^*x \to 1$ satisfying similar axioms. A monoidal category is *rigid* if every object has left and right duals.

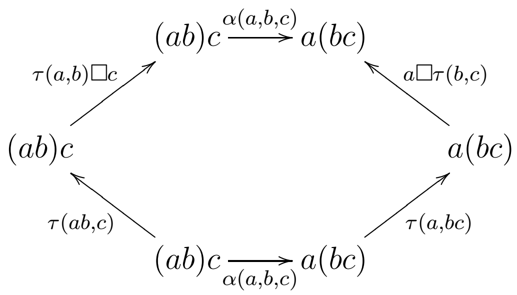
Remark. In graphical calculus, depicting morphisms as upward trajectories, birth and death are respectively:



Rigidity says zigzags can be straightened out:

The following nonstandard definition will not be used until later chapters.

Definition 4.1.8. Let $\mathcal{C}$ be a monoidal category with bifunctor $\square$ and unit 1. A *gauge automorphism* is a monoidal functor $\mathcal{C} \to \mathcal{C}$ of the form $(\mathrm{id}_C, \tau, \mathrm{id}_1)$. Explicitly, this means τ is a natural automorphism of $\square$ such that the diagram



commutes for all $a, b, c \in C$, and for all $x \in \operatorname{obj} C$,

$$\tau(1, x) = \mathrm{id}_{1\square x} \qquad\qquad \tau(x, 1) = \mathrm{id}_{x\square 1}$$

4.2 Fusion categories

Fusion categories are monoidal categories weighted down by axioms which drag them towards the realm of linear algebra.

Convention. Throughout this thesis, $\mathbb{F}$ is a field. For most physical purposes, $\mathbb{F} = \mathbb{C}$.

Definition 4.2.1. A category is $\mathbb{F}$-*linear* if it has finite biproducts and if it is enriched over the category of vector spaces over $\mathbb{F}$, i.e., if each morphism space forms a vector space such that composition of morphisms is $\mathbb{F}$-bilinear. A functor between $\mathbb{F}$-linear categories is $\mathbb{F}$-*linear* if it is likewise enriched, i.e., its restriction to each morphism space is $\mathbb{F}$-linear.

Definition 4.2.2. A *tensor category* is an $\mathbb{F}$-linear monoidal category, i.e., a monoidal category whose underlying category is $\mathbb{F}$-linear and whose bifunctor is bilinear on morphism spaces. An *oplax tensor functor* between tensor categories is an oplax monoidal $\mathbb{F}$-linear functor.

Definition 4.2.3. Let C be an $\mathbb{F}$-linear category. An object x of C is *simple* if every nonzero monomorphism into x is an isomorphism, and if x is not a zero object (biproduct of the empty tuple of objects). C is *semisimple* if every object is a biproduct of finitely many simple objects. An object of C is *strongly simple* if its endomorphism space is 1-dimensional. C is *strongly semisimple* if it is semisimple and every simple object is strongly semisimple.

Remark. Every semisimple linear category over an algebraically closed field is strongly semisimple.

Definition 4.2.4. A *fusion category* is a rigid strongly semisimple tensor category with simple monoidal unit.

Remark. According to the standard definition, fusion categories are finite, i.e., have only finitely many isomorphism classes of simple objects. In this thesis we do not assume finiteness.

4.3 Strict 2-categories

We assume familiarity with enriched categories, functors, and natural transformations (see Kelly [18]). The material in this section is in Leinster [20].

Definition 4.3.1. A *strict* 2*-category* is a category enriched over the category of categories and functors. A *strict* 2*-functor* between strict 2-categories is a functor likewise enriched.

A category consists of objects, and morphisms between objects; a 2-category consists additionally of 2-morphisms between morphisms. Standard examples of strict 2-categories are **Cat**, consisting of categories, functors, and natural transformations; and **Top**, consisting of topological spaces, continuous maps, and homotopy classes of homotopies. We met other examples in Sections 4.1 and 4.2:

Proposition 4.3.2. *Monoidal categories, oplax monoidal functors, and monoidal natural transformations form a strict* 2*-category.*

Proposition 4.3.3. *Fusion categories, oplax tensor functors, and monoidal natural transformations form a strict* 2*-category* **FCat**. *We will always assume oplax tensor functors between fusion categories take simple objects to simple objects. Note that every equivalence of fusion categories has this property.*

We will need a notion of equivalence of strict 2-categories. We first recall the notions of equivalence of categories, monoidal categories, and fusion categories.

Definition 4.3.4. Let X, Y be objects in a strict 2-category. A morphism $f \colon X \to Y$ is an *equivalence* from X to Y if there exists a morphism $g \colon Y \to X$ such that $fg \cong \mathrm{id}_Y$ and $gf \cong \mathrm{id}_X$.

Proposition 4.3.5. *Let F be a functor between categories, an oplax monoidal functor between monoidal categories, or an oplax tensor functor between fusion categories. Then F is an equivalence iff it is fully faithful (bijective on each morphism space) and essentially surjective on objects (hits every isomorphism class of objects in the target category).*

Proof. Follows from Propositions 1.1.2 and 1.2.14 of Leinster [20]. The backward implication requires the axiom of choice. □

Just as the collection $\mathrm{Fun}(C, D)$ of functors from a category C to a category D is itself a category, whose morphisms are natural transformations, so too the category $\mathrm{Fun}(\mathbf{C}, \mathbf{D})$ of strict 2-functors from a strict 2-category $\mathbf{C}$ to a strict 2-category $\mathbf{D}$ is itself a strict 2-category, whose 2-morphisms are so-called *modifications* (see Leinster [20] or Mac Lane [21]). Thus we are led to the following naive analogue of Definition 4.3.4.

Definition 4.3.6. Let $\mathbf{C}$ and $\mathbf{D}$ be strict 2-categories. A strict 2-functor $\mathbf{F} \colon \mathbf{C} \to \mathbf{D}$ is a *strict equivalence* from $\mathbf{C}$ to $\mathbf{D}$ if there exists a strict 2-functor $\mathbf{G} \colon \mathbf{D} \to \mathbf{C}$ such that $\mathbf{FG}$ is equivalent to $\mathrm{id}_{\mathbf{D}}$ and $\mathbf{GF}$ is equivalent to $\mathrm{id}_{\mathbf{C}}$.

Interestingly, Definition 4.3.6 is too strict: for **F** to be an *equivalence* of strict 2-categories, **G** should be allowed to be a mere *weak 2-functor* (Leinster [20]). Fortunately, the naive analogue of Proposition 4.3.5 turns out to be the correct notion:

Proposition 4.3.7. *A strict 2-functor between strict 2-categories is an equivalence iff it is a local equivalence (restricts to an equivalence on each morphism category), and essentially surjective on objects (hits all equivalence classes of objects).*

Proof. Similar to the proof of Proposition 4.3.5; see Leinster [20]. □

For us, the decategorification of Proposition 4.3.7 will usually suffice:

Proposition 4.3.8. *An equivalence of strict 2-categories induces bijections of objects up to equivalence, of morphisms up to isomorphism, and of 2-morphisms up to equality.*

4.4 Labeled fusion categories

The heart of this thesis is the correspondence between fusion rules and fusion categories, described here with a nonstandard formalism.

Definition 4.4.1. A *labeling* on a semisimple $\mathbb{F}$-linear category C is a map of sets $h\colon L \to \operatorname{obj} C$ inducing a bijection from L to the set of isomorphism classes of simple objects in C. Elements of L are called *labels*, and may be identified with the corresponding objects of C. A *labeled* category is one equipped with a labeling; if the category is monoidal, its monoidal unit must be a label. A *labeled* functor between labeled categories carries labels to labels.

Proposition 4.4.2. *Let $\mathcal{C}$ be a fusion category labeled by a set L. Then L has a canonical fusion rule structure (Definition 3.1.3): for $x, y \in L$, define $xy \in \mathbb{N}^L$ by $N_z^{xy} = \dim_{\mathbb{F}} \operatorname{mor}(z, x\Box y)$, where $\Box$ is the monoidal product of $\mathcal{C}$. We call $\mathcal{C}$ a fusion category* on L.

Proof. This is a standard fact. ☐

Thus every fusion category determines an isomorphism class of fusion rules. Labeling is a technical convenience picking out a representative of this isomorphism class. The inverse problem, reconstructing fusion categories from fusion rules, is stubbornly interesting. The following fundamental result is more general than stated here.

Theorem 4.4.3 (Ocneanu rigidity). *Assume the ground field is algebraically closed and has characteristic* 0. *There are only finitely many equivalence classes of fusion categories on a given finite fusion rule. Between any two finite fusion categories, there are only finitely many isomorphism classes of tensor functors.*

Proof. See Etingof, Nikshych, and Ostrik [12]. ☐

To recover fusion categories from fusion rules, we now start the journey from fusion categories to polynomial equations—a 3-lane highway paved with indices—to be continued in the next two chapters.

Lemma 4.4.4. *Let $\mathcal{C}$ and $\tilde{\mathcal{C}}$ be monoidal categories with underlying categories C and $\tilde{C}$ respectively, and let $\mathcal{S}\colon \mathcal{C} \to \tilde{\mathcal{C}}$ be an oplax monoidal functor with underlying functor S. Let $\hat{T}\colon \operatorname{obj} C \to \operatorname{obj} \tilde{C}$, and for each $x \in \operatorname{obj} C$, let $\eta(x)\colon S(x) \xrightarrow{\cong} \hat{T}(x)$. Then there* *is a unique oplax monoidal functor $\mathcal{T}\colon \mathcal{C} \to \tilde{\mathcal{C}}$ whose underlying functor agrees with $\hat{T}$, such that η is a monoidal natural isomorphism $\mathcal{S} \Rightarrow \mathcal{T}$.*

Proof. Let $\mathcal{T} = (T, T_2, T_0)$, where T agrees with $\hat{T}$ on objects and is defined on morphisms such that $\eta\colon S \to T$ is a natural isomorphism,

$$T_2 = (\eta \tilde{\Box} \eta) \cdot S_2 \cdot (\eta^{-1} \circ (\Box))\colon \quad T \circ (\Box) \Rightarrow T \tilde{\Box} T,$$

and $T_0 = S_0 \eta(1)^{-1}\colon T1 \to \tilde{1}$, where $\cdot$ denotes vertical composition of natural transformations. It is routine to check $\mathcal{T}$ is an oplax monoidal functor. $\square$

Lemma 4.4.5. *Let $\mathcal{C}$ and $\mathcal{D}$ be labeled fusion categories, and let $\mathcal{S}\colon \mathcal{C} \to \mathcal{D}$ be an oplax tensor functor which takes simple objects to simple objects. Then $\mathcal{S}$ is monoidally naturally isomorphic to a labeled oplax tensor functor.*

Proof. Let C and D be the underlying categories of $\mathcal{C}$ and $\mathcal{D}$ respectively, and define $\hat{T}\colon \operatorname{obj} C \to \operatorname{obj} D$ as follows. Let S be the underlying functor of $\mathcal{S}$, and consider $x \in \operatorname{obj} C$. If x is a label of C, then x is simple, implying $S(x)$ is simple, implying $S(x)$ is isomorphic to a unique label $\hat{T}(x)$ of D. Otherwise let $\hat{T}(x) = S(x)$. For each $x \in \operatorname{obj} C$, let $\eta(x)\colon S(x) \xrightarrow{\cong} \hat{T}(x)$. Lemma 4.4.4 yields an oplax tensor functor $\mathcal{T}$ whose underlying functor agrees with $\hat{T}$, such that η is a monoidal natural isomorphism $\mathcal{S} \Rightarrow \mathcal{T}$. By construction $\mathcal{T}$ is labeled. $\square$

Definition 4.4.6. Let **FCat** be the strict 2-category of fusion categories, oplax tensor functors which take simple objects to simple objects, and monoidal natural transformations, and let **LFCat** be the strict 2-category of labeled fusion categories, labeled oplax tensor functors, and monoidal natural transformations.

Proposition 4.4.7. *The forgetful strict* 2*-functor* **LFCat** $\to$ **FCat** *is an equivalence of strict* 2*-categories. In other words, any two equivalent fusion categories with arbitrary labels are equivalent via labeled tensor functors, and any oplax tensor functor between labeled fusion categories which takes simple objects to simple objects is monoidally naturally isomorphic to a labeled oplax tensor functor.*

Proof. Let $\mathbf{\Gamma}\colon \mathbf{LFCat} \to \mathbf{FCat}$ be the forgetful strict 2-functor. By Proposition 4.3.7, it suffices to show $\mathbf{\Gamma}$ is an essentially surjective local equivalence. By definition $\mathbf{\Gamma}$ is surjective on objects, a fortiori essentially surjective. For local equivalence, let $\mathcal{C}, \mathcal{D} \in \operatorname{obj} \mathbf{LFCat}$ and let $\mathbf{\Gamma}_{\mathcal{C}}^{\mathcal{D}}$ be the label-forgetting functor defined on the category of labeled oplax tensor functors $\mathcal{C} \to \mathcal{D}$. Then $\mathbf{\Gamma}_{\mathcal{C}}^{\mathcal{D}}$ is essentially surjective by Lemma 4.4.5, and fully faithful by definition, as there is no notion of a labeled natural transformation. By Proposition 4.3.5, $\mathbf{\Gamma}_{\mathcal{C}}^{\mathcal{D}}$ is an equivalence. Thus $\mathbf{\Gamma}$ is an equivalence. □

Chapter 5

Fusion systems

Fusion categories have so many axioms that they can be reduced to linear algebra. This process is useful for computations such as finding all equivalence classes of fusion categories on a given fusion rule, or determining whether a given fusion category has braidings. The linear algebraic essence of a fusion category is packaged in a *fusion system*, following Yamagami [31], who describes fusion rules as the "skeletons" and fusion systems as the "flesh". Yamagami proved that fusion categories and fusion systems are in 1-1 correspondence modulo equivalence. In this chapter we categorify the correspondence to an equivalence of 2-categories, as independently implicitly suggested by Kitaev [19]. As in the last chapter, we work over an arbitrary field $\mathbb{F}$.

Definition 5.0.8. A *fusion system* is a sextuple $(L, V, 1, F, \lambda, \rho)$, where L is a fusion rule; V assigns a finite-dimensional vector space $V_r^{x,y}$, called a *splitting space*, to each $x, y, r \in L$; $1 \in L$; F assigns an isomorphism $F_r^{x,y,z} \colon V_r^{x,y;z} \to V_r^{x;y,z}$ to each $x, y, z, r \in$

L, where

$$V_r^{x,y;z} = \bigoplus_u V_u^{x,y} \otimes V_r^{u,z} \qquad\qquad V_r^{x;y,z} = \bigoplus_v V_v^{y,z} \otimes V_r^{x,v}$$

and λ, ρ assign nonzero vectors $\lambda_x \in V_x^{1,x}$, $\rho_x \in V_x^{x,1}$ to each $x \in L$; satisfying the following four axioms:

Admissibility: L is a fusion rule with each multiplicity $N_r^{xy} = \dim V_r^{x,y}$.

Pentagon axiom: This diagram commutes for $w, x, y, z, p, q, u, v, r \in L$:

$$\begin{array}{ccc}
\bigoplus_{p,q} V_p^{w,x} \otimes V_q^{y,z} \otimes V_r^{p,q} & \xrightarrow{\text{canon.}} & \bigoplus_{q,p} V_q^{y,z} \otimes V_p^{w,x} \otimes V_r^{p,q} \\
{\scriptstyle \bigoplus_p \mathrm{id} \otimes F_r^{p,y,z}} \uparrow & & \downarrow {\scriptstyle \bigoplus_q \mathrm{id} \otimes F_r^{w,x,q}} \\
\bigoplus_{p,u} V_p^{w,x} \otimes V_u^{p,y} \otimes V_r^{u,z} & & \bigoplus_{q,v} V_q^{y,z} \otimes V_v^{x,q} \otimes V_r^{w,v} \\
{\scriptstyle \bigoplus_u F_u^{w,x,y} \otimes \mathrm{id}} \downarrow & & \uparrow {\scriptstyle \bigoplus_v F_v^{x,y,z} \otimes \mathrm{id}} \\
\bigoplus_{s,u} V_s^{x,y} \otimes V_u^{w,s} \otimes V_r^{u,z} & \xrightarrow[\bigoplus_s \mathrm{id} \otimes F_r^{w,s,z}]{} & \bigoplus_{s,v} V_s^{x,y} \otimes V_v^{s,z} \otimes V_r^{w,v}
\end{array}$$

Triangle axiom: $F_r^{x,1,y}(\rho_x \otimes \mu) = (\lambda_y \otimes \mu)$ for $x, y, r \in L$ and $\mu \in V_r^{x,y}$.

Rigidity: For each $r \in L$ there exist $b \in V_1^{r,\bar{r}}$ and $d \in (V_1^{\bar{r},r})^*$ such that

$$(d \otimes \rho_r^*)(F_r^{r,\bar{r},r}(b \otimes \lambda_r)) = (d \otimes \lambda_{\bar{r}}^*)((F_{\bar{r}}^{\bar{r},r,\bar{r}})^{-1}(b \otimes \rho_{\bar{r}})) = 1.$$

Definition 5.0.9. Let $\mathcal{M} = (L, V, 1, F, \lambda, \rho)$ and $\tilde{\mathcal{M}} = (\tilde{L}, \tilde{V}, \tilde{1}, \tilde{F}, \tilde{\lambda}, \tilde{\rho})$ be fusion systems. A *morphism* from $\mathcal{M}$ to $\tilde{\mathcal{M}}$ is triple (t, ξ, ξ_0), where $t \colon L \to \tilde{L}$ is a map of sets; ξ assigns a linear map $\xi_r^{x,y} \colon V_r^{x,y} \to \tilde{V}_{t(r)}^{t(x),t(y)}$ to each triple $x, y, r \in L$; and $\xi_0 \in \mathbb{F}$; satisfying the following two axioms:

Rectangle axiom: For $x, y, z, r \in L$, the following diagram commutes, writing $\tilde{x} = t(x)$, $\tilde{y} = t(y)$, etc.:

$$\begin{array}{ccc} \tilde{V}_{\tilde{r}}^{\tilde{x},\tilde{y};\tilde{z}} & \xrightarrow{\tilde{F}_{\tilde{r}}^{\tilde{x},\tilde{y},\tilde{z}}} & \tilde{V}_{\tilde{r}}^{\tilde{x};\tilde{y},\tilde{z}} \\ {\scriptstyle \bigoplus_u \xi_u^{x,y} \otimes \xi_r^{u,z}} \uparrow & & \uparrow {\scriptstyle \bigoplus_v \xi_v^{y,z} \otimes \xi_r^{x,v}} \\ V_r^{x,y;z} & \xrightarrow[F_r^{x,y,z}]{} & V_r^{x;y,z} \end{array}$$

Normalization: $\xi_0 \xi_r^{1r}(\lambda_r) = \tilde{\lambda}_{\tilde{r}}$ and $\xi_0 \xi_r^{r1}(\rho_r) = \tilde{\rho}_{\tilde{r}}$ for all $r \in L$.

Remark. Yamagami's polygonal notation [31] is a vivid way to visualize the pentagon and rectangle axioms using Pachner moves on triangulated polygons.

Definition 5.0.10. Let $S = (s, \nu, \nu_0)$ and $T = (t, \xi, \xi_0)$ be morphisms $\mathcal{M} \to \tilde{\mathcal{M}}$ of fusion systems, and let L be the label set of $\mathcal{M}$. A 2-*morphism* $\zeta \colon S \Rightarrow T$ is a map $\zeta \colon L \to \mathbb{F}$ such that

- For all $r \in L$, we require $\zeta(r) = 0$ unless $s(r) = t(r)$,
- $\zeta(x)\zeta(y)\nu_r^{x,y} = \zeta(r)\xi_r^{x,y}$ for all $x, y, r \in L$, where if one side is 0 then so must be the other,
- $\nu_0 = \xi_0 \zeta(1)$.

Proposition 5.0.11. *Fusion systems with their* 1- *and* 2-*morphisms form a strict* 2-*category* **FSys**.

Proof. Routine diagram check. □

Lemma 5.0.12. *A morphism* (t, ξ, ξ_0) *between fusion systems is an equivalence iff it is an isomorphism iff* t *is a bijection,* $\xi_r^{x,y}$ *is a vector space isomorphism for all labels* r, x, y, *and* $\xi_0 \in \mathbb{F}^\times$.

Proof. Follows easily from Definition 5.0.10. □

Lemma 5.0.13. *Let $S\colon \mathcal{M} \to \tilde{\mathcal{M}}$ be a morphism of fusion systems, L be the label set of $\mathcal{M}$, and $\zeta\colon L \to \mathbb{F}^\times$. There is a unique morphism $T\colon \mathcal{M} \to \tilde{\mathcal{M}}$ such that ζ is a 2-isomorphism from S to T.*

Proof. Follows easily from Definitions 5.0.10 and 5.0.9. □

Lemma 5.0.14. *Every equivalence of fusion systems is isomorphic to one of the form $(t, \xi, 1)$.*

Proof. Let $S = (s, \nu, \nu_0)\colon \mathcal{M} \to \tilde{\mathcal{M}}$ be an equivalence of fusion systems. Then $\nu_0 \in \mathbb{F}^\times$ by Lemma 5.0.12. Let L be the label set of $\mathcal{M}$, and define $\zeta\colon L \to \mathbb{F}^\times$ by $\zeta(1) = \nu_0$ and $\zeta(x) = 1$ for $x \neq 1$. By Lemma 5.0.13, ζ is a 2-isomorphism from S to a morphism of the required form. □

The following definition is useful for testing when two fusion systems are equivalent (for Chapter 6), and in computing twines (Chapter 7).

Definition 5.0.15. A *gauge transformation* between two fusion systems on the same label set L, rendering them *gauge equivalent*, is a morphism of the form $(\mathrm{id}_L, \xi, 1_{\mathbb{F}})$, with $\xi_r^{x,y}$ a vector space isomorphism for all $r, x, y \in L$.

The following lemma reconciles our categorical notion of equivalence of fusion systems with Yamagami's [31].

Lemma 5.0.16. *Two fusion systems are equivalent iff they are gauge equivalent up to a bijection of labels.*

Proof. Let $\mathcal{M} = (L, V, 1, F, \lambda, \rho)$ be a fusion system, $\tilde{L}$ be a set, and $t\colon \tilde{L} \to L$ be a bijection. $\mathcal{M}$ *relabeled* by t is the fusion system $\tilde{\mathcal{M}} = (\tilde{L}, \tilde{V}, \tilde{1}, \tilde{F}, \tilde{\lambda}, \tilde{\rho})$ defined as follows. For $r, x, y, z \in L$, write $\tilde{r} = t^{-1}(r)$, $\tilde{x} = t^{-1}(x)$, etc. Let $\tilde{V}_{\tilde{r}}^{\tilde{x},\tilde{y}} = V_r^{x,y}$, and define $\tilde{F}, \tilde{\lambda}, \tilde{\rho}$ similarly. Thus $\tilde{\mathcal{M}}$ is defined. Letting $\xi_{\tilde{r}}^{\tilde{x},\tilde{y}}$ be the identity on $V_r^{x,y}$, we have an equivalence $(t, \xi, 1)\colon \tilde{\mathcal{M}} \to \mathcal{M}$, called *relabeling* by t. The backward implication of Lemma 5.0.16 is now clear: a gauge transformation composed with a relabeling remains an equivalence.

For the forward implication, suppose $S = (s, \nu, \nu_0)\colon \mathcal{M} \to \mathcal{N}$ is an equivalence of fusion systems. By Lemma 5.0.14 we may take $\nu_0 = 1$. Let $R\colon \tilde{\mathcal{N}} \to \mathcal{N}$ be relabeling by s. Then $S = R(R^{-1}S)$ is a gauge transformation followed by a relabeling, as desired. □

5.1 Fusion categories and fusion systems

In this section we state the equivalence between the 2-categories of labeled fusion categories **LFCat** and of fusion systems **FSys** (Theorem 5.1.4). The constructions here will seldom be needed in later chapters: it will be enough to know that fusion categories, oplax tensor functors respecting simplicity of objects, and monoidal natural transformations are somehow faithfully captured by the auxiliary notions of fusion systems and their 1- and 2-morphisms (Corollary 5.1.5). Proposition 4.4.7 gives us the luxury of working with **LFCat** rather that **FCat**, i.e., of assuming all fusion categories and oplax tensor functors are labeled.

Definition 5.1.1. Define a strict 2-functor $\mathbf{\Psi}\colon \mathbf{LFCat} \to \mathbf{FSys}$ as follows. First let $\mathcal{C} = (C, \Box, 1, \alpha, \lambda, \rho)$ be a fusion category labeled by a set L. We construct a fusion system $\mathbf{\Psi}(\mathcal{C}) = (L, V, 1, F, \lambda', \rho')$. For $x, y, r \in L$, let $V_r^{x,y} = \mathrm{mor}_C(r, x\Box y)$. For $r, x, y, z \in L$, via the well-known isomorphisms

$$V_r^{x,y;z} \cong \mathrm{mor}_C(r, (x\Box y)\Box z) \qquad\qquad V_r^{x;y,z} \cong \mathrm{mor}_C(r, x\Box(y\Box z))$$

let $F_r^{x,y,z}$ be postcomposition by $\alpha(x, y, z)$. For $x \in L$, let $\lambda'_x = \lambda_x$ and $\rho'_x = \rho_x$. Thus $\mathbf{\Psi}$ is defined on objects.

Now suppose $\mathcal{T} = (T, T_2, T_0)\colon \mathcal{C} \to \tilde{\mathcal{C}}$ is a labeled oplax tensor functor between fusion categories labeled by sets L and $\tilde{L}$ respectively. Let

$$\mathbf{\Psi}(\mathcal{T}) = (t, \xi, \xi_0)\colon (L, V, 1, F, \lambda', \rho') \to (\tilde{L}, \tilde{V}, \tilde{1}, \tilde{F}, \tilde{\lambda}', \tilde{\rho}')$$

be the following morphism between the corresponding fusion systems. Let $t\colon L \to \tilde{L}$ agree with T. For $x, y, r \in L$, define $\xi_r^{x,y}\colon V_r^{x,y} \to \tilde{V}_{t(r)}^{t(x),t(y)}$ by $\mu \mapsto T_2(x, y)T(\mu)$ for $\mu \in V_r^{x,y}$. Define $\xi_0 \in \mathbb{F}$ by $T_0 = \xi_0 \mathrm{id}_{\tilde{1}}$. Thus $\mathbf{\Psi}$ is defined on morphisms.

Finally, suppose η is a monoidal natural transformation between labeled oplax tensor functors defined on a fusion category with label set L. Let $\mathbf{\Psi}(\eta)$ be the 2-morphism ζ between the corresponding morphisms $(s, \nu, \nu_0), (t, \xi, \xi_0)$ such that for $x \in L$,

$$\eta(x) = \begin{cases} \zeta(x)\mathrm{id}_{t(x)} & \text{if } s(x) = t(x), \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\mathbf{\Psi}\colon \mathbf{LFCat} \to \mathbf{FSys}$ is defined.

Definition 5.1.2. Let L be a set. An L-*graded vector space* is a function X assigning a vector space X_x to each $x \in L$. The *dimension* of X is the sum of the dimensions of the X_x for $x \in L$. A *graded linear map* $\mu\colon X \to Y$ of L-graded vector spaces assigns a linear map $\mu_x\colon X_x \to Y_x$ to each $x \in L$. Let $\mathrm{Grd}(L)$ be the category of finite-dimensional L-graded vector spaces and graded linear maps. Note $\mathrm{Grd}(L)$ is strongly semisimple $\mathbb{F}$-linear (Definition 4.2.3) and canonically labeled: each $x \in L$ may be regarded as an object of $\mathrm{Grd}(L)$ via

$$x_y = \begin{cases} \mathbb{F} & \text{if } y = x, \\ \{0\} & \text{otherwise.} \end{cases}$$

A map $t\colon L \to \tilde{L}$ of sets induces a functor $\mathrm{Grd}(t)\colon \mathrm{Grd}(L) \to \mathrm{Grd}(\tilde{L})$ as follows, abbreviated here as $T\colon C \to \tilde{C}$. For $X \in \operatorname{obj} C$ and $k \in \tilde{L}$, define $T(X) \in \operatorname{obj} \tilde{C}$ by $T(X)_k = \bigoplus_{r \in t^{-1}(k)} X_r$ for $k \in \tilde{L}$. For $\mu \in \operatorname{mor}_C(X, Y)$, define $T(\mu) \in \operatorname{mor}_{\tilde{C}}(T(X), T(Y))$ by $T(\mu)_k = \bigoplus_{r \in t^{-1}(k)} \mu_r$ for $k \in \tilde{L}$. Thus Grd is a functor **Set** $\to$ **Cat**.

Definition 5.1.3. Define a strict 2-functor $\mathbf{\Phi}\colon \mathbf{FSys} \to \mathbf{LFCat}$ as follows. First, given a fusion system $\mathcal{M} = (L, V, 1, F, \lambda', \rho')$, define a labeled fusion category $\mathbf{\Phi}(\mathcal{M}) = \mathcal{C} = (C, \Box, 1, \alpha, \lambda, \rho)$ as follows, following Yamagami [31] and Kitaev [19]. Let $C = \mathrm{Grd}(L)$. For $X, Y \in \operatorname{obj} C$, define $X \Box Y$ as follows. For $r \in L$, let

$$(X \Box Y)_r = \bigoplus_{x,y \in L} X_x \otimes Y_y \otimes V_r^{x,y}$$

where $\otimes = \otimes_{\mathbb{F}}$ denotes the ordinary tensor product of vector spaces. For $\tilde{X}, \tilde{Y} \in \operatorname{obj} C$ and $\mu \in \operatorname{mor}_C(X, \tilde{X})$ and $\nu \in \operatorname{mor}_C(Y, \tilde{Y})$, define $\mu \Box \nu \in \operatorname{mor}_C(X \Box Y, \tilde{X} \Box \tilde{Y})$ by

$$(\mu \Box \nu)_r = \bigoplus_{x,y \in L} \mu_x \otimes \nu_y \otimes V_r^{x,y}$$

for $r \in L$. Let the monoidal unit $1_{\mathcal{C}}$ correspond to the label 1. Via the canonical isomorphisms

$$((X \Box Y) \Box Z)_r \cong \bigoplus_{x,y,z \in L} X_x \otimes Y_y \otimes Z_z \otimes V_r^{x,y;z}$$

$$(X \Box (Y \Box Z))_r \cong \bigoplus_{x,y,z \in L} X_x \otimes Y_y \otimes Z_z \otimes V_r^{x;y,z}$$

define $\alpha(X, Y, Z) \colon (X \Box Y) \Box Z \to X \Box (Y \Box Z)$ by

$$\alpha(X, Y, Z)_r = \bigoplus_{x,y,z \in L} X_x \otimes Y_y \otimes Z_z \otimes F_r^{x,y,z}$$

Via the canonical isomorphisms

$$(1_{\mathcal{C}} \Box X)_r \cong X_r \otimes V_r^{1,r} \qquad (X \Box 1_{\mathcal{C}})_r \cong X_r \otimes V_r^{r,1}$$

define $\lambda_X \colon X \to 1_{\mathcal{C}} \Box X$ and $\rho_X \colon X \to X \Box 1_{\mathcal{C}}$ by

$$(\lambda_X)_r(v) = v \otimes \lambda'_r \qquad (\rho_X)_r(v) = v \otimes \rho'_r$$

for $v \in X_r$. Thus $\mathbf{\Phi}$ is defined on objects.

Now suppose $(t, \xi, \xi_0) \colon (L, V, 1, F, \lambda', \rho') \to (\tilde{L}, \tilde{V}, \tilde{1}, \tilde{F}, \tilde{\lambda}', \tilde{\rho}')$ is a morphism of fusion systems. Let $\mathbf{\Phi}(t, \xi, \xi_0) = \mathcal{T} = (T, T_2, T_0) \colon \mathcal{C} \to \tilde{\mathcal{C}}$ be the following labeled oplax tensor functor between the corresponding labeled fusion categories. Let $T = \operatorname{Grd}(t)$.

For T_2, let $X, Y \in \operatorname{obj} C$ and $k \in \tilde{L}$. Via

$$(T(X)\tilde{\square}T(Y))_k \overset{\text{can.}}{\cong} \bigoplus_{x,y\in L} X_x \otimes X_y \otimes \tilde{V}_k^{t(x),t(y)}$$

$$T(X\square Y)_k = \bigoplus_{r\in t^{-1}(k)} \bigoplus_{x,y\in L} X_x \otimes X_y \otimes V_r^{x,y}$$

define $T_2(X,Y)\colon T(X\square Y) \to T(X)\tilde{\square}T(Y)$ by

$$T_2(X,Y)_k = \sum_{r\in t^{-1}(k)} \bigoplus_{x,y\in L} X_x \otimes X_y \otimes \xi_r^{x,y}$$

Via the canonical isomorphism $T(1_{\mathcal{C}}) \cong 1_{\tilde{\mathcal{C}}}$, let T_0 be multiplication by ξ_0. Thus $\mathbf{\Phi}$ is defined on morphisms.

Finally, suppose ζ is a 2-morphism from (s, ν, ν_0) to (t, ξ, ξ_0). Let $\eta = \mathbf{\Phi}(\zeta)$ be the following monoidal natural transformation between the corresponding labeled oplax tensor functors, with underlying functors S and T respectively. For $X \in \operatorname{obj} C$, define $\eta(X) \in \operatorname{mor}_{\tilde{C}}(S(X), T(X))$ as follows. For $k \in \tilde{L}$, to define $\eta(X)_k\colon S(X)_k \to T(X)_k$, since

$$S(X)_k = \bigoplus_{p\in s^{-1}(k)} X_p \qquad\qquad T(X)_k = \bigoplus_{q\in t^{-1}(k)} X_q$$

it suffices to define a linear map $\mu\colon X_p \to X_q$ for each $p \in s^{-1}(k)$ and $q \in t^{-1}(k)$. Let μ be multiplication by $\zeta(p)$ if $p = q$, and 0 otherwise. Thus $\mathbf{\Phi}\colon \mathbf{FSys} \to \mathbf{LFCat}$ is defined.

Theorem 5.1.4. *The strict 2-functors $\mathbf{\Psi}$ and $\mathbf{\Phi}$ of Definitions 5.1.1 and 5.1.3 form an equivalence between the strict 2-category* **LFCat** *of labeled fusion categories and the strict 2-category* **FCat** *of fusion systems.*

Proof. Routine diagram check. Requires the axiom of choice, to fix biproducts of all objects of all fusion categories. □

Corollary 5.1.5.

1. *Up to equivalence, fusion categories are in 1-1 correspondence with fusion systems.*

2. *Let $\mathcal{C}$ and $\mathcal{D}$ be fusion categories, corresponding to fusion systems $\mathcal{M}$ and $\mathcal{N}$ respectively. Up to isomorphism, oplax tensor functors $\mathcal{C} \to \mathcal{D}$ taking simple objects to simple objects are in 1-1 correspondence with morphisms $\mathcal{M} \to \mathcal{N}$.*

3. *Let $F, G\colon \mathcal{C} \to \mathcal{D}$ be such functors, corresponding to morphisms $f, g\colon \mathcal{M} \to \mathcal{N}$ respectively. Monoidal natural transformations $F \Rightarrow D$ are in 1-1 correspondence with 2-morphisms $f \Rightarrow g$.*

Proof. Theorem 5.1.4, Proposition 4.4.7, and Proposition 4.3.8. Via Lemma 5.0.16, the first statement alternatively follows from Proposition 1.1 and Lemma 2.2 of Yamagami [31]. □

Chapter 6

6j fusion systems

Fusion systems are the linear algebraic data of fusion categories, but for computations we must get our hands dirty in the ground field by picking bases of splitting spaces. It is convenient to encapsulate the result in a third and most concrete manifestation of a fusion category, which we call a *6j fusion system*, a point in the affine variety cut out by the pentagon equation on a given fusion rule.

It would be straightforward to continue in the spirit of the last two chapters by defining the 2-category of 6j fusion systems, but for notational simplicity we only define what will be needed for computations. In particular, we only treat the multiplicity-free case, since most of the fusion rules we consider are multiplicity-free. As always, $\mathbb{F}$ is a field.

Definition 6.0.6. Let L be a multiplicity-free fusion rule. A *6j fusion system* on L is map $f\colon L^6 \to \mathbb{F}$, assigning a *6j symbol* f^{xyz}_{urv} to each sextuple (x, y, z, u, r, v), such that for $w, x, y, z, p, u, r, v, q \in L$,

Admissibility: $f^{xyz}_{urv} = 0$ unless (x, y, z, u, r, v) is *admissible*, i.e., $u \in xy$ and $v \in yz$ and $r \in uz \cap xv$.

Invertibility: Each *recoupling matrix* $F^{xyz}_r = (f^{xyz}_{urv})_{v,u}$ is invertible, where v, u range over all elements making (x, y, z, u, r, v) admissible.

Pentagon axiom: P^{wxyz}_{purvq} : $\quad f^{wxq}_{prv} f^{pyz}_{urq} = \sum_{s\in L} f^{xyz}_{svq} f^{wsz}_{urv} f^{wxy}_{pus}$

Triangle axiom: F^{x1y}_r is the identity matrix, 1×1 or 0×0.

Rigidity: $f^{r\bar{r}r}_{1r1} = ((F^{\bar{r}r\bar{r}}_{\bar{r}})^{-1})_{11} \neq 0$.

Definition 6.0.7. Let f and $\tilde{f}$ be 6j fusion systems on a multiplicity-free fusion rule L. A *gauge transformation* from f to $\tilde{f}$, rendering them *gauge equivalent*, is a map $\xi \colon L^3 \to \mathbb{F}$, written $(x, y, r) \mapsto \xi^{xy}_r$, such that for $x, y, z, u, r, v \in L$,

Invertibility: $\xi^{xy}_r \neq 0$ iff $r \in xy$.

Rectangle axiom: G^{xyz}_{urv} : $\quad f^{xyz}_{urv} \xi^{yz}_v \xi^{xv}_r = \xi^{xy}_u \xi^{uz}_r \tilde{f}^{xyz}_{urv}$

Normalization: $\xi^{1r}_r = \xi^{r1}_r = 1$.

We say f and $\tilde{f}$ are *equivalent* if they are gauge equivalent up to relabeling by an automorphism of L.

Definition 6.0.8. Let f be a 6j fusion system on a multiplicity-free fusion rule L, and let $\xi, \tilde{\xi}$ be gauge automorphisms of f, i.e., gauge transformations from f to itself. A 2-*isomorphism* from ξ to $\tilde{\xi}$ is a function $\zeta \colon L \to \mathbb{F}^\times$ such that $\zeta(x)\zeta(y)\xi^{x,y}_r = \zeta(r)\tilde{\xi}^{x,y}_r$ for all $x, y, r \in L$.

Theorem 6.0.9. *Let L be a multiplicity-free fusion rule.*

1. *Up to equivalence, fusion categories on L are in 1-1 correspondence with 6j fusion systems on L.*

2. *Let $\mathcal{C}$ be a fusion category on L, and let f be a corresponding 6j fusion system on L. Up to isomorphism, gauge automorphisms of $\mathcal{C}$ (Definition 4.1.8) are in 1-1 correspondence with gauge automorphisms of f.*

3. *Let $\mathcal{T}$ be a gauge automorphism of $\mathcal{C}$, and ξ a corresponding gauge automorphism of f. Automorphisms of $\mathcal{T}$ (Definition 4.1.6) are in 1-1 correspondence with automorphisms of ξ (Definition 6.0.8).*

Proof. It is straightforward to convert between fusion systems and 6j fusion systems via bases of splitting spaces $V_r^{x,y}$. Corollary 5.1.5 is a bridge between fusion systems and fusion categories. □

We pause to record two lemmas for later use.

Lemma 6.0.10. $F_r^{1xy} = F_r^{xy1} = \mathrm{Id}$ *for F as in Definition 6.0.6 and $x, y, r \in L$.*

Proof. $P^{\bar{r},1,x,y}_{\bar{r},\bar{y},1,r,r}$ and $P^{x,y,1,\bar{r}}_{r,r,1,\bar{x},\bar{r}}$ and the triangle axiom. □

Lemma 6.0.11. *If f is a 6j fusion system and ξ satisfies the invertibility and normalization axioms of a gauge transformation, then $\tilde{f}$ defined by the rectangle axiom is a 6j fusion system, and ξ is a gauge transformation from f to $\tilde{f}$.*

Proof. Routine. □

6.1 Fusion categories and group cohomology

It is well-known that third group cohomology classifies fusion categories on groups. In this section we see where second and first cohomology fit into the picture: they classify gauge automorphisms and automorphisms thereof (Theorem 6.1.2).

The next definition's generality will be needed in Section 6.2.

Definition 6.1.1. Let S be a group, U an S-bimodule, and $n \in \{0,1,2,3\}$. An *n-cochain* is a function $S^n \to U$. We identify 0-cochains with elements of U. The *left and right coboundary operators* $\grave{\delta}, \acute{\delta}\colon U^{S^n} \to U^{S^{n+1}}$ are defined as follows: for an n-cochain h_n, define $\grave{\delta} h_n, \acute{\delta} h_n\colon S^{n+1} \to U$ by

$$\grave{\delta} h_0(a) = \frac{{}^a h_0}{h_0} \qquad \acute{\delta} h_0(a) = \frac{h_0^a}{h_0}$$

$$\grave{\delta} h_1(a,b) = \frac{h_1(a)\, {}^a h_1(b)}{h_1(ab)} \qquad \acute{\delta} h_1(a,b) = \frac{h_1^b(a) h_1(b)}{h_1(ab)}$$

$$\grave{\delta} h_2(a,b,c) = \frac{h_2(a,bc)\, {}^a h_2(b,c)}{h_2(a,b) h_2(ab,c)} \qquad \acute{\delta} h_2(a,b,c) = \frac{h_2(a,bc) h_2(b,c)}{h_2^c(a,b) h_2(ab,c)}$$

$$\grave{\delta} h_3(a,b,c,d) = \frac{h_3(a,b,c) h_3(a,bc,d)\, {}^a h_3(b,c,d)}{h_3(a,b,cd) h_3(ab,c,d)} \qquad \acute{\delta} h_3(a,b,c,d) = \cdots$$

Here U is written multiplicatively; left and right exponentiation of cochains denotes the S-actions on U. If it is known that $\grave{\delta} h_n = \acute{\delta} h_n$, we write δh_n.

A *normalized* cochain is 1 whenever any argument is 1. An n-cochain h is an *n-cocycle* if $\grave{\delta} h = 1$ and an *n-coboundary* if $h = \grave{\delta} k$ for some k. $H^n(S,U)$ is the abelian group of normalized n-cochains modulo normalized n-coboundaries.

Remark. Our nonstandard normalization condition is harmless: by Lemma 15.7.1 of Hall [16] every cocycle is cohomologous to a normalized cocycle, and by Lemma 15.7.2 every normalized coboundary is the coboundary of a normalized cochain.

Theorem 6.1.2. *View $\mathbb{F}^\times$ as a trivial module over a group G. Then*

1. *$H^3(G, \mathbb{F}^\times)$ is in 1-1 correspondence with 6j fusion systems on G up to gauge equivalence. $H^3(G, \mathbb{F}^\times)/\mathrm{aut}G$ is in 1-1 correspondence with fusion categories (or systems) on G up to equivalence.*

2. *$H^2(G, \mathbb{F}^\times)$ is in 1-1 correspondence with isomorphism classes of gauge automorphisms of any fusion category (or system) on G (Definition 6.0.8).*

3. *$H^1(G, \mathbb{F}^\times)$ is in 1-1 correspondence with automorphisms of any gauge automorphism of any fusion category (or system) on G.*

Proof. For the first statement, identify maps $f\colon G^6 \to \mathbb{F}$ satisfying the admissibility axiom of Definition 6.0.6 with 3-cochains, and maps $\xi\colon G^3 \to \mathbb{F}$ satisfying the invertibility axiom of Definition 6.0.7 with 2-cochains, via

$$f^{a,b,c}_{u,r,v} = \delta_{u,ab}\delta_{r,abc}\delta_{v,bc}f(a,b,c), \qquad\qquad \xi^{a,b}_{r} = \delta_{r,ab}\xi(a,b).$$

If f is a normalized 3-cocycle, $\delta f(r, \bar{r}, r, \bar{r}) = 1$ implies $f(r, \bar{r}, r)f(\bar{r}, r, \bar{r}) = 1$, i.e., f is rigid. Therefore every normalized 3-cocycle is a 6j fusion system on G, and conversely by Lemma 6.0.10. Moreover ξ is a gauge transformation from f to $\tilde{f}$ iff ξ is normalized and $\tilde{f} = f\delta\xi$. Our identification of 6j fusion systems on G with normalized 3-cocycles thus descends to a 1-1 correspondence between 6j fusion systems on G up to gauge equivalence and $H^3(G, \mathbb{F}^\times)$, which descends to the second claimed correspondence via Theorem 6.0.9.

The second and third statements are similar, but easier. □

6.2 Feudal fusion categories

In this section, using the formalism of 6j fusion systems, we characterize fusion categories on feudal fusion rules (Section 3.2). A string of technical definitions precedes the characterization (Theorem 6.2.5), which describes third cohomology of groups of even order as a special case (Corollary 6.2.7). For $\mathbb{Z}_4$ and the Tambara-Yamagami and fermionic Moore-Read fusion rules (Example 2.2.2), we check Theorem 6.2.5 agrees with common wisdom and Tambara-Yamagami [28] and Bonderson [3], respectively. Unfortunately Theorem 6.2.5 does not readily indicate which feudal fusion rules have fusion categories; without extra work the most we can say is that the adjoint subrule must be abelian (Corollary 6.2.11).

Definition 6.2.1. A *symmetric bicharacter* on a finite group A over a ring B is a map $\chi\colon A \times A \to B^\times$ such that

$$\chi(b,a) = \chi(a,b), \qquad \chi(ab,c) = \chi(a,c)\chi(b,c).$$

We say χ is *nondegenerate* if $\sum_b \chi(a,b) = 0$ for $a \neq 1$.

Definition 6.2.2. An *involutory ambidextrous algebra* over a group S is a ring B with two operations

$$B \to B\colon \mu \mapsto \bar{\mu}$$

$$S \times B \times S \to B\colon (a,\mu,b) \mapsto {}^a\mu^b$$

such that $\mu \mapsto \bar{\mu}$ is an involution (ring antiautomorphism of order two), $\mu \mapsto {}^a\mu^b$ is a ring endomorphism for $a, b \in S$, and

$$^a({}^b\mu^c)^d = {}^{ab}\mu^{cd}, \qquad \overline{{}^a\mu^b} = {}^{\bar{b}}\bar{\mu}^{\bar{a}}$$

for $a, b, c, d \in S$ and $\mu \in B$.

Convention. Let B be an involutory ambidextrous algebra over a group S, and let X be a set. Then B^X inherits the involutory ambidextrous S-algebra structure of B: for $\epsilon \in B^X$ and $a, b \in S$, define ${}^a\epsilon^b, \bar{\epsilon} \in B^X$ by ${}^a\epsilon^b(x) = {}^a\epsilon(x)^b$ and $\bar{\epsilon}(x) = \overline{\epsilon(x)}$ for $x \in X$. For $\mu, \nu \in B$ and $a \in S$, we write $\mu^a\nu$ for $\mu({}^a\nu)$, not $(\mu^a)\nu$.

Definition 6.2.3. Let B be an involutory ambidextrous algebra over a group S. Let $\chi, \upsilon \colon S \times S \to B^\times$ and $\tau \in B^\times$.

- χ is a *τ-quasisymmetric υ-biderivation* if υ is normalized and for $a, b, c \in S$,

$$\bar{\chi}(b, a) = {}^{\bar{a}}\chi^{\bar{b}}(a, b)\frac{{}^{\bar{a}}\tau^{\bar{b}} \cdot \tau}{{}^{\bar{a}}\tau\tau^{\bar{b}}} \qquad \frac{\upsilon}{\upsilon^c}(a, b)\chi(ab, c) = \chi(a, c)^a\chi(b, c).$$

- Suppose the set A of elements of S acting trivially on B is finite. The triple (χ, υ, τ) is an *überderivation* on S over B if χ is a τ-quasisymmetric υ-biderivation such that the symmetric bicharacter $\chi|_{A\times A}$ is nondegenerate, and $|A|\tau\bar{\tau} = 1_B$.

- Let $\mathrm{fix}(S)$ be the elements of B fixed under the S-actions. A *gauge transformation* from (χ, υ, τ) to another überderivation $(\tilde{\chi}, \tilde{\upsilon}, \tilde{\tau})$, rendering them *gauge equivalent*, is a triple $(\theta, \phi, \varsigma) \in \mathrm{fix}(S)^{S\times S} \times (B^\times)^S \times (B^\times)$, with ϕ normalized, such that for $a, b \in S$,

$$\frac{\tilde{\chi}(a,b)}{\chi(a,b)} = \frac{\phi(a)^a \bar{\phi}^b(b)^a \varsigma^b \cdot \varsigma}{\phi^b(a)\bar{\phi}^b(b)^a \varsigma\varsigma^b} \qquad \frac{\tilde{\upsilon}}{\upsilon} = \frac{\grave{\delta}\phi}{\theta} \qquad \frac{\tilde{\tau}}{\tau} = \frac{\bar{\varsigma}}{\varsigma}$$

- Let $(\theta, \phi, \varsigma), (\tilde{\theta}, \tilde{\phi}, \tilde{\varsigma})$ be gauge transformations between a pair of überderivations on S over B. A 2-*isomorphism* from $(\theta, \phi, \varsigma)$ to $(\tilde{\theta}, \tilde{\phi}, \tilde{\varsigma})$ is a pair $(\zeta_0, \zeta_1) \in \operatorname{fix}(S)^S \times (B^\times)$ such that

$$\tilde{\phi}(a) = \frac{{}^a\zeta_1}{\zeta_1}\zeta_0(a)\phi(a) \qquad \tilde{\varsigma} = \zeta_1\bar{\zeta}_1\varsigma$$

Definition 6.2.4. Let L be a feudal fusion rule with serfs S and lords M. Let $B = \mathbb{F}^M$, with ring structure inherited from $\mathbb{F}$. Then B is an involutory ambidextrous S-algebra: for $\mu \in B$ and $a, b \in S$, define ${}^a\mu^b, \bar{\mu} \in B$ by ${}^a\mu^b(m) = \mu(\bar{a}m\bar{b})$ and $\bar{\mu}(m) = \mu(\bar{m})$ for $m \in M$. For a 6j fusion system f on L, a gauge transformation ξ of 6j fusion systems on L, or a 2-isomorphism ζ of such gauge transformations, let

$$\Psi f = (\chi, \upsilon, \tau) \qquad \Psi\xi = (\theta, \phi, \varsigma) \qquad \Psi\zeta = (\zeta_0, \zeta_1)$$

where for $a, b \in S$ and $m \in M$,

$$\chi(a,b)(m) = f^{a,\bar{a}m\bar{b},b}_{m\bar{b},m,\bar{a}m} \qquad \theta(a,b)(m) = \xi^{a,b}_{ab}$$

$$\upsilon(a,b)(m) = f^{a,b,\bar{b}\bar{a}m}_{ab,m,\bar{a}m} \qquad \phi(a)(m) = \xi^{a,\bar{a}m}_{m} \qquad \zeta_0(a) = \zeta(a)$$

$$\tau(m) = f^{m,\bar{m},m}_{1,m,1} \qquad \varsigma(m) = \xi^{m,\bar{m}}_{1} \qquad \zeta_1(m) = \zeta(m)$$

Theorem 6.2.5. *Let $\mathcal{Y}$ be the strict 2-category of 6j fusion systems on a feudal fusion rule with lords M, and $\mathcal{X}$ the strict 2-category of überderivations on serfs over $\mathbb{F}^M$; in each category morphisms are gauge transformations, composed via multiplication in $\mathbb{F}$. Then $\Psi\colon \mathcal{Y} \to \mathcal{X}$ is an equivalence, surjective on the nose.*

Proof. Subsection 6.2.2. □

Corollary 6.2.6. *Let L be a properly feudal fusion rule, or a $\mathbb{Z}_2$-graded group all of whose automorphisms are graded, with serfs S and lords M. Then fusion categories (or 6j fusion systems) on L up to equivalence are in 1-1 correspondence with überderivations on S over $\mathbb{F}^M$ up to gauge equivalence and up to simultaneously permuting S and M by an automorphism of L.*

Proof. Theorems 6.2.5 and 6.0.9. □

Corollary 6.2.7. *Let G be a group and S an index 2 subgroup. Then $H^3(G, \mathbb{F}^\times)$ is in 1-1 correspondence with überderivations on S over $\mathbb{F}^{G \setminus S}$ up to gauge equivalence.*

Proof. Theorems 6.2.5 and 6.1.2. □

Example 6.2.8 (well-known)*.* If $\mathbb{F}$ has square roots, $H^3(\mathbb{Z}_4, \mathbb{F}^\times)$ is in 1-1 correspondence with 4th roots of unity in $\mathbb{F}$.

Proof. See Subsection 6.2.1. □

Example 6.2.9 (Tambara-Yamagami [28])*.* Let $L = A \sqcup \{m\}$ be a Tambara-Yamagami fusion rule. Then fusion categories on L up to equivalence are in 1-1 correspondence with pairs (χ, τ) up to relabeling χ by an automorphism of A, where $\chi \colon A \times A \to \mathbb{F}^\times$ is a nondegenerate symmetric bicharacter and $\tau = \pm |A|^{-1/2} \in \mathbb{F}$.

Proof. This is a special case of Corollary 6.2.6. □

Example 6.2.10 (Bonderson [3])*.* If $\mathbb{F}$ has square roots, fermionic Moore-Read fusion categories up to equivalence are in 1-1 correspondence with 4th roots of -1 in $\mathbb{F}$.

Proof. See Subsection 6.2.1. ☐

Corollary 6.2.11. *If there is a fusion category on a feudal fusion rule with adjoint subrule A and lords M, then*

- *A is abelian.*
- *The characteristic of $\mathbb{F}$ does not divide $|A|$.*
- *If $|M|$ is odd or $m = \bar{m}$ for some $m \in M$, then $\sqrt{|A|} \in \mathbb{F}$.*

Proof. Let (χ, υ, τ) be an überderivation on serfs over $B = \mathbb{F}^M$. The two desired conditions on $|A|$ follow from $|A|\tau\bar{\tau} = 1_B$. Since $\chi|_{A\times A}$ is a bicharacter,

$$\sum_{c \in A} \chi(ab\bar{a}\bar{b}, c) = |A| 1_B$$

for $a, b \in A$. Since $|A| \neq 0$ in $\mathbb{F}$ and $\chi|_{A\times A}$ is nondegenerate, $ab\bar{a}\bar{b} = 1$. ☐

Remark. The results of this section would still hold if we did not require fusion categories and 6j fusion systems to be rigid.

6.2.1 Proof of Examples 6.2.8 and 6.2.10

Lemma 6.2.12. *Consider a feudal fusion rule with two lords $M = \{m_1, m_2\}$. Suppose $\mathbb{F}$ has square roots, and let $\tau_0(m_1) = \tau_0(m_2) = \pm|A|^{-1/2} \in \mathbb{F}^\times$. Then any überderivation on serfs over $\mathbb{F}^M$ is gauge equivalent to one of the form (χ, υ, τ_0). If two such überderivations are gauge equivalent, they are related by a gauge transformation of the form $(\theta, \phi, 1)$.*

Proof. Let $(\chi, \upsilon, \tau) \in \operatorname{obj} \mathcal{X}$, where $\mathcal{X}$ is as in Theorem 6.2.5. Choose $\varsigma \in \mathbb{F}^M$ such that ${}^{\varsigma(m_1)}/_{\varsigma(m_2)} = \pm\sqrt{{}^{\tau(m_1)}/_{\tau(m_2)}}$. Let $\tilde{\tau}(m) = {}^{\varsigma(\bar{m})\tau(m)}/_{\varsigma(m)}$ for $m \in M$. Then $\tilde{\tau}$ is constant on M. By Lemma 6.0.11 and Theorem 6.2.5 there exist $\theta, \phi, \varsigma, \tilde{\chi}, \tilde{\upsilon}$ such that $(\tilde{\chi}, \tilde{\upsilon}, \tilde{\tau}) \in \operatorname{obj} \mathcal{X}$ and $(\theta, \phi, \varsigma) \in \operatorname{mor}_{\mathcal{X}}((\chi, \upsilon, \tau), (\tilde{\chi}, \tilde{\upsilon}, \tilde{\tau}))$. Since $\tilde{\tau}^2 \equiv {}^1/_{|A|}$ and ${}^{\bar{\varsigma}}/_{\varsigma}$ has sign freedom, w.l.o.g. $\tilde{\tau} = \tau_0$.

Now let $\chi, \upsilon, \tilde{\chi}, \tilde{\upsilon}$ be arbitrary such that $(\chi, \upsilon, \tau_0), (\tilde{\chi}, \tilde{\upsilon}, \tau_0) \in \operatorname{obj} \mathcal{X}$ are related by some gauge equivalence $(\theta, \phi, \varsigma)$. Then $\bar{\varsigma} = \varsigma$, implying

$$\frac{\tilde{\chi}(a,b)}{\chi(a,b)} = \frac{\phi(a)^a \bar{\phi}^b(b)}{\phi^b(a)\bar{\phi}^b(b)}$$

Thus $(\theta, \phi, 1) \in \operatorname{mor}_{\mathcal{X}}((\chi, \upsilon, \tau), (\tilde{\chi}, \tilde{\upsilon}, \tilde{\tau}))$. ☐

Convention. When $|M| = 2$, identify $\mathbb{F}^M$ with $\mathbb{F}^2$ naturally, and $\mathbb{F}$ with the diagonal.

Proof of Example 6.2.10. Let $S = \{1, \psi, \alpha, \alpha'\}$ and $M = \{\sigma, \sigma'\}$ (Example 2.2.2), and suppose (χ, υ, τ) is an überderivation on S over $B = \mathbb{F}^M$. By Lemma 6.2.12 we may take τ constant. Writing χ and ${}^{\upsilon}/_{\bar{\upsilon}}$ as matrices over B indexed by S, we find

$$\chi = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & p & -p \\ 1 & p & x & r \\ 1 & -p & \bar{r} & x \end{pmatrix} \qquad {}^{\upsilon}/_{\bar{\upsilon}} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & p^2 & -{}^{pr}/_{x} & -{}^{px}/_{r} \\ 1 & -{}^{\bar{p}r}/_{x} & {}^{x^2}/_{p} & xr \\ 1 & -{}^{\bar{p}x}/_{r} & x\bar{r} & -{}^{x^2}/_{p} \end{pmatrix}$$

for some $x \in \mathbb{F}^\times$ and $p, r \in B^\times$; the only requirements are $x^4 = p\bar{p} = -1$ and $r\bar{r} = -x^2$. By Lemma 6.0.11 and Theorem 6.2.5 the gauge equivalence class of f is uniquely determined by x, which is invariant under fusion rule automorphisms. Then invoke Corollary 6.2.6. ☐

Proof of Example 6.2.8. Regard $\mathbb{Z}_4 = \{\pm 1, \pm i\}$ as feudal with lords $M = \{\pm i\}$. Then an überderivation (χ, υ, τ) on $\{\pm 1\}$ over $\mathbb{F}^M$ is uniquely determined by $p, q, \tau \in \mathbb{F}^M$, where $p = \chi(-1,-1)$ and $q = \upsilon(-1,-1)$. By Lemma 6.2.12 we may take τ constant. Then the only requirements on p, q are $p \in \mathbb{F}^\times$ and $p^2 = {}^q/_{\bar{q}}$, i.e., $p^4 = 1$ and $q_1 = p^2 q_2$ where $(q_1, q_2) = q$. The gauge equivalence class of (χ, υ, τ) is uniquely determined by p. Then invoke Corollary 6.2.7. $\square$

6.2.2 Proof of Theorem 6.2.5

We generalize Tambara and Yamagamis' proof [28] of Example 6.2.9. We ignore 2-morphisms, as they are easily dealt with.

Convention. We identify $\mathbb{F}^\times$ with the set of constant functions $M \to \mathbb{F}^\times$, and freely use Lemma 2.2.3.

Let L be a feudal fusion rule with serfs S, lords M, and adjoint subrule A, and let $B = \mathbb{F}^M$. We write $f, \tilde{f}$ for arbitrary 6j fusion systems on L with recoupling matrices $F, \tilde{F}$; a, b, c, d, e for arbitrary serfs; and m, l for arbitrary lords. We identify f (likewise $\tilde{f}$) with the collection of eight functions

$$\alpha \colon S \times S \times S \to \mathbb{F}^\times$$

$$\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3 \colon S \times S \to B^\times$$

$$\gamma \colon S \times S \to B$$

defined according to the following convention:

$$\alpha(a,b,c) = F^{a,b,c}_{abc}$$

$$\alpha_1(a,b)(m) = F^{m\bar{b}\bar{a},a,b}_{m} \qquad \alpha_2(a,b)(m) = F^{a,\bar{a}m\bar{b},b}_{m} \qquad \alpha_3(a,b)(m) = F^{a,b,\bar{b}\bar{a}m}_{m}$$

$$\beta_1(a,b)(m) = F^{a,m,\bar{m}\bar{a}b}_{b} \qquad \beta_2(a,b)(m) = F^{m,a,\bar{a}\bar{m}b}_{b} \qquad \beta_3(a,b)(m) = F^{b\bar{a}\bar{m},m,a}_{b}$$

$$\gamma(a,b)(m) = f^{m\bar{a},a\bar{m}b,\bar{b}m}_{b,m,a}$$

By Lemma 6.0.10, $\alpha, \alpha_1, \alpha_2, \alpha_3$ are normalized and $\beta_i(1,-) \equiv 1$ for $i = 1,2,3$.

Lemma 6.2.13. *Every nonempty recoupling matrix is* 1×1 *or* $(\gamma(a,b))_{a\in A', b\in A''}$ *for some cosets* A', A'' *of* A *in* S.

We write ξ for an arbitrary gauge transformation from f to $\tilde{f}$, identified with the collection of four functions

$$\theta\colon S \times S \to \mathbb{F}^\times, \qquad \phi, \psi, \omega\colon S \to B^\times$$

defined according to the following convention:

$$\theta(a,b)(m) = \xi^{a,b}_{ab} \qquad \omega(a)(m) = \xi^{m,\bar{m}a}_{a}$$

$$\phi(a)(m) = \xi^{a,\bar{a}m}_{m} \qquad \psi(a)(m) = \xi^{m\bar{a},a}_{m}$$

By Definition 6.0.7, θ, ϕ, ψ are normalized. It is routine to gather the rectangle axiom for ξ and the pentagon axiom for f into

$$
\begin{aligned}
G^{000} &= G^{a,b,c}_{ab,abc,bc} : && \tilde{\alpha} = \alpha\delta\theta \\
G^{100} &= G^{m\bar{b}\bar{a},a,b}_{m\bar{b},m,ab} : && \tilde{\alpha}_1\acute{\delta}\psi = \theta\alpha_1 \\
G^{010} &= G^{a,\bar{a}m\bar{b},b}_{m\bar{b},m,\bar{a}m} : && \tilde{\alpha}_2(a,b)\phi^b(a)\psi(b) = \phi(a)^a\psi(b)\alpha_2(a,b) \\
G^{001} &= G^{a,b,\bar{b}\bar{a}m}_{ab,m,\bar{a}m} : && \tilde{\alpha}_3\theta = \alpha_3\grave{\delta}\phi \\
G^{011} &= G^{a,m,\bar{m}\bar{a}b}_{am,b,\bar{a}b} : && \tilde{\beta}_1(a,b)^{\bar{a}}\phi(a)^{\bar{a}}\omega(b) = \omega(\bar{a}b)\theta(a,\bar{a}b)\beta_1(a,b) \\
G^{101} &= G^{m,a,\bar{a}\bar{m}b}_{ma,b,\bar{m}b} : && \tilde{\beta}_2(a,b)\psi^{\bar{a}}(a)\omega^{\bar{a}}(b) = {}^b\bar{\phi}(a)\omega(b)\beta_2(a,b) \\
G^{110} &= G^{b\bar{a}\bar{m},m,a}_{b\bar{a},b,ma} : && \tilde{\beta}_3(a,b)\theta(b\bar{a},a)\bar{\omega}^{b\bar{a}}(b\bar{a}) = \bar{\omega}^{b\bar{a}}(b)\psi^{\bar{a}}(a)\beta_3(a,b) \\
G^{111} &= G^{m\bar{a},a\bar{m}b,\bar{b}m}_{b,m,a} : && \tilde{\gamma}(a,b)\omega^a(b)\phi(b) = {}^b\bar{\omega}^a(a)\psi(a)\gamma(a,b) \\
P^{0000} &= P^{a,b,c,d}_{ab,abc,abcd,bcd,cd} : && 1 = \delta\alpha \\
P^{0001} &= P^{a,b,c,\bar{c}\bar{b}\bar{a}m}_{ab,abc,m,\bar{a}m,\bar{b}\bar{a}m} : && 1 = \alpha\grave{\delta}\alpha_3 \\
P^{1000} &= P^{m\bar{c}\bar{b}\bar{a},a,b,c}_{m\bar{c}\bar{b},m\bar{c},m,abc,bc} : && \acute{\delta}\alpha_1 = \alpha \\
P^{0010} &= P^{a,b,\bar{b}\bar{a}m\bar{c},c}_{ab,m\bar{c},m,\bar{a}m,\bar{b}\bar{a}m} : && \alpha_3(a,b)\alpha_2(ab,c) = {}^a\alpha_2(b,c)\alpha_2(a,c)\alpha_3^c(a,b) \\
P^{0100} &= P^{a,\bar{a}m\bar{c}\bar{b},b,c}_{m\bar{c}\bar{b},m\bar{c},m,\bar{a}m,bc} : && \alpha_1(b,c)\alpha_2(a,bc) = \alpha_2^c(a,b)\alpha_2(a,c)^a\alpha_1(b,c)
\end{aligned}
$$

$$
\begin{aligned}
P^{0011} &= P^{a,b,m,\bar{m}\bar{b}\bar{a}c}_{ab,abm,c,\bar{a}c,\bar{b}\bar{a}c}: && \alpha(a,b,\bar{b}\bar{a}c)\beta_1(ab,c) = {}^{\overline{ab}}\alpha_3(a,b)^{\bar{b}}\beta_1(a,c)\beta_1(b,\bar{a}c) \\
P^{1100} &= P^{c\bar{b}\bar{a}\bar{m},m,a,b}_{c\bar{b}\bar{a},c\bar{b},c,mab,ab}: && \alpha(c\bar{b}\bar{a},a,b)\beta_3(ab,c) = \beta_3(a,c\bar{b})\beta_3^{\bar{a}}(b,c)\alpha_1^{\overline{ab}}(a,b) \\
P^{0101} &= P^{a,m,b,\bar{b}\bar{m}\bar{a}c}_{am,amb,c,\bar{a}c,\bar{m}\bar{a}c}: && \beta_1(a,c)^{\bar{a}}\beta_2(b,c) = \beta_2(b,\bar{a}c)\beta_1^{\bar{b}}(a,c)^{\bar{a}}\alpha_2^{\bar{b}}(a,b) \\
P^{1010} &= P^{c\bar{b}\bar{m}\bar{a},a,m,b}_{c\bar{b}\bar{m},c\bar{b},c,amb,mb}: && \overline{{}^{b\bar{c}}\beta_2^{a}}(a,c)\beta_3(b,c) = {}^{\bar{a}}\alpha_2^{\bar{b}}(a,b)^{\bar{a}}\beta_3(b,c)\overline{{}^{b\bar{c}}\beta_2^{a}}(a,c\bar{b}) \\
P^{0110} &= P^{a,m,\bar{m}\bar{a}c\bar{b},b}_{am,c\bar{b},c,\bar{a}c,\bar{m}\bar{a}c}: && \beta_1(a,c)\overline{\beta_3^{b\bar{c}a}}(b,c) = \overline{\beta_3^{b\bar{c}a}}(b,\bar{a}c)\alpha(a,\bar{a}c\bar{b},b)\beta_1(a,c\bar{b}) \\
P^{1001} &= P^{m,a,b,\bar{b}\bar{a}\bar{m}c}_{ma,mab,c,\bar{m}c,\bar{a}\bar{m}c}: && \beta_2(a,c)\beta_2^{\bar{a}}(b,c) = {}^{c}\bar{\alpha}_3(a,b)\beta_2(ab,c)\alpha_1^{\overline{ab}}(a,b) \\
P^{0111} &= P^{a,\bar{a}m\bar{c},c\bar{m}b,\bar{b}m}_{m\bar{c},b,m,\bar{a}m,c}: && \alpha_2(a,c)\gamma(c,b) = {}^{a}\gamma(c,\bar{a}b)\alpha_3(a,\bar{a}b)^{a}\beta_1^{c}(a,b) \\
P^{1110} &= P^{m\bar{c},c\bar{m}b,\bar{b}m\bar{a},a}_{b,m\bar{a},m,c,\bar{b}m}: && \alpha_2(b,a)\gamma(c,b) = {}^{b}\beta_3^{a}(a,c)\alpha_1(c\bar{a},a)\gamma^{a}(c\bar{a},b) \\
P^{1011} &= P^{m\bar{c}\bar{a},a,c\bar{m}b,\bar{b}m}_{m\bar{c},b,m,ac,c}: && \alpha_1(a,c)\gamma(c,b) = \beta_2^{ac}(a,b)\gamma(ac,b)^{b}\bar{\beta}_1{}^{c}(a,ac) \\
P^{1101} &= P^{m\bar{c},c\bar{m}b,a,\bar{a}\bar{b}m}_{b,ba,m,c,\bar{b}m}: && \gamma(c,b)\alpha_3(b,a) = {}^{b}\bar{\beta}_2{}^{c}(a,c)\gamma(c,ba)^{b}\bar{\beta}_3{}^{c}(a,ba) \\
P^{1111} &= P^{a\bar{m},m,\bar{m}e,\bar{e}m\bar{a}b}_{a,a\bar{m}e,b,m\bar{a}b,d}: && \delta_{b,ad}{}^{e}\bar{\beta}_1(a,b)\beta_3(d,b) = \sum_{c\in eA} {}^{e}\bar{\gamma}^{a}(c,a)\bar{\beta}_2^{a}(c,b)\gamma^{\bar{d}}(d,c)
\end{aligned}
$$

where P^{1111} assumes $\bar{b}ad \in A$.

Definition 6.2.14. f is *normal* if $\beta_1(-,1) = \beta_2(-,1) \equiv 1$.

Lemma 6.2.15. *Every 6j fusion system on L is gauge equivalent to a normal one.*

Proof. Given arbitrary f, we construct ξ making $\tilde{f}$ normal. By G^{011},

$$\tilde{\beta}_1(a,1) = \frac{\omega(\bar{a})\theta(a,\bar{a})\beta_1(a,1)}{{}^{\bar{a}}\omega(1)^{\bar{a}}\phi(a)}$$

so we can choose ω making $\tilde{\beta}_1(a,1) = 1$. By G^{101},

$$\tilde{\beta}_2(a,1) = \frac{\bar{\phi}(a)\omega(1)\beta_2(a,1)}{\psi^{\bar{a}}(a)\omega^{\bar{a}}(1)}$$

so we can choose ψ making $\tilde{\beta}_2 = 1$. Then any choice of normalized ϕ, θ makes $\tilde{f}$ a 6j fusion system on L by Lemma 6.0.11. ☐

Lemma 6.2.16. Ψ *restricts to an isomorphism on the category of normal 6j fusion systems on L.*

Proof. It is routine to check $\mathcal{Y}$ and $\mathcal{X}$ are categories. Let $\mathcal{Z}$ be the category of normal 6j fusion systems, a full subcategory of $\mathcal{Y}$, and let $\Xi = \Psi|_{\mathcal{Z}}$. Let $f \in \operatorname{obj}\mathcal{Z}$, and $(\chi, \upsilon, \tau) = \Xi f = (\alpha_2, \alpha_3, \gamma(1,1))$. Then

$$\alpha = \frac{1}{\grave{\delta}\upsilon} \colon S^3 \to \mathbb{F}^\times \qquad (P^{0001})$$

$$\beta_1(b,a) = \frac{\alpha(\bar{a}, b, \bar{b}a)}{{}^{\bar{b}a}\upsilon(\bar{a}, b)} \qquad (P^{0011})$$

$$\beta_2(b,a) = \frac{\upsilon(a,\bar{a})}{\upsilon^b(a,\bar{a})}\chi^{\bar{b}}(a,b) \qquad (P^{0101})$$

$$\alpha_1(b,c) = \frac{1}{\bar{\upsilon}^{bc}(b,c)} \qquad (P^{1001}, P^{0100})$$

$$\chi \text{ is an } \upsilon\text{-biderivation} \qquad (P^{0010}, P^{0100})$$

$$\gamma(a,1) = \tau\bar{\upsilon}(\bar{a}, a) \qquad (P^{1011})$$

$$\gamma(c,a) = {}^{a}\!\left(\frac{\tau\bar{\upsilon}(\bar{c}, c)}{\upsilon^c(\bar{a}, a)}\right)\frac{1}{\chi(a,c)} \qquad (P^{0111})$$

$$\tau \in B^\times \qquad \text{(invertibility)}$$

$$\beta_3(a,c) = \frac{\bar{\upsilon}(a,\bar{c})\tau^{\bar{a}}}{\alpha(a,\bar{c},c)\alpha(a\bar{c}, c\bar{a}, a)\tau} \qquad (P^{1110})$$

$$\tau\bar{\tau} \in \mathbb{F}^\times \qquad (P^{1101})$$

$$\chi \text{ is } \tau\text{-quasisymmetric} \qquad (P^{1010})$$

$$\tau\bar{\tau} \equiv {}^{1}\!/\!{}_{|A|} \qquad (P^{1111})$$

$$\chi|_{A\times A} \text{ is nondegenerate} \qquad (P^{1111})$$

Thus Ξ is well-defined and injective on the level of objects.

To check Ξ is surjective on the level of objects, suppose $(\chi, \upsilon, \tau) \in \mathcal{X}$. We must show f defined via the above equations, with $\alpha_2 = \chi$ and $\alpha_3 = \upsilon$, is in $\operatorname{obj} \mathcal{Z}$. It suffices to show f is a 6j fusion system on L. By construction f satisfies admissibility. To check invertibility, let $p, q \in S$. Since $\tau\bar{\tau} \equiv {}^1\!/\!{}_{|A|}$ and $\chi|_{A\times A}$ is a nondegenerate symmetric bicharacter, the matrix $({}^q\tau/\chi(y,x))_{x,y\in A}$ has inverse $({}^q\bar{\tau}\chi(y,x))_{x,y\in A}$. Since

$$\gamma(px, yq) = \frac{{}^q\bar{\upsilon}(\bar{x}\bar{p}, px)}{\chi(q, px)} \frac{{}^q\tau}{\chi(y,x)} \frac{\upsilon(y,q)}{\chi(y,p)\upsilon^p(y,q)^q\upsilon^p(\bar{q}\bar{y}, yq)}$$

for $x, y \in A$, the matrix $(\gamma(px, yq))_{x,y\in A}$ is invertible over B. Equivalently, for all $m \in M$ the recoupling matrix $F_m^{m\bar{p},p\bar{m}q,\bar{q}m} = (\gamma(u,v)(m))_{u\in pA, v\in qA}$ is invertible over $\mathbb{F}$. Therefore f satisfies the invertibility axiom by Lemma 6.2.13. It is routine to verify the pentagon axiom, and trivial to verify the triangle axiom. Rigidity means $\alpha(a, \bar{a}, a)\alpha(\bar{a}, a, \bar{a}) = 1$ and $\tau(m) = ((F_{\bar{m}}^{\bar{m},m,\bar{m}})^{-1})_{1,1}$ for all $a \in S$ and $m \in M$. The first condition follows from $\delta\alpha(a, \bar{a}, a, \bar{a}) = 1$. The second follows from

$$F_{\bar{m}}^{\bar{m},m,\bar{m}} = \left(\frac{\bar{\upsilon}(\bar{x}, x)\tau}{\chi(y,x)\upsilon(\bar{y}, y)}(\bar{m})\right)_{x,y\in A}$$

$$(F_{\bar{m}}^{\bar{m},m,\bar{m}})^{-1} = \left(\frac{\upsilon(\bar{x}, x)\bar{\tau}\chi(y,x)}{\bar{\upsilon}(\bar{y}, y)}(\bar{m})\right)_{x,y\in A}$$

as χ is normalized and $\tau(m) = \bar{\tau}(\bar{m})$. Thus f is a 6j fusion system on L, showing Ξ is surjective on the level of objects.

Suppose $\xi \in \operatorname{mor}_{\mathcal{Z}}(f, \tilde{f})$. Letting $(\tilde{\chi}, \tilde{\upsilon}, \tilde{\tau}) = \Xi\tilde{f}$, we have

$$\omega(a) = \frac{{}^a\phi(\bar{a})^a\varsigma}{\theta(\bar{a}, a)} \qquad (G^{011})$$

$$\psi(a) = \frac{\bar{\phi}^a(a)\varsigma^a}{\varsigma} \qquad (G^{110})$$

$$\frac{\tilde{\upsilon}}{\upsilon} = \frac{\grave{\delta}\phi}{\theta} \tag{G^{001}}$$

$$\frac{\tilde{\chi}(a,b)}{\chi(a,b)} = \frac{\phi(a)^a\psi(b)}{\phi^b(a)\psi(b)} = \frac{\phi(a)^a\bar{\phi}^b(b)^a\varsigma^b \cdot \varsigma}{\phi^b(a)\bar{\phi}^b(b)^a\varsigma\varsigma^b} \tag{G^{010}}$$

$$\frac{\tilde{\tau}}{\tau} = \frac{\bar{\varsigma}}{\varsigma} \tag{G^{111}}$$

Since $\theta \colon M \to \mathbb{F}^\times$, we see Ξ is well-defined and injective on each morphism space. It is routine to check Ξ is surjective on each morphism space. Therefore Ξ is an isomorphism. □

Proof of Theorem 6.2.5. The two preceding lemmas suffice. □

Chapter 7

Twines and fermionic Moore-Read

Fusion categories may admit various extra stuctures manufacturing topological invariants defined on various sorts of tangles. One of the main such structures is *braiding*, yielding representations of object-colored braids. Braided fusion categories apply to physics by interpreting braids as trajectories (Chapter 8). They enjoy Ocneanu rigidity: there are only finitely many braidings on a given finite fusion category over a favorable field. But in cases where the full structure of braiding is unnecessary or unavailable, it may be fruitful to consider only pure braids. To this end Bruguieres [6] introduced *entwined* or *purely braided* strict monoidal categories. For entwined monoidal categories we consider two notions of equivalence, a naturally discovered categorical equivalence and *homothetic* equivalence. Under the former, Ocneanu rigidity holds, and second group cohomology classifies entwined fusion categories on groups. Transporting twines to fusion systems enables us to calculate them on fermionic Moore-Read fusion categories, where we find no nontrivial twines up to categorical equivalence but a unique nontrivial twine up to homothetic equivalence.

7.1 Braided categories

A monoidal structure on a category defines, for any tuple of objects $x_1, \ldots, x_n$, an object $x_1 \Box \cdots \Box x_n$ well-defined up to unique structural natural isomorphisms which can be swept under the rug. In a *braided* monoidal category, given a braid b on n strands, there is a natural isomorphism

$$\beta(b, x_1, \ldots, x_n) \colon x_1 \Box \cdots \Box x_n \to x_{\pi(1)} \Box \cdots \Box x_{\pi(n)} \tag{7.1}$$

where π is the underlying permutation of b. The correspondence β must respect composition and juxtaposition of braids. It determines, and is determined by, a natural isomorphism

$$\sigma \colon (\Box) \Rightarrow (\Box) \circ \mathrm{Flip}$$

where $\mathrm{Flip} \colon C \times C \to C \times C$ is the functor swapping the two coordinates. By a coherence theorem of Joyal and Street, σ furnishes a braiding iff it satisfies the *hexagon axiom.* Some references are Joyal and Street [17]; Mac Lane [21]; Bakallov and Kirillov [2]; and Drinfeld, Gelaki, Nikshych, and Ostrik [9].

Braided categories are used in category theory, representation theory, knot theory, and quantum topology. As sketched in Chapter 8, braided fusion categories with extra structures are used to model anyons in the fractional quantum Hall effect, such as might be used for topological quantum computation.

7.2 Entwined categories

Only commutative fusion rules may admit braided fusion categories. For noncommutative fusion rules, in general an isomorphism of the form (7.1) can only be defined when the underlying permutation π is trivial, i.e., the braid b is pure. Specializing to pure braids is of independent interest. An *entwined* category is like a braided category, except that the correspondence β from braids to isomorphisms is only defined for pure braids.

Entwined categories are a recent invention of Bruguieres [6]. They may see applications in anyon theory (Chapter 8) and invariants of ribbon links and tangles. Like a braiding, a twine is completely pinned down by a natural isomorphism of bifunctors via a coherence theorem motivating the following axiomatization.

Bruguieres [6] uses strict monoidal categories and functors (strict meaning all structural isomorphisms are equalities); the following definitions are the natural adaptations to general monoidal categories and oplax monoidal functors.

Definition. Let $\mathcal{C} = (C, \Box, 1, \alpha, \lambda, \rho)$ be a monoidal category. A *twine* or *pure braiding* on $\mathcal{C}$ is a natural automorphism τ of $\Box$ such that $(\mathrm{id}_C, \tau, \mathrm{id}_1)$ is a gauge automorphism of $\mathcal{C}$ (Definition 4.1.8) satisfying the *dodecagon axiom*: the diagram

$$\begin{array}{ccc}
((ab)c)d & \longrightarrow & a(b(cd)) \\
{\scriptstyle \tau(ab,c)\Box d}\uparrow & & \uparrow{\scriptstyle a\Box\tau(b,cd)} \\
((ab)c)d & & a(b(cd)) \\
\uparrow & & \uparrow \\
a((bc)d) & & (a(bc))d \\
{\scriptstyle a\Box(\tau^{-1}(b,c)\Box d)}\uparrow & & \uparrow{\scriptstyle (a\Box\tau^{-1}(b,c))\Box d} \\
a((bc)d) & & (a(bc))d \\
\uparrow & & \uparrow \\
a(b(cd)) & & ((ab)c)d \\
{\scriptstyle a\Box\tau(b,cd)}\uparrow & & \uparrow{\scriptstyle \tau(ab,c)\Box d} \\
a(b(cd)) & \longleftarrow & ((ab)c)d
\end{array}$$

commutes for all objects a, b, c, d, where the unlabeled arrows are canonical associations. An *entwined* or *purely braided* monoidal category is one equipped with a twine.

Observation. *Every fusion category has a* trivial *twine given by the trivial gauge transformation.*

Definition. Let C and C' be entwined monoidal categories. A *strictly entwined colax monoidal functor* from C to C' is a colax monoidal functor (F, F_2, F_0) such that for all $a, b \in \operatorname{obj} C$, this square commutes:

$$\begin{array}{ccc}
Fa\Box' Fb & \xrightarrow{\tau'(Fa,Fb)} & Fa\Box' Fb \\
{\scriptstyle F_2(a,b)}\uparrow & & \uparrow{\scriptstyle F_2(a,b)} \\
F(a\Box b) & \xrightarrow[F\tau(a,b)]{} & F(a\Box b)
\end{array}$$

Definition 7.2.1. Let $\mathcal{C} = (C, \Box, 1, \alpha, \lambda, \rho)$ and $\tilde{\mathcal{C}} = (\tilde{C}, \tilde{\Box}, \tilde{1}, \tilde{\alpha}, \tilde{\lambda}, \tilde{\rho})$ be monoidal categories with twines τ and $\tilde{\tau}$ respectively. A *colax entwined functor* is a colax monoidal functor $\mathcal{T} = (T, T_2, T_0)\colon \mathcal{C} \to \tilde{\mathcal{C}}$ equipped with a monoidal natural transformation

$\epsilon\colon \mathcal{T}\circ(\mathrm{id}_C,\tau,\mathrm{id}_1) \Rightarrow (\mathrm{id}_{\tilde{C}},\tilde{\tau},\mathrm{id}_{\tilde{1}})\circ\mathcal{T}$. Written out in full, ϵ is a natural transformation $T \Rightarrow T$ such that the diagram

$$\begin{array}{ccccc}
T\tilde{\Box}T & \xrightarrow{\epsilon\tilde{\Box}\epsilon} & T\tilde{\Box}T & \xleftarrow{\tilde{\tau}\circ(T\times T)} & T\tilde{\Box}T \\
{\scriptstyle T_2}\uparrow & & & & \uparrow{\scriptstyle T_2} \\
T\circ(\Box) & \xleftarrow[T\circ\tau]{} & T\circ(\Box) & \xrightarrow[\epsilon\circ(\Box)]{} & T\circ(\Box)
\end{array}$$

commutes in $\mathrm{Fun}(C\times C,\tilde{C})$, and $\epsilon(1)=\mathrm{id}_{T(1)}$.

Proposition 7.2.2. *Entwined categories, colax entwined functors, and monoidal natural transformations form a strict* 2*-category.*

Proof. Routine diagram check. ☐

Definition 7.2.3. Two twines on a monoidal category $\mathcal{C}$ are *equivalent* if the identity monoidal functor on $\mathcal{C}$ can be promoted to an equivalence (Definition 4.3.4) in the strict 2-category of Proposition 7.2.2.

Proposition 7.2.4. *Two twines on the same monoidal category $\mathcal{C}$ are equivalent iff they are isomorphic as monoidal functors, i.e., related by a monoidal natural isomorphism.*

This chapter's narrow computational exploration will not suffice to judge whether Definition 7.2.3 is good; some future refinement might prove more useful.

7.2.1 Entwined fusion categories

The notions of entwined monoidal categories so far introduced carry over unchanged to the realm of fusion categories, where a virtue of Definition 7.2.3 is immediately discovered:

Corollary 7.2.5. *Ocneanu rigidity holds for entwined fusion categories: there are only finitely many equivalence classes of twines on a given finite fusion category.*

Proof. Proposition 7.2.4 and Ocneanu rigidity for tensor functors (Theorem 4.4.3). □

Entwined fusion categories may be suitable for describing certain fractional quantum Hall theories when braidings are unavailable. For instance, the fermionic Moore-Read fusion rule is significant physically, but lacks braidings (Bonderson [3]), and so needs an alternative to the usual structure of braided fusion categories.

7.2.2 Entwined fusion systems

Definition 7.2.6. For ring elements X, Y, Z, write $[X, Y, Z] = XYZ - ZYX$. We say X, Y, Z *triple commute* if $[X, Y, Z] = 0$.

Definition 7.2.7. A *twine* ξ on a fusion system $\mathcal{S} = (L, V, 1, F, \lambda, \rho)$ assigns an automorphism $\xi_r^{x,y}$ of $V_r^{x,y}$ to each triple $r, x, y \in L$, such that $(\mathrm{id}_L, \xi, 1)$ is a gauge automorphism of $\mathcal{S}$ (Definition 5.0.15) satisfying the following *dodecagon axiom.* Let P_0, P_1, P_2 be the left, bottom left or bottom right, and right vertices of the pentagon equation (5.0.8), respectively. Let $A_i\colon P_i \to P_{i+1}$ be the canonical associators, for $i = 1, 2$. Let

$$\Xi_0 = \bigoplus_{p,u} \mathrm{id} \otimes \xi_u^{p,y} \otimes \mathrm{id}, \qquad \Xi_1 = \bigoplus_{s,u} \xi_s^{x,y} \otimes \mathrm{id} \otimes \mathrm{id}, \qquad \Xi_2 = \bigoplus_{q,v} \mathrm{id} \otimes \xi_v^{x,q} \otimes \mathrm{id}$$

Then $D_r^{wxyz}: \qquad [A_1 \Xi_0 A_1^{-1}, \Xi_1^{-1}, A_2^{-1} \Xi_2 A_2] = 0.$

Definition 7.2.8. Two twines on the same fusion system are *equivalent* if they are isomorphic as gauge automorphisms (Definition 5.0.10).

Proposition 7.2.9. *Let $\mathcal{C}$ be a fusion category, and $\mathcal{S}$ a corresponding fusion system. Then up to equivalence, twines on $\mathcal{C}$ are in 1-1 correspondence with twines on $\mathcal{S}$.*

Proof. By Corollary 5.1.5, up to isomorphism, gauge automorphisms of $\mathcal{C}$ are in 1-1 correspondence with gauge automorphisms of $\mathcal{S}$. It is straightforward to check that the two versions of the dodecagon axiom for fusion categories and fusion systems are equivalent. □

Proposition 7.2.10. *Let $\mathcal{C}$ be a fusion category whose fusion rule is a group G. Then twines on $\mathcal{C}$ up to equivalence are in 1-1 correspondence with $H^2(G, \mathbb{F}^\times)$.*

Proof. Since G is a group, the splitting spaces P_i in Definition 7.2.7 have rank 1, so the dodecagon axiom is trivially satisfied. Then see Theorem 6.0.9. □

7.3 Fermionic Moore-Read twines

7.3.1 Entwined 6j fusion systems

Definition 7.3.1. Let f be a 6j fusion system on a multiplicity-free fusion rule L. A *twine* on f is a gauge automorphism ξ (Definition 6.0.7) satisfying the following *dodecagon axiom*: Let $w, x, y, z, r \in L$, and let A, X, B, Y, C be the square matrices indexed by $L \times L$ with $((s', t'), (s, t))$-entries

$$\delta_{s',s}\delta_{t',t}\xi_t^{sy} \qquad f_{trt'}^{ws'z} f_{sts'}^{wxy} \qquad \delta_{s',s}\delta_{t',t}\xi_s^{xy} \qquad f_{sts'}^{xyz} \qquad \delta_{s',s}\delta_{t',t}\xi_t^{xs}$$

respectively. Then $D_r^{wxyz}: \qquad [XAX^{-1}, B^{-1}, Y^{-1}CY] = 0.$

Definition 7.3.2. Two twines $\xi, \tilde{\xi}$ on a 6j fusion system on a multiplicity-free fusion rule L are *equivalent* if they are related by a 2-isomorphism, i.e., if there exists $\zeta \colon L \to \mathbb{F}^\times$ such that $\zeta(x)\zeta(y)\xi_r^{x,y} = \zeta(r)\tilde{\xi}_r^{x,y}$ for $x, y, r \in L$.

Proposition 7.3.3. *Let $\mathcal{C}$ be a fusion category on a multiplicity-free fusion rule L, and let f be a corresponding 6j fusion system on L. Up to equivalence, twines on $\mathcal{C}$ are in 1-1 correspondence with twines on f.*

Proof. Follows from Proposition 7.2.9 and Theorem 6.0.9. □

7.3.2 Feudal twines

In this subsection we study twines on feudal 6j fusion systems. Unfortunately the formulas we wring out of the dodecagon axiom are opaque; their only application here is for fermionic Moore-Read twines, in the next subsection.

Lemma 7.3.4. *Let B be an invertible ambidextrous algebra over a group S. Then $(\theta, \phi, \varsigma) \in \mathrm{fix}(S)^{S\times S} \times (B^\times)^S \times (B^\times)$ is a gauge automorphism (Definition 6.2.3) of any überderivation on S over $\mathbb{F}^B$ iff ϕ is normalized and for $a, b \in S$,*

$$1 = \frac{\phi(a)^a \bar{\phi}^b(b)^a \varsigma^b \cdot \varsigma}{\phi^b(a)\bar{\phi}^b(b)^a \varsigma\varsigma^b} \qquad\qquad \theta = \grave{\delta}\phi \qquad\qquad \bar{\varsigma} = \varsigma$$

Proof. Immediate from Definition 6.2.3, recorded here for convenience. □

The following convention and definition are purely technical, only used in the following lemma.

Convention. In this subsection we use three special notational conveniences. Let $A \trianglelefteq S$ be groups and B a ring. An underscore _ within an expression denotes an arbitrary element of A. For instance, if $\phi\colon S \to B$ and $b \in S$, we write $\phi(\_b)$ for the function $A \to B$ taking each $x \in A$ to $\phi(xb)$. If $\chi\colon S \times S \to B$, then for any $y \in A$, we write $\chi_y = \chi(y, \_)\colon A \to B$. Finally, for any $f, g\colon A \to B$, we write $f \cdot_A g = \sum_{x \in A} f(x)g(x)$.

Definition 7.3.5. Let B be an invertible ambidextrous bimodule over a group S, and suppose the set A of elements of S acting trivially on B is finite. Suppose (χ, υ, τ) is an überderivation on S over B (Definition 6.2.3). A *twine* on (χ, υ, τ) is a gauge transformation $(\theta, \phi, \varsigma)$ from (χ, υ, τ) to itself satisfying the following five axioms: for all $a, b, c, d, e \in S$ and $r \in A$,

$$D^{1110}\colon \qquad \phi(\_b) \cdot_A \chi_r = 0 \qquad \text{or} \qquad \phi(e)\phi(era) = \phi(er)\phi(ea)$$

$$D^{0111}\colon \qquad {}^a\phi(e)\phi(are) = \phi(ae)\,{}^a\phi(re) \qquad \text{or} \qquad \chi_r \cdot_A \bar{\phi}^c(c\_) = 0$$

$$D^{1011}\colon \qquad \chi_r \cdot_A \frac{1}{\phi^d}(\_b) = 0 \qquad \text{or} \qquad {}^a\phi(d)\phi(adr) = \phi(ad)\,{}^a\phi(dr)$$

$$D^{1101}\colon \qquad \phi(rd)\phi(da) = \phi(d)\phi(rda) \qquad \text{or} \qquad \chi_r \cdot_A \frac{1}{{}^d\bar{\phi}^c}(c\_) = 0$$

$$\begin{aligned} D^{1111}\colon \qquad & \sum_z {}^{\bar{a}}(\chi_{z/x} \cdot_A \phi(\_a))\bar{\phi}^e(ez)({}^c\bar{\chi}_{z/y} \cdot_A \phi^e(c\_)) \\ &= \sum_z {}^{\bar{a}}(\chi_{y/z} \cdot_A \phi(\_a))\bar{\phi}^e(ez)({}^c\bar{\chi}_{x/z} \cdot_A \phi^e(c\_)) \end{aligned}$$

Lemma 7.3.6. *Let L be a feudal fusion rule with serfs S and lords M, and let f be a 6j fusion system on L, corresponding to an überderivation (χ, υ, τ) on S over $\mathbb{F}^M$ as in Theorem 6.2.5. Then twines on f are in 1-1 correspondence with twines on (χ, υ, τ), via the equivalence of that theorem.*

Proof. Just trudging through the dodecagon axiom (Definition 7.3.1), using the feudal pentagons of Subsection 6.2.2. Fortunately all but five of the dodecagons are 1-dimensional and thus automatically satisfied. □

7.3.3 Equivalence of twines

Lemma 7.3.7. *Let* (χ, υ, τ) *be an überderivation on the fermionic Moore-Read fusion rule* $\{1, \alpha, \psi, \alpha', \sigma, \sigma'\}$ *(Example 2.2.2), corresponding to a fusion category as in Example 6.2.10. W.l.o.g. by Lemma 6.2.12, take* $\tau \in \mathbb{F}$. *Then* $(\theta, \phi, \varsigma) \in \mathrm{fix}(S)^{S\times S} \times (B^\times)^S \times (B^\times)$ *is a gauge automorphism of* (χ, υ, τ) *iff* ϕ *is normalized and*

$$\theta = \grave{\delta}\phi, \qquad \phi(\psi) \in \mathbb{F}^\times, \qquad \frac{\phi(\alpha)}{\bar{\phi}(\alpha)} = \frac{\phi(\alpha')}{\bar{\phi}(\alpha')} \in \{1, -1\}.$$

Moreover, $(\theta, \phi, \varsigma)$ *is a twine iff in addition* $\frac{\phi(\alpha')}{\phi(\alpha)} = \phi(\psi) \in \{1, -1\}$.

Proof. With only four serfs and two lords, it is straightforward to work out the formulas of Lemma 7.3.4 and Definition 7.3.5. Then invoke Lemma 7.3.6. □

Theorem 7.3.8. *Every twine on a fermionic Moore-Read fusion category is equivalent in the eyes of Definition 7.2.3 to the trivial twine.*

Proof. Let $(\theta, \phi, \varsigma)$ be as in Lemma 7.3.7. Let $S = \{1, \alpha, \psi, \alpha'\}$ be the serfs of the fermionic Moore-Read fusion rule (Example 2.2.2). By Theorem 6.2.5, a 2-isomorphism (Definition 6.0.8) from the trivial twine to the gauge automorphism corresponding to $(\theta, \phi, \varsigma)$ is given by $\zeta_0 \colon S \to \mathbb{F}^\times$ and $\zeta_1 \in (\mathbb{F}^\times)^M$ such that

$$\phi(\psi) = \zeta_0(\psi) \qquad \phi(\alpha) = \zeta_0(\alpha)\frac{\overline{\zeta_1}}{\zeta_1} \qquad \phi(\alpha') = \zeta_0(\alpha')\frac{\overline{\zeta_1}}{\zeta_1} \qquad \varsigma = \zeta_1\overline{\zeta_1}$$

Due to the constraints on ϕ, these equations can always be satisfied. Thus $(\theta, \phi, \varsigma)$ is equivalent to the trivial twine. $\square$

It is not clear that Definition 7.2.3 is the right notion of equivalence of twines. Its virtues are categoricity and Ocneanu rigidity. Here is an alternative notion.

Definition 7.3.9. A twine ξ on a 6j fusion system on a multiplicity-free fusion rule L is *homothetically trivial* if it factors through the universal grading of L, i.e., ξ_z^{xy} depends only on the images of x, y, z in the universal grading group (Theorem 2.1.9). Two twines on a 6j fusion system are *homothetically equivalent* if their pointwise quotient in $\mathbb{F}$ is homothetically trivial.

Note that a twine is homothetically trivial iff it is homothetically equivalent to the trivial twine. This definition is motivated by what sort of representations of pure braid groups are afforded by an entwined fusion category. With a homothetically trivial twine, an n-strand pure braid acts as a homothety on each splitting space from one simple object to n simple objects.

Theorem 7.3.10. *Up to homothetic equivalence, there is a unique nontrivial twine on any fusion category on the fermionic Moore-Read fusion rule of Example 2.2.2. Using a 6j fusion system f with $f_{1,\sigma,1}^{\sigma,\sigma',\sigma} = f_{1,\sigma',1}^{\sigma',\sigma,\sigma'}$, a representative ξ of the nontrivial* *homothetic equivalence class of twines is given by*

$$\begin{array}{llll}
\xi_{\sigma}^{1,\sigma} = 1 & \xi_{\sigma'}^{1,\sigma'} = 1 & \xi_{1}^{\sigma,\sigma'} = -i & \xi_{1}^{\sigma',\sigma} = -i \\
\xi_{\sigma}^{\psi,\sigma} = -1 & \xi_{\sigma'}^{\psi,\sigma'} = -1 & \xi_{\psi}^{\sigma,\sigma'} = i & \xi_{\psi}^{\sigma',\sigma} = i \\
\xi_{\sigma}^{\alpha,\sigma'} = 1 & \xi_{\sigma'}^{\alpha,\sigma} = -1 & \xi_{\alpha}^{\sigma,\sigma} = i & \xi_{\alpha}^{\sigma',\sigma'} = -i \\
\xi_{\sigma}^{\alpha',\sigma'} = -1 & \xi_{\sigma'}^{\alpha',\sigma} = 1 & \xi_{\alpha'}^{\sigma,\sigma} = -i & \xi_{\alpha'}^{\sigma',\sigma'} = i
\end{array}$$

with ξ symmetric in the superscripts and $\xi_{ab}^{a,b} = 1$ for all serfs $a, b \in \{1, \alpha, \psi, \alpha'\}$. In the notation of Lemma 7.3.7, this representative ξ is given by

$$\phi(\psi) = (-1, -1), \qquad \phi(\alpha) = (1, -1), \qquad \phi(\alpha') = (-1, 1), \qquad \varsigma = -i.$$

Proof. The constraint on f is synonymous, via Definition 6.2.4, with the innocuous assumption $\tau \in \mathbb{F}$ of Lemma 7.3.7. The theorem follows straightforwardly from Lemma 7.3.7, as we now sketch. Recall from Example 2.2.2 that the universal grading of the fermionic Moore-Read fusion rule identifies ψ with 1 and α' with α while leaving the two lords distinct. Then a twine $(\theta, \phi, \varsigma)$ as in Lemma 7.3.7 is homothetically trivial iff $\phi(\psi) = 1$. The homothetically nontrivial twines, with $\phi(\psi) = -1$, are all homothetically equivalent. □

7.4 Structures on fermionic Moore-Read

7.4.1 Pure twists

A key notion for braided fusion categories is a *twist*, needed to define a ribbon structure, which gives invariants of ribbon tangles. A twist determines a twine via the so-called suspender formula, stated here categorically:

Definition 7.4.1. A *twist* on an entwined monoidal category, fusion category or system is a 2-isomorphism from the automorphism corresponding to the twine to the identity.

Definition 7.4.2. Let $\mathcal{C} = (C, \Box, 1, \alpha, \lambda, \rho)$ be a rigid entwined monoidal category with twine τ. A twist θ on $\mathcal{C}$ is *self-dual* if

$$\theta^2(x) = x \xrightarrow{\lambda} 1x \xrightarrow{b\Box x} (xx^*)x \xrightarrow{\alpha} x(x^*x) \xrightarrow{x\Box\tau^{-1}} x(x^*x) \xrightarrow{x\Box d} x1 \xrightarrow{\rho^{-1}} x \tag{7.2}$$

for all objects x, where x^* is a right dual of x (Definition 4.1.7).

This notion of a twist in the absence of braiding is due to Bruguieres [6], who showed that self-dual twisted rigid monoidal categories give invariants of ribbon string links. A twist corresponds to a 2π rotation of a ribbon strand. For modeling fermionic fractional quantum Hall theories such as fermionic Moore-Read, it may be useful to consider only rotations that are multiples of 4π:

Definition 7.4.3. A *pure twist* on an entwined monoidal category is a 2-isomorphism from the automorphism corresponding to the square of the twine to the identity.

Definition 7.4.4. A pure twist on a rigid entwined monoidal category is *self-dual* if it plays the role of θ^2 in equation (7.2).

Definition 7.4.5. Let f be a 6j fusion system on a multiplicity-free fusion rule L, with twine ξ and twist $\zeta\colon L \to \mathbb{F}^\times$. We say ζ is *self-dual* if $\zeta(x) = 1/\xi_1^{x,\bar{x}}$ for $x \in L$.

Lemma 7.4.6. *Definitions 7.4.4 and 7.4.5 agree.*

Proof. Similar to the corresponding correspondence for rigidity. □

Proposition 7.4.7. *A fermionic Moore-Read twine ξ given by $(\theta, \phi, \varsigma)$ as in Lemma 7.3.7 admits a self-dual pure twist iff $\phi(\psi) = \frac{\bar{\phi}}{\phi}(\alpha)$, or equivalently*

$$\xi_\sigma^{\psi,\sigma} = \frac{\xi_\sigma^{\alpha,\sigma'}}{\xi_{\sigma'}^{\alpha,\sigma}}$$

If it exists, the pure twist ζ is given by $\zeta_0 \equiv 1$ and $\zeta_1 \equiv 1/\varsigma$, or equivalently

$$\zeta(a) = 1, \qquad\qquad \zeta(m) = 1/\xi_1^{\sigma,\sigma'} = 1/\xi_1^{\sigma',\sigma}$$

for any serf $a \in \{1, \alpha, \psi, \alpha'\}$ or lord $m \in \{\sigma, \sigma'\}$. The representative twine of Theorem 7.3.10 has a self-dual pure twist of 1 *on serfs and i on lords.*

Proof. A self-dual pure twist ζ is given by $\zeta_0\colon S \to \mathbb{F}^\times$ and $\zeta_1\colon M \to \mathbb{F}^\times$, where S is the serfs and M the lords, such that

$$\zeta_0(1) = \zeta_0(\psi) = 1, \qquad \zeta_0(\alpha) = \phi(\psi)\frac{\phi}{\bar{\phi}}(\alpha'), \qquad \zeta_0(\alpha') = \phi(\psi)\frac{\phi}{\bar{\phi}}(\alpha), \qquad \zeta_1 \equiv \frac{1}{\varsigma}$$

with the condition $\phi(\alpha) = \bar{\phi}(\alpha')$, which is equivalent to $\phi(\psi) = \frac{\bar{\phi}}{\phi}(\alpha)$. □

Remark. To build well-behaved invariants for a suitable class of ribbon tangles, Bruguieres [6] imposes additional axioms on twisted spherical monoidal categories, defining *turban* categories. Perhaps analogues of these axioms may be found for purely twisted entwined spherical monoidal categories, and may be needed for physical applications.

7.4.2 Pivotality and sphericity

Pivotality is a key property of rigid monoidal categories, generalizing the natural isomorphism between a finite-dimensional vector space and its double dual. See for instance Boyarchenko [4] or Wang [29]. We skip to the corresponding notion for 6j fusion systems.

Definition 7.4.8 (Wang [29])**.** A *pivotal structure* on a 6j fusion system on a multiplicity-free fusion rule L is a function $t\colon L \to \mathbb{F}^\times$ such that

$$t(1) = 1,$$

$$t(\bar{a}) = t(a)^{-1},$$

$$t(a)^{-1}t(b)^{-1}t(c) = F_1^{a,b,\bar{c}}F_1^{b,\bar{c},a}F_1^{\bar{c},a,b}$$

for all $a, b, c \in L$ with $c \in ab$. We say t is *spherical* if $t(a)^2 = 1$ for all a.

Lemma 7.4.9. *Let f be a normal (Definition 6.2.14) feudal 6j fusion system. Let (χ, υ, τ) be the corresponding überderivation on $\mathbb{F}^M$ over the serfs S, where M is the lords. A pivotal structure on f is given by a pair $(t_0, t_1) \in (\mathbb{F}^\times)^S \times (\mathbb{F}^M)^\times$ with*

$$t_0(1) = 1, \qquad t_0(\bar{a}) = t_0(a)^{-1}, \qquad t_1\overline{t_1} \equiv 1,$$

$$\delta t_0(a,b)^{-1} = \alpha(a,b,\bar{b}\bar{a})\alpha(b,\bar{b}\bar{a},a)\alpha(\bar{b}\bar{a},a,b),$$

$$t_0(a) = \frac{t_1}{{}^{a}t_1\bar{\beta}_3(a,1)} = \frac{t_1}{\beta_3^a(a,1)t_1^a} = {}^{a}\overline{t_1}t_1\beta_3(\bar{a},1)$$

where as in the proof of Lemma 6.2.16,

$$\alpha = \frac{1}{\grave{\delta}\upsilon} \colon S^3 \to \mathbb{F}^\times \qquad\qquad \beta_3(a,1) = \frac{\tau^{\bar{a}}}{\alpha(a,\bar{a},a)\tau}$$

Proof. Straight from Definition 7.4.8. □

Proposition 7.4.10. *Over a field with square roots and four distinct 4th roots of unity, a fermionic Moore-Read fusion category has four pivotal structures, two of them spherical. In a nice gauge—a normal feudal gauge whose corresponding überderivation (χ, υ, τ) has $\tau \in \mathbb{F}$ and υ symmetric—a pivotal structure $t \colon L \to \mathbb{F}^\times$ is given by*

$$t(1) = t(\psi) = 1, \qquad t(\alpha) = t(\alpha') = t(\sigma)^2 xr, \qquad t(\sigma)^4 = 1, \qquad t(\sigma') = t(\sigma)^{-1}$$

where $\{1, \alpha, \psi, \alpha', \sigma, \sigma'\}$ is the fermionic Moore-Read fusion rule (Example 2.1.2) and $xr = \pm 1$ is determined by the 6j symbols as in the proof of Example 6.2.10.

Proof. Routine consequence of Lemma 7.4.9. □

7.4.3 Summary

#	structures	underlying structure
1	fusion rule	hypergroup
4	fusion categories	fusion rule
1	twine	fusion category
0	braidings	fusion category
0 or 1	self-dual pure twists	twine
4	pivotal structures	fusion category
2	spherical structures	fusion category

Table 7.1: Structures on the fermionic Moore-Read hypergroup, each considered up to equivalence fixing the underlying structure. Twines are considered up to monoidal natural isomorphism of tensor functors.

Chapter 8

Fractional quantum Hall physics

Here we sketch the roles of translation invariant polynomials, fusion rules, fusion categories, and twines in fractional quantum Hall physics and topological quantum computation. A decent introduction to these miraculous realms is beyond the scope of this thesis; we merely aim to motivate our mathematics. For reference, see Wang [29].

In the fractional quantum Hall effect (FQHE), a two-dimensional layer of electrons sandwiched between two semiconductors is subjected to a strong perpendicular magnetic field and cooled near absolute zero. The electrons are described by a wavefunction which is a Gaussian multiplied by a complex polynomial. Properties of such polynomials are studied in Sections 8.1 and 8.2. As an example, the Pfaffian wavefunction

$$\mathrm{Pf}\left(\frac{1}{z_i - z_j}\right)\prod_{i<j}(z_i - z_j)^2$$

(dropping the Gaussian) for $2n$ electrons at $z_1, \ldots, z_{2n}$, where Pf denotes the Pfaffian polynomial of a skew-symmetric $2n \times 2n$ matrix, is believed to model the *Moore-Read* state.

Under the right conditions, certain defects or vortices or localized patterns arise in the electron liquid, called *quasiparticles*. At an emergent level of description, these quasiparticles take on a life of their own. For a given theory, corresponding to some range of physical conditions, there is a finite set of quasiparticle types, called *labels*. Quasiparticles can fuse together, or split apart into other quasiparticles. Regarding the absence of a quasiparticle as the trivial label 1, the combinatorial laws of fusing and splitting are described by a fusion rule, as discussed at the start of Chapter 3.

A category $\mathcal{C}$ could describe a physical theory as follows: objects correspond to possible states; morphisms correspond to physical processes. For a quantum-mechanical theory such as in the FQHE, morphism spaces are Hilbert spaces, and states can be superposed yielding a biproduct on objects; thus $\mathcal{C}$ is $\mathbb{C}$-linear (Definition 4.2.1). Unlike in ordinary physical theories such as govern the underlying electrons, in a FQH state quasiparticle positions are immaterial; a state is determined by a multiset of labels. As a quasiparticle cannot change its label and lacks degrees of freedom other than changing phase, $\mathcal{C}$ is semisimple, labeled by the quasiparticle types (Definition 4.4.1). Juxtaposition or coexistence of quasiparticles yields a monoidal structure on $\mathcal{C}$. The monoidal unit 1 is simple, being the trivial label. Each label has a dual label; the electron sea may birth a quasiparticle along with its dual, and reversely a quasiparticle and its dual may annihilate, making $\mathcal{C}$ *rigid* (Definition 4.1.7). Thus labeled fusion categories describe FQH quasiparticles.

In a theory with n spatial dimensions and one time dimension, a worldline or history of events can be regarded as an $(n+1)$-dimensional sculpture. If the events are m (quasi)particles dancing about without touching, the sculpture is an m-strand braid in an $(n+1)$-dimensional space. The physical theory governing the dancers is *topological* if it is invariant under local deformations, i.e., only the braid's topology matters. Topological theories only exist as emergent descriptions of special condensed matter regimes such as in the FQHE, called *topological phases of matter*, a hot topic for experimentalists and theorists.

For $n \geq 3$, it is well-known that braiding in $(n+1)$ dimensions is uninteresting: topologically, all that matters is how particles are permuted. In particular, transposing or swapping two identical particles is an order 2 operation; for a quantum mechanical theory, it is trivial or hits the wavefunction with a minus sign. These are the precise meanings of *boson* and *fermion*, describing protons and electrons respectively. But for $n = 2$, as in the FQHE, interesting braid action is possible. In particular, transposing two identical particles may introduce a complex phase other than ± 1; such particles are called *abelian anyons*. Even more exotic are *nonabelian anyons*, whose transposition effects a unitary transformation between Hilbert spaces (ground state degeneracies) of dimension > 1. Nonabelian anyons provide the topological interest of the FQHE and the source of its quantum computational power.

Braiding of FQH quasiparticles is usually modeled by a braided fusion category (Section 7.1). Actually there are structures other than braiding, such as sphericity and twists, leading to *modular* categories. Given objects $x_1, \ldots, x_n$ in a braided fusion category with bifunctor $\Box$, an n-strand braid b with underlying permutation π induces a morphism

$$x_1 \Box \cdots \Box x_n \to x_{\pi(1)} \Box \cdots \Box x_{\pi(n)}$$

For any object x, postcomposition with this morphism gives a linear map

$$\mathrm{mor}(x, x_1 \Box \cdots \Box x_n) \to \mathrm{mor}(x, x_{\pi(1)} \Box \cdots \Box x_{\pi(n)})$$

of complex vector spaces. Thus a braided fusion category affords representations of colored braid groups. Some of these representations are of considerable interest within pure mathematics. But they may also come to exist in our world, for instance by braiding nonabelian anyons, one possible mechanism for quantum computing using the FQHE.

A quantum algorithm is a family of unitary transformations $T\colon V \to W$ of finite-dimensional Hilbert spaces. Computational input is encoded as a state vector in V; the transformation T is applied; and the result in W is measured. The challenge of quantum algorithm design is to cook up a T which solves some interesting problem more efficiently than known classical solutions. For practical purposes, the most exciting quantum algorithms are simulations of quantum physical systems, which would be useful in the science and engineering of small things like molecules. But building quantum computers is hard because they tend to be terribly sensitive to noise in the hardware or environment. Topological quantum computation is the strategy of building a quantum computer using topological phases of matter, which are "deaf" to local perturbations.

Some FQH theories do not seem to fit the usual framework of a braided fusion category (Read and Wang [25]). One reason may be that as a FQH anyon is a pattern made of many electrons, it should not notice an extra electron, which as a fermion might gum up a braiding with a minus sign. This difficulty might be circumvented with twines (pure braidings), which is why we studied them in Chapter 7.

8.1 Translation invariant polynomials

A polynomial $p(z_1, \ldots, z_n)$ is *translation invariant* if

$$p(z_1 + c, \ldots, z_n + c) = p(z_1, \ldots, z_n)$$

for all c. A wavefunction for n electrons in the FQHE is usually a translation invariant complex polynomial multiplied by a Gaussian (Wen and Wang [30]). For classifying such wavefunctions, the Gaussian may be ignored. One might expect the polynomial to be antisymmetric, as electrons are fermions. But under the strange conditions of the FQHE, sometimes we can model them as bosons, yielding a symmetric polynomial. Thus we are led to study translation invariant (anti)symmetric polynomials. Antisymmetric polynomials need no special treatment, being symmetric polynomials multiplied by the Vandermonde determinant $\prod_{i<j}(z_i - z_j)$. Translation invariant symmetric polynomials are studied in this section and the next over a field $\mathbb{F}$ of characteristic 0. For intended applications, $\mathbb{F} = \mathbb{C}$.

In this section we give a simple description of the ring of all translation invariant symmetric polynomials (Corollary 8.1.3). We begin with just translation invariance. Let $T \subseteq \mathbb{F}[z_1, \ldots, z_n]$ be the ring of translation invariant polynomials. Imagining $z_1, \ldots, z_n$ as the locations of n identical particles, and $x_1, \ldots, x_n$ as the corresponding center of mass coordinates, our main theorem says that T written in terms of $x_1, \ldots, x_n$ is $\mathbb{F}[x_1, \ldots, x_n]$ modulo one degree of freedom.

Theorem 8.1.1. *Let $\rho\colon \mathbb{F}[x_1, \ldots, x_n] \to T$ be the surjective algebra homomorphism*

$$x_i \mapsto z_i - z_{\text{avg}}$$

where $z_{\text{avg}} = \frac{1}{n}(z_1 + \cdots + z_n)$. Then $\ker \rho = (x_{\text{avg}})$.

Proof. Subsection 8.1.1. ☐

Now let $R \subseteq T$ be the ring of translation invariant symmetric polynomials in $z_1, \ldots, z_n$, and $S \subseteq \mathbb{F}[x_1, \ldots, x_n]$ be the ring of symmetric polynomials in $x_1, \ldots, x_n$.

Corollary 8.1.2. *Let $\sigma\colon S \to R$ agree with ρ. Then σ is a surjective algebra homomorphism, with kernel $(x_1 + \cdots + x_n)$.*

Proof. It suffices to show $\rho(S) = R$. Clearly $\rho(S) \subseteq R$. Given $p(z_1, \ldots, z_n) \in R$, translation invariance yields

$$p(z_1, \ldots, z_n) = p(z_1 - z_{\text{avg}}, \ldots, z_n - z_{\text{avg}}) = \rho(p(x_1, \ldots, x_n)).$$

Thus $\rho(S) = R$. ☐

Since $\mathbb{F}$ has characteristic 0, any element of S can be written uniquely as a polynomial in the power sum symmetric polynomials $x_1^k + \cdots + x_n^k$, where $1 \leq k \leq n$.

In other words, the algebra homomorphism $\theta\colon \mathbb{F}[w_1, \dots, w_n] \to S$ defined by $\theta(w_k) = x_1^k + \cdots + x_n^k$ is an isomorphism. Note that we could define a different isomorphism θ using elementary symmetric polynomials or complete homogeneous symmetric polynomials. In any case, $\sigma\theta\colon \mathbb{F}[w_1, \dots, w_n] \to R$ is a surjective algebra homomorphism, with kernel (w_1).

Corollary 8.1.3. *The algebra homomorphism* $\mathbb{F}[w_2, \dots, w_n] \to R$ *given by*

$$w_k \mapsto (z_1 - z_{\text{avg}})^k + \cdots + (z_n - z_{\text{avg}})^k$$

is an isomorphism.

Next we consider the vector space R^d of all polynomials in R which are homogeneous of degree d. Let f be the above isomorphism. Since $f(w_k)$ is homogeneous of degree k, we obtain a basis for R^d, namely all

$$w_\lambda = \prod_{k=2}^{n} f(w_k)^{\lambda_k} \tag{8.1}$$

where λ is any partition of d into integers between 2 and n, and λ_k is the multiplicity of k in λ. Simon, Rezayi, and Cooper [27] prove directly that these w_λ form a basis of R^d, whereas we have deduced this fact from the ring structure of R. Although [27] defines w_λ using elementary symmetric polynomials rather than power sum symmetric polynomials, this difference is purely cosmetic. Since the dimension m_d of R^d is the number of partitions of d into integers between 2 and n, a generating function for m_d is

$$\sum_{d=0}^{\infty} m_d t^d = \prod_{s=2}^{n} \frac{1}{1 - t^s}$$

Finally, we describe the vector space $A \subset \mathbb{F}[z_1, \ldots, z_n]$ of translation invariant antisymmetric polynomials. It is well-known that any antisymmetric polynomial can be written uniquely as $q\Delta$, where q is a symmetric polynomial and Δ is the Vandermonde determinant $\prod_{i<j}(z_i - z_j)$. Since Δ is translation invariant, we have $A = R\Delta$, defining a vector space isomorphism $R \to A$, which sends each basis (8.1) to a basis for the vector space of homogeneous translation invariant antisymmetric polynomials of degree $d + n(n-1)/2$.

8.1.1 Proof of Theorem 8.1.1

We factor ρ into two maps which are easier to study:

$$\overset{\rho}{\overbrace{X \xrightarrow[\tau]{} Y \xrightarrow[\pi]{} T}}$$

Let $Y = \mathbb{F}[y_1, \ldots, y_n]$, and define τ, π by

$$\tau(x_i) = \frac{1}{n}\sum_{j=0}^{n-1}(n-1-j)y_{i+j} \qquad\qquad \pi(y_i) = z_i - z_{i+1}$$

where index addition is modulo n. Then

$$\pi\tau(x_i) = \frac{1}{n}\left((n-1)z_i - \sum_{j\neq i} z_j\right) = z_i - z_{\text{avg}}$$

showing $\pi\tau = \rho$. It suffices to show $\tau^{-1}(\ker \pi) = (x_{\text{avg}})$. Since $\tau(x_{\text{avg}}) \propto y_{\text{avg}}$, this follows from Lemmas 8.1.4 and 8.1.6 below.

Lemma 8.1.4. *τ is an isomorphism.*

Proof. Let $\hat{\tau}\colon \mathbb{F}x_1 + \cdots + \mathbb{F}x_n \to \mathbb{F}y_1 + \cdots + \mathbb{F}y_n$ be the linear map which extends to τ. Then the matrix M of $\hat{\tau}$ with respect to the evident bases is the $n \times n$ circulant matrix with first column vector

$$v = \frac{1}{n}(n-1, n-2, \dots, 0).$$

Then $M^\top$ is the circulant matrix with first row v. Since $\mathrm{char}\mathbb{F} = 0$, the entries of v form a strictly decreasing sequence of nonnegative reals. Therefore $M^\top$ is nonsingular by Theorem 3 of Geller, Kra, Popescu, and Simanca [14]. Hence $\hat{\tau}$ is an isomorphism. Therefore τ is an isomorphism by Observation 8.1.5. □

Observation 8.1.5. *Suppose $f : \mathbb{F}[a_1, \dots, a_n] \to \mathbb{F}[b_1, \dots, b_n]$ is an algebra homomorphism between polynomial rings which restricts to a linear map*

$$\hat{f} : \mathbb{F}a_1 + \cdots + \mathbb{F}a_n \to \mathbb{F}b_1 + \cdots + \mathbb{F}b_n$$

If $\hat{f}$ is an isomorphism, then so is f.

Proof. The universal property of polynomial rings. □

Lemma 8.1.6. $\ker \pi = (y_{\mathrm{avg}})$.

Proof. Let $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ be the chain map

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (y_{\mathrm{avg}}) & \longrightarrow & Y & \xrightarrow{\pi} & T & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \alpha_1} & & \uparrow{\scriptstyle \alpha_2} & & \uparrow{\scriptstyle \alpha_3} & & \\
0 & \longrightarrow & (y_1) & \longrightarrow & Y & \xrightarrow[\pi']{} & Y' & \longrightarrow & 0
\end{array}$$

where $Y' = \mathbb{F}[y_2, \dots, y_n]$, and $\pi', \alpha_3, \alpha_2, \alpha_1$ are the algebra homomorphisms such that π' kills y_1 and fixes the other variables, α_3 and π agree, α_2 sends y_1 to y_{avg} and fixes the other variables, α_1 and α_2 agree, and the unlabeled nonzero maps are inclusions of ideals. We want the top sequence to be exact. Since the bottom sequence is exact, it suffices to check α is a chain isomorphism.

Since α is a chain map, it suffices to show each component is an isomorphism. By Observation 8.1.5, α_2 is an isomorphism. Then so is α_1. For α_3, let $\beta\colon Y' \to Y'$ be the algebra homomorphism given by $\beta(y_i) = y_i + \cdots + y_n$ for $2 \le i \le n$. Again by Observation 8.1.5, β is an isomorphism. Then it suffices to show $\gamma = \alpha_3\beta$ is an isomorphism.

Note $\gamma\colon Y' \to T$ and $\gamma(y_i) = z_i - z_1$ for $2 \le i \le n$. Since

$$p(z_1, \ldots, z_n) = p(0, z_2 - z_1, \ldots, z_n - z_1) = \gamma(p(0, y_2, \ldots, y_n))$$

for any $p(z_1, \ldots, z_n) \in T$, the homomorphism γ is surjective. If

$$0 = \gamma(q(y_2, \ldots, y_n)) = q(z_2 - z_1, \ldots, z_n - z_1)$$

then $0 = \pi'(q(y_2 - y_1, \ldots, y_n - y_1)) = q(y_2, \ldots, y_n)$, showing γ is injective. Thus γ is an isomorphism. □

8.2 Haldane's conjecture

In this section we find a counterexample to Haldane's conjecture [15] that every homogeneous translation invariant symmetric polynomial satisfies a certain physically convenient property (Proposition 8.2.7). We prove the conjecture for polynomials of at most three variables, construct a minimal counterexample, and discuss whether a weakened version of the conjecture holds.

Every symmetric polynomial is a unique linear combination of symmetrized monomials, which physicists like to call *boson occupation states*. We identify symmetrized monomials with multisets of natural numbers:

$$[l_1, \ldots, l_n] = \sum_{\sigma \in \mathrm{Sym}(n)} z_{\sigma(1)}^{l_1} \cdots z_{\sigma(n)}^{l_n}$$

For instance, the multiset $[5, 0, 0]$ corresponds to the symmetrized monomial $2z_1^5 + 2z_2^5 + 2z_3^5$. *Squeezing* a symmetrized monomial $[l_1, \ldots, l_n]$ means decrementing l_i and incrementing l_j for any pair of indices i, j such that $l_i > l_j + 1$. The *squeezing order* is a partial order on symmetrized monomials: put $s > t$ iff t can be obtained from s by repeated squeezing. For a symmetric polynomial p, let $B(p)$ be the set of all symmetrized monomials with nonzero coefficient in p. We view $B(p)$ as a poset under the squeezing order and refer to it as the *squeezing poset* of p.

Definition 8.2.1. A symmetric polynomial is *Haldane* if its squeezing poset has a maximum.

Conjecture 8.2.2 (Haldane [15])**.** *Every homogeneous translation invariant symmetric polynomial is Haldane.*

Remark. Since squeezing preserves homogeneous degree, Haldane polynomials are homogeneous. Many homogeneous symmetric polynomials are not Haldane, such as $[3, 3, 0] + [4, 1, 1]$, but these might not be translation invariant.

Proposition 8.2.3. *Haldane's conjecture holds for polynomials of* ≤ 3 *variables.*

Proof. The conjecture is vacuously true for univariate polynomials. Every bivariate symmetrized monomial of homogeneous degree d has the form $[a, b]$, with $a + b = d$. These are linearly ordered under squeezing, so Haldane's conjecture is automatic in the bivariate case.

For the trivariate case, define $\tau\colon \mathbb{F}[z_1, z_2, z_3] \longrightarrow \mathbb{F}[z_1, z_2, z_3, t]$ by

$$\tau(p)(z_1, z_2, z_3, t) = p(z_1 + t, z_2 + t, z_3 + t),$$

so that p is translation invariant iff $\tau(p) = p$. Define linear endomorphisms τ_i of $\mathbb{F}[z_1, z_2, z_3]$ by $\tau(p) = \sum_{i=0}^{d} \tau_i(p)t^i$, so that p is translation invariant iff $\tau_i(p) = 0$ for all $i > 0$. Then

$$\tau_1([a, b, c]) = a[a-1, b, c] + b[a, b-1, c] + c[a, b, c-1]$$

for all $a, b, c > 0$. Now suppose $[a, b, c]$ is a maximal element of the squeezing poset of some $p \in R_3^d$, with $a \geq b \geq c > 0$. Then $[a+1, b, c-1]$ and $[a, b+1, c-1]$ are not in $B(p)$. The above equation then implies that the coefficient of $[a, b, c]$ in p equals c times the coefficient of $[a, b, c-1]$ in $\tau_1(p)$. Thus $\tau_1(p) \neq 0$, contradicting the translation invariance of p. Therefore every maximal element of $B(p)$ has the form $[a, b, 0]$, with $a + b = d$. These are linearly ordered under squeezing; their maximum is the maximum of $B(p)$. $\square$

Any two symmetrized monomials written as weakly decreasing sequences of natural numbers can be compared lexicographically. The lexicographic order $>_{\text{lex}}$ on symmetrized monomials linearizes the squeezing order. Let R_n^d be the vector space of translation invariant symmetric n-variate polynomials of homogeneous degree d, and let L_n^d be the set of lexicographic maxima of squeezing posets of polynomials in R_n^d. Note $|L_n^d| \leq \dim R_n^d$.

Definition 8.2.4. A symmetrized monomial s is *completely squeezable* if $s >_{\text{lex}} t$ implies $s > t$, for all symmetrized monomials t.

Lemma 8.2.5. *If every element of L_n^d is completely squeezable, then Haldane's conjecture holds for R_n^d. If Haldane's conjecture holds for R_n^d, then L_n^d is linearly ordered under squeezing.*

Proof. The first statement is immediate. For the second, suppose $m_1, m_2 \in L_n^d$ are incomparable. Let $p_1, p_2 \in R_n^d$ such that m_i is the lexicographic maximum of $B(p_i)$ for $i = 1, 2$. W.l.o.g. assume m_1 is lexicographically bigger than m_2, and let c_i be the coefficient of m_2 in p_i. Choose a scalar $c \neq -c_1/c_2$, and let $q = p_1 + cp_2$. Then $q \in R_n^d$ and $m_1, m_2 \in B(q)$. Since m_1 is the lexicographic maximum of $B(q)$, it is maximal in $B(q)$ under squeezing. Since m_1 and m_2 are incomparable, q is not Haldane. □

Proposition 8.2.6. *Haldane's conjecture holds for R_4^d with $d < 14$ but fails for R_4^{14}.*

Proof. It is a straightforward computational linear algebraic exercise to compute L_n^d using the basis for R_n^d given by formula (8.1). Since every symmetrized monomial of the form $[a, b, 0, \ldots, 0]$ is completely squeezable, as is $[6, 4, 2, 0]$, we see that every element of L_4^d, $d < 14$, is completely squeezable (Table 8.1). But L_4^{14} is not linearly ordered under squeezing: $[8, 4, 2, 0]$ and $[7, 7, 0, 0]$ are incomparable. Then apply Lemma 8.2.5.

□

d	L_4^d
0	$\emptyset$
1	$\emptyset$
2	$\{[2,0,0,0]\}$
3	$\{[3,0,0,0]\}$
4	$\{[4,0,0,0],[2,2,0,0]\}$
5	$\{[5,0,0,0]\}$
6	$\{[6,0,0,0],[4,2,0,0],[3,3,0,0]\}$
7	$\{[7,0,0,0],[5,2,0,0]\}$
8	$\{[8,0,0,0],[6,2,0,0],[5,3,0,0],[4,4,0,0]\}$
9	$\{[9,0,0,0],[7,2,0,0],[6,3,0,0]\}$
10	$\{[10,0,0,0],[8,2,0,0],[7,3,0,0],[6,4,0,0],[5,5,0,0]\}$
11	$\{[11,0,0,0],[9,2,0,0],[8,3,0,0],[7,4,0,0]\}$
12	$\{[12,0,0,0],[10,2,0,0],[9,3,0,0],[8,4,0,0],[7,5,0,0],[6,6,0,0],[6,4,2,0]\}$
13	$\{[13,0,0,0],[11,2,0,0],[10,3,0,0],[9,4,0,0],[8,5,0,0]\}$
14	$\{[14,0,0,0],[12,2,0,0],[11,3,0,0],[10,4,0,0],[9,5,0,0],[8,6,0,0],$ $[8,4,2,0],[7,7,0,0]\}$

Table 8.1: Enumeration of lexicographic maxima.

It is a straightforward computational linear algebraic exercise to construct a non-Haldane polynomial in R_4^{14} by following the proof of Lemma 8.2.5. We get

$$\begin{aligned} p = \;& 3[8,4,2,0]-3[8,4,1,1]-3[8,3,3,0]+6[8,3,2,1]-3[8,2,2,2] \\ &+3[7,7,0,0]-42[7,6,1,0]+46[7,5,2,0]+80[7,5,1,1]-22[7,4,3,0] \\ &-188[7,4,2,1]+112[7,3,3,1]+8[7,3,2,2]+77[6,6,2,0]+70[6,6,1,1] \\ &-182[6,5,3,0]-700[6,5,2,1]+112[6,4,4,0]+168[6,4,3,1]+1078[6,4,2,2] \\ &-728[6,3,3,2]+5[5,5,4,0]+1072[5,5,3,1]+246[5,5,2,2]-722[5,4,4,1] \\ &-2976[5,4,3,2]+1808[5,3,3,3]+1805[4,4,4,2]-1130[4,4,3,3]. \end{aligned}$$

Proposition 8.2.7. *The polynomial p is a minimal counterexample to Haldane's conjecture.*

Proof. One checks by computer that p is translation invariant. Since it is symmetric and homogeneous but lacks a maximum (Figure 8.1), it breaks Haldane's conjecture. By Proposition 8.2.6 it is minimal with respect to arity and homogeneous degree. $\square$

Remark. The counterexample of Proposition 8.2.7 is not minimal with respect to homogeneous degree. For instance, the smallest pentavariate counterexamples have homogeneous degree 10.

Remark 8.2.8. We might weaken Haldane's conjecture by hoping R_n^d has a basis of Haldane polynomials. Computer evidence suggests $|L_n^d| = \dim R_n^d$. Writing $L_n^d = \{l_1, \dots, l_k\}$, we could then obtain a special basis $\{p_1, \dots, p_k\}$ of R_n^d satisfying $B(p_i) \cap L_n^d = \{l_i\}$. Perhaps it would be a Haldane basis or could be used to construct one.

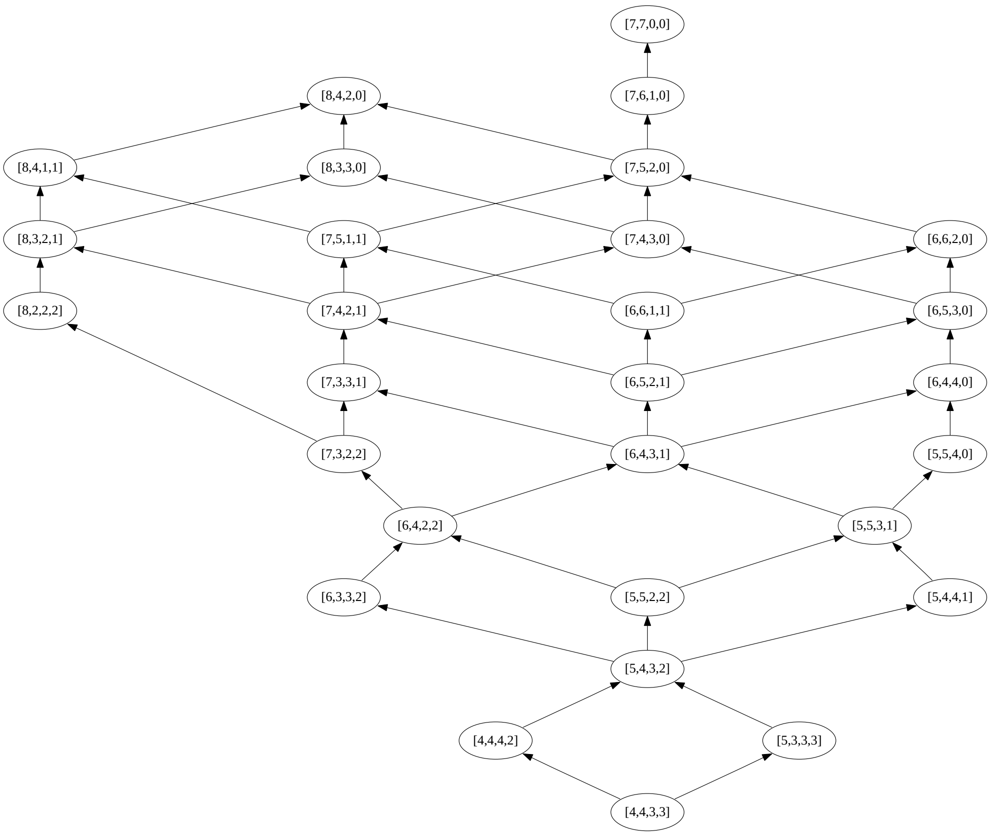


Figure 8.1: Hasse diagram of $B(p)$. Arrows point from smaller to bigger elements.

Chapter 9

Open problems

1. Classify absolutely regular hypergroups of nilpotence class 2, perhaps in terms of groups and posets as in Section 2.3.

2. Classify fusion rules of nilpotence class 2, perhaps in terms of the underlying ARHs and group 2-cocycles over $\mathbb{Z}_+$ as in Section 3.3.

3. Determine which nilpotent fusion rules of simple current index 2 admit fusion categories, and which of these fusion categories admit braidings or twines, perhaps using Theorem 6.2.5 and Lemma 7.3.6.

4. Decide the best definition of equivalence of entwined monoidal categories.

5. Determine how to model fractional quantum Hall quasiparticle motion in the absence of braiding, such as for fermionic Moore-Read. We know of three potentially suitable weakenings of the notion of a braided fusion category: projectively braided fusion categories, whose pentagons and hexagons need only commute up

to phase; twines; and braided fusion categories enriched not over the category of vector spaces, but over some other category such as super vector spaces. Projectivization does not seem promising. We conjecture that entwined fusion categories will provide useful models which will ultimately be refined by enriched braided fusion categories.

6. Determine whether homogeneous translation invariant symmetric polynomials enjoy Haldane bases (Remark 8.2.8).

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