APPENDIX A

Algebraic Topology

This appendix is a brief review of basic algebraic topology. The idea is to make explicit the foundations on which geometric group theory is built and to establish standard notation and terminology.

A.1. Cell complexes and Euler characteristics

The notion of a cell complex is flexible enough to construct complicated spaces, but restrictive enough to avoid pathological examples such as the topologist's sine curve or the Hawaiian earring. The most basic cell complexes are the simplicial ones.

Definition A.1.1 (Simplicial complexes). An *abstract simplicial complex* is a collection $\mathcal{S}$ of finite subsets of a fixed set V such that $\tau \subset \sigma \in \mathcal{S}$ implies $\tau \in \mathcal{S}$. The elements of V are called *vertices* and the elements of $\mathcal{S}$ are called *simplices* because of the shapes they produce in the geometric realization. Let U be a real vector space with a basis whose elements are indexed by V. To each $\sigma \in \mathcal{S}$ we associate the subset of U formed by all nonnegative linear combinations

$$\sum_{v \in \sigma} \lambda_v v \quad \text{with} \quad \sum_{v \in \sigma} \lambda_v = 1.$$

If σ has n elements, then this set is an ordinary $(n-1)$-simplex. (For $n = 0, 1, 2$ and 3, an n-simplex is a point, an interval, a triangle and a tetrahedron.) The union of the simplices associated to each $\sigma \in \mathcal{S}$ is the (topological) *geometric realization.* For convenience we use $\mathcal{S}$ for both the abstract simplicial complex and for its geometric realization, and we use σ for both a finite subset of V and the topological simplex it contributes to $\mathcal{S}$.

Because of their concrete description, simplicial complexes are nice to work with, but there are situations where they are unnaturally restrictive. A more flexible construction involves iteratively attaching cells.

Definition A.1.2 (Attaching spaces along subspaces). If X_0 and X_1 are topological spaces, A is a subspace of X_1 and $f : A \to X_0$ is a continuous map, then we can form a quotient of $X_0 \sqcup X_1$ by identifying each point $a \in A$ with its image $f(a) \in X_0$. The resulting space X is denoted $X_0 \sqcup_f X_1$ and it is described as the space X_0 with X_1 *attached along A via f*. See Figure 1.

Definition A.1.3 (Cell complexes). The notion of a *cell complex* or *CW complex* (terms we use interchangeably) is defined inductively, dimension by dimension. A 0-*dimensional cell complex* is an arbitrary set of points called 0-*cells* with the discrete topology. An n-*dimensional cell complex* or n-*complex* X is constructed by

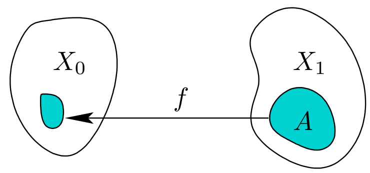

FIGURE 1. A schematic representation of spaces and maps used to construct $X = X_0 \sqcup_f X_1$.

attaching a disjoint union of n-discs along their boundary spheres to an already constructed $(n-1)$-dimensional cell complex X^{n-1}. In particular, let $E^n = \coprod \mathbb{D}^n$ be a disjoint union of n-discs and for each n-disc fix a continuous map $f : \partial\mathbb{D}^n \to X^{n-1}$, called the *attaching map*. There is then an induced map $F : \partial E^n \to X^{n-1}$ and the complex $X = X^{n-1} \sqcup_F E^n$ is an n-dimensional cell complex. For any $j < n$, the space X^j embeds into X and thus X^j can be viewed as a subspace of X; it is called the *j-skeleton of* X. Since unadorned superscripts often indicate dimension, we use $X^{(j)}$ to denote the j-skeleton of a cell complex X.

The interiors of the n-discs map homeomorphically into X and these images are called the *n-cells of* X. Since the points of X can be partitioned into $X^{(n-1)}$ and the n-cells of X, by induction, the set X can be viewed as a disjoint union of its j-cells, $0 \leq j \leq n$. For convenience, we often refer to 0-cells and 1-cells as *vertices* and *edges* respectively, and 1-complexes as *graphs*. A cell complex is *finite* if it has only finitely many cells.

Infinite dimensional cell complexes can also be constructed. Given cell complexes $X^0 \subset X^1 \subset \cdots \subset X^k \subset \cdots$ where each X^k is a k-dimensional cell complex constructed by attaching k-discs along their boundary to the previous complex in the list, we let X denote the union of these nested spaces and declare $U \subset X$ to be an open subset of X iff $U \cap X^k$ is open in X^k for all $k \geq 0$.

REMARK A.1.4 (Dimension -1). The inductive construction described above could actually have started one step earlier by declaring the empty topological space to be a (-1)-dimensional cell complex X^{-1}. The 0-dimensional cell complexes are constructed by attaching a disjoint union of 0-discs along their boundary spheres to this (-1)-dimensional cell complex. Because $\mathbb{D}^0$ is the entire space $\mathbb{R}^0$, it is open in the topology of $\mathbb{R}^0$ and thus its boundary is empty. This is completely consistent with the idea that $\partial\mathbb{D}^0 = \mathbb{S}^{-1}$ since $\mathbb{S}^{-1}$, by definition, is the set of vectors in $\mathbb{R}^0$ of length 1, which is, once again, empty. As a consequence, $E^0 = \coprod \mathbb{D}^0$ is a set of points with the discrete topology, ∂E^0 is the empty set, and the only choice we have for $F : \partial E^0 \to X^{-1}$ is the empty map between empty spaces. The resulting space $X = X^{-1} \sqcup_F E^0$ is then a set of points with the discrete topology. This convention is often useful. For example, one can define the k-cells of X as the images of the interiors of the k-discs under their attaching maps with no need to single out the 0-cells of X for separate treatment.

A *subcomplex* of a cell complex X is a union of j-cells that is closed in the topology of X. The various skeleta are obvious examples of subcomplexes, but there are many others. The fact that cell complexes are well-behaved is illustrated by the following theorem:

THEOREM A.1.5 (Cell complex properties). *Every cell complex X is normal and Hausdorff. It is connected iff it is path-connected iff its* 1*-skeleton is connected. It is compact iff it has only finitely many cells. And every compact subspace of X is contained in some finite subcomplex.*

As a consequence of Theorem A.1.5, the image of any k-disc, being compact, is contained in some finite subcomplex. Historically, this property was called *closure-finite* since the finite subcomplex contains the closure of the corresponding k-cell. Cell complexes were originally defined in a way that relied heavily on the *closure-finite* property and the use of the *weak topology* on the union. Hence the name *CW complex.*

DEFINITION A.1.6 (Euler characteristics). Let X be a finite cell complex and let c_i denote the number of i-cells that X contains. The (ordinary) *Euler characteristic* of X is equal to $\sum_{i\geq 0}(-1)^i c_i$ and it is denoted $\chi(X)$. The *reduced Euler characteristic* of X is a slight modification of the Euler characteristic where we consider the empty set as a (-1)-dimensional cell of X. When viewed in this way $c_{-1} = 1$ and the alternating sum over the cells of X yields $\sum_{i\geq -1}(-1)^i c_i = \chi(X) - 1$. The reduced Euler characteristic is denoted $\widetilde{\chi}(X)$. Despite the redundancy, it is useful to have both $\chi(X)$ and $\widetilde{\chi}(X)$ available. For example, $\widetilde{\chi}(\mathbb{S}^n) = (-1)^n$ and $\chi(X \times Y) = \chi(X) \times \chi(Y)$. Neither pattern can be stated as cleanly in the other notation.

There is great freedom in the definition of a cell complex, as the nature of the attaching maps is not very restrictive. In particular, it is not the case that every cell complex is homeomorphic to a simplicial complex (Exercise 5). In this book we often restrict ourselves to a simpler situation where the spaces are always homeomorphic to simplicial complexes (Exercise 6).

DEFINITION A.1.7 (Cellular maps and combinatorial complexes). A map $Y \to X$ between cell complexes is *cellular* if its restriction to each cell of Y is a homeomorphism onto a cell of X. A cell complex X is *combinatorial* if a cell structure can be imposed on the domain of each attaching map of each k-cell of X so that the result is a cellular map between cell complexes. In the literature, combinatorial cell complexes are also known as regular cell complexes.

A.2. Fundamental groups and van Kampen's theorem

Next we shift our attention from spaces to maps.

DEFINITION A.2.1 (Homotopic maps). Two maps $g, h : X \to Y$ are *homotopic* if there is a map $F : X \times I \to Y$ (a *homotopy*) such that $g = f_0$ and $h = f_1$ where $f_t : X \to Y$ is the map defined by the equation $f_t(x) = F(x,t)$. We write $g \cong h$ when g and h are homotopic maps. When $g : X \to Y$ is homotopic to a constant map (i.e. a map whose image is a single point of Y), then g is *null-homotopic.* If A is a subspace of X and there is a homotopy $F : X \times I \to Y$ such that $F(a,s) = F(a,t)$ for all $s, t \in I$, then g and h are *homotopic relative to* A.

Recall that a *based space* is a pair (X, x) where X is a topological space and x is a point of X and a *based map* is a map from (Y, y) to (X, x) is a map $f : Y \to X$ with $f(y) = x$. Such a map is denoted $f : (Y, y) \to (X, x)$.

DEFINITION A.2.2 (Fundamental groups). The *fundamental group* of a cell complex X based at a point x is the set of equivalence classes of paths in X that start and end at x where two paths are considered equivalent if they are homotopic relative to their endpoints. The multiplication of two such classes is defined by taking the equivalence class of the concatenation of representatives. If x and $\hat{x}$ are two points in the same connected component of X, then any path connecting them induces an isomorphism $\pi_1(X,x) \approx \pi_1(X,\hat{x})$. We should note, however, that the exact isomorphism usually depends on the choice of a connecting path. Since fundamental groups of connected cell complexes are well-defined up to isomorphism, basepoints are occasionally suppressed.

One of the key properties of this construction is its functoriality.

PROPOSITION A.2.3 (Functorality). *There is a functor from the category of based topological spaces to the category of groups such that the image of (X,x) is $\pi_1(X,x)$ and the image of the map $f:(X,x)\to(Y,y)$ is the group homomorphism $f_*:\pi_1(X,x)\to\pi_1(Y,y)$. In particular, if $f=gh$ as based maps, then the corresponding group homomorphisms satisfy $f_*=g_*h_*$, and if f is the identity map then f_* is the identity group homomorphism.*

To illustrate the benefits of functoriality, consider a retraction onto a subspace. Let A be a subspace of X and let $i:A\to X$ be the inclusion map. Recall that a map $r:X\to A$ is called a *retraction* if $ri=\mathbf{1}_A$ and it is a *deformation retraction* if, in addition, ir is homotopic to $\mathbf{1}_X$ relative to the subspace A.

PROPOSITION A.2.4 (Retractions and fundamental groups). *If A is a connected subspace of a connected space X, $i:A\to X$ is the inclusion map and $r:X\to A$ is a retraction, then r_* is surjective and i_* is injective. In particular, $\pi_1(A,a)$ can be viewed as a subgroup of $\pi_1(X,i(a))$.*

PROOF. Pick $a\in A$. By Proposition A.2.3, $r_*i_*=\mathbf{1}_G$ where $G=\pi_1(A,a)$. The rest follows from the fact that $\mathbf{1}_G$ is a bijection. □

add connecting text

PROPOSITION A.2.5. *For every subcomplex A of a cell complex X there is a small open neighborhood N of A such that N deformation retracts to A. In particular, N is homotopy equivalent to A.*

PROPOSITION A.2.6. *Let X be a cell complex. Then every map $f:\mathbb{S}^1\to X$ can be homotoped to a map $\hat{f}:\mathbb{S}^1\to X^{(1)}$. In particular, if $\iota:X^{(1)}\hookrightarrow X$ is the inclusion of the 1-skeleton into X, then the induced map $\iota_*:\pi_1(X^{(1)})\to\pi_1(X)$ is a surjection.*

DEFINITION A.2.7 (Wedge products). Let $\{(X_\alpha,x_\alpha)\}$ be a collection of based spaces. The *wedge product* of this collection is the quotient of their disjoint union in which all of the base points have been identified:

$$\bigvee X_\alpha = \coprod_\alpha X_\alpha/\{x_\alpha\sim x_\beta\}.$$

The resulting space is denoted $\vee_\alpha X_\alpha$ or $X\vee Y$ when only two spaces are involved.

If X is a cell complex with basepoint x, and X can be expressed as a union of subcomplexes A_α, each of which contains x, then there is a map

$$\phi:\bigvee A_i\to X$$

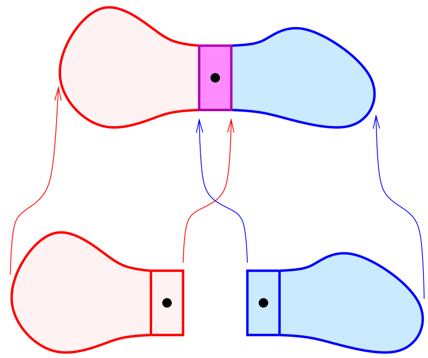

FIGURE 2. A decomposition into subcomplexes induces a map from the wedge product of the subcomplexes

defined by making ϕ an isomorphism when restricted to any A_i.

THEOREM A.2.8 (van Kampen's Theorem). *Let X be a cell complex that can be expressed as a union of path-connected subcomplexes $X = \bigcup A_\alpha$, where for all pairs of distinct indices $A_\alpha \cap A_\beta = C$, for a fixed, path-connected subcomplex C. The resulting map from the wedge product of the pieces, $\phi : \bigvee A_\alpha \to X$ induces a surjection on the level of fundamental groups: $\phi_* : \pi_1(\bigvee A_\alpha) \twoheadrightarrow \pi_1(X)$. Further, if for each index α we let ι_α denote the induced map $\pi_1(C) \to \pi_1(A_\alpha)$, then the kernel of ϕ_* is the normal subgroup generated by $\{\iota_\alpha(c)\iota_\beta(c^{-1}) \mid c \in \pi_1(C)\}$.*

COROLLARY A.2.9. *Let X be a cell complex and let $\iota : X^{(2)} \hookrightarrow X$ be the inclusion of its 2-skeleton. Then the induced map $\iota_* : \pi_1(X^{(2)}) \to \pi_1(X)$ is an isomorphism.*

PROOF. In higher dimensions, when you attach cells it is along 1-connected subspaces, so the kernel is trivial. □

A.3. Group actions and covering spaces

In the prologue we began our study of the fundamental group of the complement of the trefoil knot, $G \approx \pi_1(\mathbb{S}^3 \setminus K)$, by forming a cell complex $\mathcal{D}$ with $G \approx \pi_1(\mathcal{D})$. In order to study the structure of G we needed to understand not just the structure of $\mathcal{D}$, but how G acts on $\widetilde{\mathcal{D}}$.

DEFINITION A.3.1 (Group actions). A *left action* of a group G on a mathematical structure X is a group homomorphism from G to $\text{AUT}(X)$, the group of all invertible structure preserving maps under function composition. Thus, if X is a topological space, $\text{AUT}(X)$ is the group of all homeomorphisms from X to itself. More explicitly, a left group action of G on X is a function $a : G \times X \to X$ such that (1) for each $g \in G$, the restriction $g\cdot : X \to X$ defined by $g \cdot (x) = a(g, x)$ is a homeomorphism from X to itself, (2) $g \cdot (h \cdot (x)) = (gh) \cdot (x)$ for all $g, h \in G$ and for all $x \in X$, and (3) the identity element of G restricts to the identity homeomorphism.

Left group actions are denoted $G \curvearrowright X$, which is read as "G acts on X". The word "left" is usually suppressed since the sidedness of the action is implied by the way that functions are denoted. In order to define a *right action* of G on X we would need to use algebraist notation (i.e. we would have to write $(x)f$ instead of

$f(x)$ to describe the function f applied to the point x). The few occasions where algebraist notation for functions and right group actions are needed are clearly indicated.

DEFINITION A.3.2 (Proper group actions). Let $G \curvearrowright X$, where X is a topological space. The group G is acting *properly discontinuously* on X if for every point $x \in X$ there is a neighborhood U of x such that $\{g \in G \mid g(U) \cap U \neq \emptyset\}$ is finite. The action is *free* if the open set U can always be chosen so that this set contains only the identity element of G. The *stabilizer* of a point $x \in X$ is the subgroup $\mathrm{Stab}(x) = \{g \in G \mid g(x) = x\}$, and a group action is *proper* if all of the point stabilizers are finite.

REMARK A.3.3 (Group actions and categories). The notion of a group action depends on the category used to define $\mathrm{AUT}(X)$. Consider the group $\mathrm{AUT}(\mathbb{S}^1)$. When $\mathbb{S}^1$ is viewed as a cell complex, the natural maps are cellular maps and $\mathrm{AUT}(\mathbb{S}^1)$ is either a cyclic group of order 2 (in the one 0-cell case) or a finite dihedral group whose order depends on the number of 0-cells in $\mathbb{S}^1$; when $\mathbb{S}^1$ is viewed as a metric space, the natural maps to are isometries and $\mathrm{AUT}(\mathbb{S}^1)$ becomes the Lie group $O(2)$; and when $\mathbb{S}^1$ is viewed purely as a topological space, $\mathrm{AUT}(\mathbb{S}^1)$ contains all homeomorphisms from $\mathbb{S}^1$ to itself, which is quite a large group.

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Suppose $\cdot : G \times X \to X$ is a left group action of a group G on a space X. Because group elements are invertible, every map $g\cdot : X \to X$ is necessarily one-to-one and onto. Moreover, when X has any additional structure (such as a cell structure, or an orientation on its 1-skeleton, etc.), we shall assume that the action of G preserves this additional structure. In the case of a cell structure, this means that each map $g\cdot$ induces a bijection from the i-cells of X to the i-cells of X.

DEFINITION A.3.4 (Quotients). Given an action $G \curvearrowright X$, the *quotient* of the action is the quotient space formed by identifying $g \cdot x$ with x for each $x \in X$ and $g \in G$. It is denoted $G\backslash X$. A *fundamental domain* for an action $G \curvearrowright X$ is a path connected, closed subset $\mathcal{F} \subset X$ such that $G \cdot \mathcal{F} = X$ with no proper subset of $\mathcal{F}$ satisfying these conditions. When X is a cell complex one can always find a fundamental domain that is a subcomplex, but this is not required. Note that given a fundamental domain $\mathcal{F}$ there is an induced surjection $\mathcal{F} \twoheadrightarrow G\backslash X$. A group action $G \curvearrowright X$ is *cocompact* if $G\backslash X$ is compact, or equivalently, if there is a compact fundamental domain.

PROPOSITION A.3.5 (Free actions have quotients). *If $G \curvearrowright X$ is a free left action of a group G on a cell complex X where, by convention, the action respects the cell structure, then there is a well-defined cell structure on its quotient $G\backslash X$.*

A.3.1. Covering spaces. A map $f : Y \to X$ between path-connected topological spaces X and Y is called a *covering map* when for every $x \in X$ there exists an open set U containing x such that $f^{-1}(U)$ can be written as a disjoint union of open sets U_α where f restricted to each U_α is a homeomorphism. When $f : Y \to X$ is a covering map then Y is called a *cover of X*. A covering map must be a local homeomorphism, but in general this is not sufficient (Exercise 9). For cell complexes, however, the two concepts are equivalent.

PROPOSITION A.3.6 (Recognizing covers). *If X and Y are connected cell complexes, then $f : Y \to X$ is a covering map if and only if f is a local homeomorphism.*

If $f : Y \to X$, $g : Z \to X$, and $h : Z \to Y$ are maps such that $f \circ h = g$, then h is called a *lift of g through f*. When f and g are based maps, then we additionally require h to be a based map taking the base point of Z to the basepoint of Y. The definition of a cover is designed to facilitate the creation of lifts.

THEOREM A.3.7 (Map lifting). *Let (X, x), (Y, y) and (Z, z) be path connected based spaces, let $f : (Y, y) \to (X, x)$ be a cover and let $g : (Z, z) \to (X, x)$ be an arbitrary map. When Z is a cell complex there exists a based map $h : (Z, z) \to (Y, y)$ such that $f \circ h = g$ iff $g_*(\pi_1(Z, z)) \subset f_*(\pi_1(Y, y))$. Moreover, when such a map exists, it is unique.*

Special cases of Theorem A.3.7 have their own names. When Z is a 1-cell, it is called *path lifting* and when Z is a 2-cell it is called *homotopy lifting*. In both cases the condition is trivially satisfied since $g_*(\pi_1(Z, z))$ is the trivial subgroup of $\pi_1(X, x)$. Homotopy lifting is used to show that if f is cover then f_* is injective.

PROPOSITION A.3.8 (Covers and subgroups). *If $f : Y \to X$ is a covering with $f(y) = x$, then $f_* : \pi_1(Y, y) \to \pi_1(X, x)$ is an injection. In particular, the fundamental group of Y at y can be viewed as a subgroup of the fundamental group of X at x.*

Let $f : Y \to X$ be a covering and let $f(y) = x$. The right stabilizers of f (i.e. the maps $g : Y \to Y$ such that $f \circ g = f$), are called *deck transformations* and they form a *group of deck transformations* under composition. When the group of deck transformations of f acts transitively on the preimages of x, then f is called a *regular covering* and Y is a *regular cover of X*. Regular covers correspond to normal subgroups.

PROPOSITION A.3.9 (Regular covers and normal subgroups). *If $f : Y \to X$ is a covering with $f(y) = x$, then Y is a regular cover of X iff $f_*(\pi_1(Y, y))$ is a normal subgroup of $\pi_1(X, x)$. Moreover, when Y is a regular cover of X the quotient of $\pi_1(X, x)$ by $f_*(\pi_1(Y, y))$ is isomorphic to the group of deck transformations.*

If $f : Y \to X$ is a covering, X and Y are connected spaces, and Y is simply connected, then Y is called the *universal cover of X*. An easy application of Theorem A.3.7 shows that universal covers are unique (up to the natural notion of equivalence defined by lifts in both directions whose compositions are identity maps).

THEOREM A.3.10 (Fundamental theorem of covering spaces). *If X is connected topological space that has a universal cover $\widetilde{X}$, then there is a natural bijection between the connected covers of X and the subgroups of $\pi_1(X, x)$.*

(indicate the proof since this uses the quotient by the H-action defined earlier) Cell complexes, as usual, are extremely well behaved.

PROPOSITION A.3.11 (Recognizing universal covers). *Every connected cell complex has a universal cover. Moreover, if X and Y are connected cell complexes, then Y is the universal cover of X iff Y is simply connected and there exists a local homeomorphism $f : Y \to X$.*

A.4. Homotopy invariants and Whitehead's theorem

(homotopy type, contractibility, n-connected)

Definition A.4.1 (Homotopy equivalences). A map $f : X \to Y$ is a *homotopy equivalence* if there exists a map $g : Y \to X$ such that both compositions are homotopic to the appropriate identity map. In symbols this requires $fg \cong \mathbf{1}_Y$ and $gf \cong \mathbf{1}_X$. Two spaces X and Y are *homotopy equivalent* and have the same *homotopy type* if there exists a homotopy equivalence $f : X \to Y$. A *homotopy invariant* of a space X is something defined using X where the resulting answer or object depends only on the homotopy type of X.

Proposition A.4.2 (Fundamental groups are homotopy invariants). *If $f : X \to Y$ is a homotopy equivalence and $f(x) = y$, then $f_* : \pi_1(X, x) \to \pi_1(Y, y)$ is an isomorphism. In particular, connected spaces with the same homotopy type have isomorphic fundamental groups.*

Remark A.4.3. It is a basic result from algebraic topology that the Euler characteristic of a finite cell complex only depends on its topology and not on details of its cellular structure. Using cellular homology the alternating sum of the c_i is easily seen to be equal to the alternating sum of the betti numbers of X. But since all homology theories agree on finite cell complexes, and singular homology is insensitive to the cell structure of X, the Euler characteristic only depends on the topology of X.

Theorem A.4.4 (Invariance of $\chi(X)$). *Euler characteristic is a homotopy invariant. If X and Y are homotopy equivalent spaces and $\chi(X)$ and $\chi(Y)$ can be defined, then $\chi(X) = \chi(Y)$.*

Homology and cohomology are also homotopy invariants, and we will on occasion make use of them. However we do not review their definitions and basic properties as topics such as group cohomology are not a central focus of this book.

There are two common ways to modify a cell complex without changing its homotopy type. One is to collapse a contractible subcomplex and the other is to replace an attaching map with an alternate map homotopic to it. This section is devoted to an application of the first; discussion of the second is postponed until Section 1.1. For a proof of the following results see Chapter 0 in [**16**].

Theorem A.4.5 (Collapsing contractible subcomplexes). *If A is a contractible subcomplex of a cell complex X, then the quotient map $X \to X/A$ is a homotopy equivalence.*

Theorem A.4.6 (Modifying the attaching maps). *If A is a subcomplex of a cell complex X_1 and $f, g : A \to X_0$ are homotopic maps, then the spaces $X_0 \sqcup_f X_1$ and $X_0 \sqcup_g X_1$ are homotopy equivalent.*

Theorem A.4.7 (Contractibility). *If X is a connected topological space, then the following conditions are equivalent.*

1. *X has the homotopy type of a point (i.e. X is contractible)*
2. *the identity map $\mathbf{1} : X \to X$ is null-homotopic*
3. *every map $Y \to X$ is null-homotopic*

A space satisfying these conditions is said to be contractible*, and contractibility is a homotopy invariant.*

Proof. Exercise 14. □

Theorem A.4.7 is true for arbitrary topological spaces. For connected *cell complexes*, it is sufficient to show that every map $Y \to X$ where Y is compact is null-homotopic. In fact, it is sufficient to show that for every $n \geq 0$ and each map $\mathbb{S}^n \to X$ is null-homotopic. That this is implied by the above is clear. That it is sufficient to show contractibility is a part of a nontrivial theorem due to J.H.C. Whitehead.

THEOREM A.4.8 (Whitehead's theorem). *A cell complex X is contractible iff for every $n \geq 0$, each map $\mathbb{S}^n \to X$ is null-homotopic.*

PROPOSITION A.4.9. *The nested union of n-connected cell complexes is n-connected. More specifically, if $A_0 \subset A_1 \subset \cdots \subset A_k \subset \cdots$ is a nested sequence of n-connected subcomplexes of a cell complex X and $A = \cup_{k \geq 0} A_k$, then A itself is n-connected. As a consequence, the nested union of contractible cell complexes is contracible.*

PROOF. Any map $f : \mathbb{S}^m \to A$ with $m \leq n$ is contained in a finite subcomplex B. Since each cell of B is contained in some A_i and there are only finitely many cells in B, all of B is contained in some A_i. The fact that A_i in n-connected now implies that f is homotopic to a constant map inside $A_i \subset A$. Thus A is n-connected. The final assertion follows by Theorem A.4.8. □

There is a family of homotopy invariant properties that sits between being connected and being contractible. The most common is being *simply connected*, that is being path connected and having trivial fundamental group (although some do not require simply connected spaces to be path connected).

DEFINITION A.4.10 (Connectivity). A topological space X is *n-connected* if for all $k \leq n$, each map $\mathbb{S}^k \to X$ is null-homotopic. Being 0-connected is the same as path connected, and 1-connected is the same as *simply connected*.

REMARK A.4.11 ($\pi_n(X, x)$). For those familiar with the definition of the higher homotopy groups, $\pi_n(X, x)$, it is easy to prove that our definition of n-connectivity is equivalent to the condition that $\pi_i(X, x)$ is trivial for $i \leq n$. See Exercise XXX. (add an exercise where we hint how to make a tail and let it wiggle.

A.5. Classifying spaces and Hurewicz's theorem

In the prologue we illustrated how one can understand certain facts about certain groups G via their actions on contractible complexes $\widetilde{X}$. In that particular case it was helpful that the action was free and the complex was contractible. An *Eilenberg-MacLane space* for a group G is a cell complex whose fundamental group is G and whose universal cover is contractible. Such a space is also referred to as a $K(G, 1)$ and as a *classifying space for G*.

THEOREM A.5.1 (Eilenberg-MacLane spaces). *For every group G there exists a connected cell complex X whose universal cover is contractible and whose fundamental group is G. Moreover, if X and Y have contractible universal covers and isomorphic fundamental groups, then X and Y are homotopy equivalent.*

The proof that any two $K(G, 1)s$ are homotopy equivalent (*Hurewicz's Theorem*) is a bit too long of a distraction for us. (See Theorem 1B.8 in [**16**].) The existence claim can be viewed as a topological variation of Cayley's Theorem. Cayley's Theorem states that every group can be faithfully represented as a group of permutations. The proof constructs an action of G on its own elements via left multiplication. The proof we use below extends this action of G on its own elements, and in the end yields a faithful representation of G as a group of deck transformations of a contractible topological space.

PROOF OF EXISTENCE. To prove the existence of $K(\pi, 1)$s, start with a vertex set of the form $G \times \mathbb{N}$ and think of the second coordinate as describing the "column" to which the vertex belongs. Extend this vertex set to a simplicial complex by declaring that any finite set of vertices drawn from distinct columns forms the vertex set of a simplex. Call this complex X and notice that it is just the countable join of discrete sets of vertices, each of cardinality $|G|$. In particular, if $|G| = |H|$, then the simplicial complexes built from G and H are the same. Standard tools from algebraic topology, like the Künneth formulas, prove that X is contractible.

Since G acts on itself by left multiplication (Cayley's Theorem), it also acts on $G \times \mathbb{N}$ by left multiplication applied to the first coordinate. This action preserves columns and any n-tuple of vertices coming from distinct columns will be taken to another n-tuple of vertices coming from (the same) distinct columns. As the action is free when restricted to any column, the action of G on X is also free. Thus the quotient $G \backslash X$ is a $K(G, 1)$. □

These facts enable one to apply homotopy invariants in the study of groups. We say that a homotopy invariant assertion is true of a group G iff it is true of any (and thus every) Eilenberg-MacLane space for G. In particular, one can declare the homology and cohomology groups of a group to be the homology and cohomology groups of any $K(G, 1)$.

DEFINITION A.5.2 (Finite type). A group G is of *finite type* if it admits a finite $K(G, 1)$-complex. Equivalently, a group G is of finite type if there is a free, cocompact action of G on a finite dimensional, contractible cell complex.

DEFINITION A.5.3 (Euler characteristics of groups). If G is a group of finite type, then the *Euler characteristic* of G is Euler characteristic of any finite $K(G, 1)$. (If you happen to have a non-finite $K(G, 1)$, when G is in fact of finite type, then the Euler characteristic can still be computed by taking the alternatinig sum of the betti numbers, which are homotopy invariants.) For example, the fundamental group G of the complement of the trefoil knot has a $K(G, 1)$ described in the prologue. This complex has two vertices, five edges, and three faces, hence $\chi(G) = 0$.

PROPOSITION A.5.4. *Let G be a group with a finite $K(G, 1)$, X. If H is a finite index subgroup of G, then $\chi(H)$ exists, and*

$$\chi(H) = [G : H] \cdot \chi(G)$$

PROOF. The cover $\overline{X}$ of X whose fundamental group is H is a $K(H, 1)$. Since it is a $[G : H]$-fold cover, if X contains c_i i-cells, then $\overline{X}$ contains $\overline{c}_i = [G : H] \cdot c_i$ i-cells. Thus

$$\chi(\overline{X}) = \sum (-1)^i \overline{c}_i = [G : H] \sum (-1)^i c_i = [G : H] \chi(X).$$

□

We leave the following corollary as a (fun) exercise.

COROLLARY A.5.5. *Let S_g be the closed orientable surface of genus g, and fix two integers g and h, both greater than 1. Then $\pi_1(S_g)$ is a finite index subgroup of $\pi_1(S_h)$ if and only if $g-1$ is a multiple of $h-1$.*

PROPOSITION A.5.6. *Let G be a group of finite type. Then G contains no non-trivial finite subgroup.*

PROOF. Let X be a finite $K(G,1)$, of dimension d, and let $\widetilde{X}$ be its universal cover. Assume to the contrary that G has a non-trivial finite subgroup, and therefore that G has a subgroup isomorphic to a finite cyclic group $\mathbb{Z}_n$. Since G acts freely on $\widetilde{X}$, $\mathbb{Z}_p \curvearrowright \widetilde{X}$. It follows that $H_i(\mathbb{Z}_n) = H_i(\mathbb{Z}_n \backslash \widetilde{X})$ and in particular, $H_i(\mathbb{Z}_n) = 0$ for all $i > d$. But $H_{2j+1}(\mathbb{Z}_n) \approx \mathbb{Z}_n$ for all $j \geq 0$. □

Avoiding groups with torsion is often overly restrictive. For example, consider the group G of isometries of the Euclidean plane, generated by reflections in the sides of an equilateral triangle. The This action is not free, but it is cocompact and proper. As we will see, such actions are often more than sufficient for one to derive deep facts about the group.

Exercises

Cell complexes

1. Let X be a cell complex, let x and y be 0-cells of X and let A be a connected finite subcomplex containing x and y with a minimum number of cell. Prove that A is the image of an embedded interval $f : I \to X$ starting at x and ending at y.
2. Let X be a 1-complex that contains a 1-disc where both endpoints are attached to the same 0-cell. Use a retraction to show that X is not simply-connected.
3. Let X be a 1-complex and let $f : \mathbb{S}^1 \to X$ be an embedding. Show that X is not simply-connected by collapsing all but one 1-cell of the image of f and applying the previous exercise.
4. Let $f : [0, 1] \to [0, 1]$ be an infinitely oscillating function like that shown on the left in Figure 3. Use this function as part of an attaching map (as indicated on the right in Figure 3) to create a 2-complex with three vertices, three edges, and a single 2-cell. Show that this "shower curtain complex" is not homeomorphic to any simplicial complex.

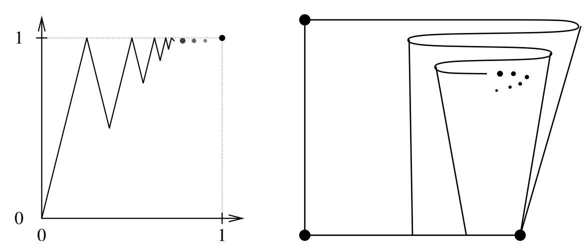

FIGURE 3. The "shower curtain complex" is not homeomorphic to any simplicial complex.

5. Prove that every combinatorial cell complex is homeomorphic to a simplicial complex.
6. (Classifying compact surfaces) (sketch out how to classify compact surfaces) Here are the main steps.
   a. show that you only need a single 2-cell.
   b. make all moebius edges adjacent
   c. isolate crossing annular edges
   d. remove noncrossed annular edges
   e. eliminate the mixed case
   f. use the abelianizations to distinguish the remaining cases
7. Prove Corollary A.5.5.
8. Let Y be the image of the map $g : \mathbb{R} \to \mathbb{R}^3$ defined by $g(t) = (\cos t, \sin t, e^t)$ and turn Y into a path connected topological space by giving it the subspace topology. There is a map $f : Y \to \mathbb{S}^1$ that comes from projecting onto the first two coordinates. Prove that f is a local homeomorphism but not a covering map.
9. Show that every one-relator Artin group is a torus knot group. Which torus knot groups arise in this way?

Group actions and covering spaces

10. Add an exercise that covers the Hawaiian earring.(Figure 4) (fix: do not have universal covers. Identifying a single point in a simply connected metric space with a single in another simply connected metric space does not need to result in a simply connected space. This space, which is the union of the circles centered at $(0, 1/n)$ and tangent to the x-axis...)

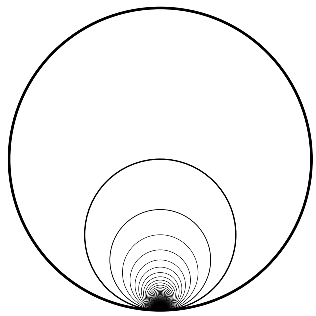

FIGURE 4. Hawaiian earring.

11. (Normal subgroups) Let H be a subgroup of G, let $A := G/H$ be the set of left H-cosets, let $\kappa := |A|$ be the index of H in G, and let $\kappa!$ denotes the size of SYM_A (the bijections $A \to A$ under composition). Prove that there is a normal subgroup N of G contained in H whose index in G is at most $\kappa!$.
12. (Infinite Index) Let A be a set and let $G = \text{SYM}_A$ be the group of all permutations (i.e. bijections) $f : A \to A$ under function composition. Choose an element $a \in A$ and let H be the subgroup of permutations that fix a. Prove that index of H in G is $\kappa = |A|$ and that the only normal subgroup of G in H is the trivial subgroup (index $\kappa!$).

Homotopy invariants and Whitehead's theorem

13. (Contractibility) Prove that the characterizations of contractibility list in Theorem A.4.7 are equivalent.