NOTES FOR OCTOBER 6, 2004

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In mathematics, the simple ideas usually come last.
Jacques Hadamard (1865-1963)

1. Review

Proposition 1.1. *In a metric space (A, d) every open d-ball $B_\epsilon(a)$ can be rewritten as the union of open d-balls $B_{\delta(b)}(b)$, one centered at each point $b \in B_\epsilon(a)$.*

Proposition 1.2. *Two metrics d and d' are equivalent on A if and only if every for all $a \in A$, every open d-ball centered at a contains an open d' centered at a and vice versa.*

Corollary 1.3. *If d and d' are equivalent metrics, then every set which is d-open is also d'-open and vice versa. Conversely, it is easy to see that if every d-open set is also d'-open, then by the previous proposition, d and d' are equivalent.*

Theorem 1.4. *A function $f : (A, d) \to (B, d')$ between metric spaces is continuous if and only if for each open set U in B, $f^{-1}(U)$ is open in A.*

Warning 1.5. The *image* of an open set need not be open !!!

There is a notion of *Lipschitz equivalence* for metrics which implies that they are equivalent, but I won't say too much about that.

2. Topological spaces

There the familiar defining properties: everything and nothing are open sets, and the collection is closed under (arbitrary) unions and finite intersections. Clearly given any collections of subsets, you can finite the "smallest" topology with these as open sets.

Example 2.1 (Zariski topology)**.** Call a subset of $\mathbb{C}^n$ closed if it is the solution set of a system of polynomial equations. This collection of sets contains the empty set and all of $\mathbb{C}^n$ and it is closed under finite union and arbitrary intersections. Thus the complements of these sets form a topology on $\mathbb{C}^n$, known as the *Zariski topology*.

There are some closely related structures which you need to keep separate from the notion of a topology. A lattice of sets is a collection of subsets closed under finite unions and finite intersections. A σ-algebra is a collection of subsets closed under countable unions and countable intersections (as well as complementation) and a complete lattice is a collection of subsets closed under arbitrary unions and arbitrary intersections. Every complete lattice is an example of a topology, and every topology is an example of a lattice. Notice that the usual topology on $\mathbb{R}^n$ is not a complete lattice.

Date: October 6, 2004.