

COMPUTING REFLECTION LENGTH IN AN AFFINE COXETER GROUP

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ABSTRACT. In any Coxeter group, the conjugates of elements in its Coxeter generating set are called reflections, and the reflection length of an element is its length with respect to this expanded generating set. In this article we give a simple formula that computes the reflection length of any element in any affine Coxeter group and we provide a simple uniform proof.

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INTRODUCTION

In any Coxeter group W , the conjugates of elements in its standard Coxeter generating set are called *reflections*, and we write R for the set of all reflections in W . The reflections generate W and the associated *reflection length* function $\ell_R: W \rightarrow \mathbb{Z}_{\geq 0}$ records the length of w with respect to this expanded generating set. When W is spherical, i.e., finite, reflection length can be given an intrinsic, geometric definition, as follows. Define a *root subspace* to be any space spanned by a subset of the corresponding root system and define the *dimension* $\dim(w)$ of an element w in W to be the minimum dimension of a root subspace that contains the move-set of w . (See Section 1.5 for the details.)

Theorem ([Car72, Lemma 2]). *Let W be a spherical Coxeter group and let $w \in W$. Then $\ell_R(w) = \dim(w)$.*

When W is infinite, much less is known, [MP11, Dus12, MST15], even in the affine case. In this article, we give a simple, analogous formula that computes the

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reflection length of any element in any affine Coxeter group and we provide a simple uniform proof.

Theorem A (Formula). *Let W be an affine Coxeter group and let $p: W \rightarrow W_0$ be the projection onto its associated spherical Coxeter group. For any element $w \in W$, its reflection length is*

$$\ell_R(w) = 2 \cdot \dim(w) - \dim(p(w)) = 2d + e,$$

where $e = \dim(p(w))$ and $d = \dim(w) - \dim(p(w))$.

We call $e = e(w) = \dim(p(w))$ the *elliptic dimension* of w and we call $d = d(w) = \dim(w) - \dim(p(w))$ the *differential dimension* of w . Both statistics are geometrically meaningful. For example, $e(w) = 0$ if and only if w is a translation (that is, if it sends every point x to $x + \lambda$ for some fixed vector λ), and $d(w) = 0$ if and only if w is elliptic (that is, if it fixes a point); see Proposition 1.31.

The translation and elliptic versions of our formula were already known (see [MP11] and [Car72], respectively), and these special cases suggest that the formula in Theorem A might correspond to a carefully chosen factorization of w as a product of a translation and an elliptic element. This is indeed the case.

Theorem B (Factorization). *Let W be an affine Coxeter group. For every element $w \in W$, there is a translation-elliptic factorization $w = t_\lambda u$ such that $\ell_R(t_\lambda) = 2d(w)$ and $\ell_R(u) = e(w)$. In particular, $\ell_R(w) = \ell_R(t_\lambda) + \ell_R(u)$ for this factorization of w .*

The proof of Theorem B relies on a nontrivial technical result recently established by Vic Reiner and the first author [LR16, Corollary 1.4]. Elements in an affine Coxeter group typically have many different potential translation-elliptic factorizations, and the most common way to find one is to view the group as a semidirect product. For an affine Coxeter group W that naturally acts cocompactly on an n -dimensional euclidean space, the set of translations in W forms a normal abelian subgroup T isomorphic to \mathbb{Z}^n , and the quotient $W_0 = W/T$ is the spherical Coxeter group associated with W . An identification of W as a semidirect product $T \rtimes W_0$ corresponds to a choice of an inclusion map $i: W_0 \hookrightarrow W$ that is a section of the projection map $p: W \rightarrow W_0$. There is a unique point x fixed by the subgroup $i(W_0)$, and every element has a unique factorization $w = t_\lambda u$ where t_λ is a translation in T and u is an elliptic element in the copy of W_0 that fixes x . However, for some elements w none of the translation-elliptic factorizations that come from an identification of W as a semidirect product satisfy Theorem B; see Example 2.4.

In the particular case of the symmetric group \mathfrak{S}_n (the spherical Coxeter group of type A), reflection length also has a natural combinatorial characterization: $\ell_R(w) = n - c(w)$, where $c(w)$ is the number of *cycles* of the permutation w [Dén59]. When W is the affine symmetric group $\tilde{\mathfrak{S}}_n$, we show that the differential and elliptic dimensions of an element can also be given a combinatorial interpretation, leading to the very similar formula in Theorem 4.25. While this combinatorial formula leads to a practical implementation, it is computationally infeasible for large n . As we show in Appendix A, computing reflection length is essentially equivalent to the NP-complete problem `SubsetSum`.

Structure. The article has five sections. In Section 1, we discuss the necessary background on spherical and affine Coxeter groups, including the definition of the

dimension of an element. In Section 2, we use the ideas of Section 1 to prove our main theorems. In Section 3, we discuss our understanding of the local distribution of reflection length. In Section 4, we restrict our attention to the affine symmetric groups and develop a combinatorial understanding of reflection length. In Section 5, we point to topics for further study. Finally, in Appendix A, we describe an algorithm for computing reflection length in the affine symmetric group.

1. DIMENSIONS IN AFFINE COXETER GROUPS

In this section we recall the relevant background on reflection length and basic notions for spherical and affine Coxeter groups. We then develop a notion of the dimension of an element in an affine Coxeter group. We assume the reader is familiar with Coxeter groups at the level of Humphreys [Hum90].

1.1. Reflection length. This section reviews basic facts about reflection length in arbitrary Coxeter groups.

Definition 1.1 (Reflection length). Given a Coxeter group W with standard Coxeter generating set S , a *reflection* is any element conjugate to an element in S . Let R denote the set of all reflections in W ; note that R is infinite whenever the Coxeter group W is infinite. The *reflection length* $\ell_R(w)$ of an element w is the minimum integer k such that there exist reflections $r_i \in R$ with $w = r_1 r_2 \cdots r_k$. The identity has reflection length 0.

Remark 1.2 (Prior results). When W is a spherical Coxeter group, Carter's lemma mentioned in the introduction establishes that the reflection length of an element $w \in W$ may be computed in terms of the dimension of its move-set or fixed space (see Section 1.2 below for the definitions): $\ell_R(w) = \dim(\text{MOV}(w)) = \dim(V) - \dim(\text{FIX}(w))$. When W is an affine Coxeter group acting cocompactly on an n -dimensional euclidean space, the second and third authors have shown that $\ell_R(w)$ is bounded above by $2n$ for every element w in W and established an exact formula for the reflection length of a translation [MP11, Theorem A and Proposition 4.3]. Milićević, Thomas, and the fourth author found better bounds and explicit formulas in some special cases as part of their work on affine Deligne–Lusztig varieties [MST15, Theorem 11.3]. Finally, when W is neither spherical nor affine, Duzsenko has shown that the reflection length function is always unbounded [Dus12].

Remark 1.3 (Basic properties). If W is a reducible Coxeter group $W = W_1 \times W_2$, the reflection length of an element $w \in W$ is just the sum of the reflection lengths of its factors [MP11, Proposition 1.2]. Thus, it is sufficient to study *irreducible* Coxeter groups, for which there is a well-known classification [Hum90, Section 6.1]. Since the set R of reflections is a union of conjugacy classes, the length function $\ell_R: W \rightarrow \mathbb{Z}_{\geq 0}$ is constant on conjugacy classes. This implies, in particular, that $\ell_R(uv) = \ell_R(vu)$. The function ℓ_R can also be viewed as the natural combinatorial distance function on the Cayley graph of W with respect to the generating set R , which means that ℓ_R satisfies the triangle inequality $\ell_R(u) - \ell_R(v) \leq \ell_R(uv) \leq \ell_R(u) + \ell_R(v)$. Finally, for every Coxeter group there is a homomorphism onto $\mathbb{Z}/2\mathbb{Z}$ that sends every reflection to the nonidentity element. As a consequence, reflection length has a parity restriction: $\ell_R(uv) = \ell_R(u) + \ell_R(v) \pmod{2}$. These facts combine to show that $\ell_R(rw) = \ell_R(wr) = \ell_R(w) \pm 1$ for every element $w \in W$ and reflection $r \in R$.

One standard result about reflection factorizations is that they can be rewritten in many different ways.

Lemma 1.4 (Rewriting reflection factorizations). *Let $w = r_1 r_2 \cdots r_k$ be a reflection factorization of an element w of a Coxeter group. For any selection $1 \leq i_1 < i_2 < \cdots < i_m \leq k$ of positions, there is a length- k reflection factorization of w whose first m reflections are $r_{i_1} r_{i_2} \cdots r_{i_m}$ and another length- k reflection factorization of w where these are the last m reflections.*

Proof. Because the set of reflections is closed under conjugation, we have for any reflections r and r' that the elements $r'' = rr'r$ and $r''' = r'rr'$ are also reflections, satisfying $rr' = r''r = r'r'''$. Thus in any reflection factorization one may replace a consecutive pair rr' of factors with the pair $r''r$ (moving r to the right one position) or $r'r'''$ (moving r' to the left one position) without changing the length of the factorization. Iterating these rewriting operations allows one to move any subset of reflections into the desired positions. \square

Remark 1.5 (Hurwitz action). The individual moves in the proof of Lemma 1.4 are called *Hurwitz moves*. Globally, they correspond to an action, called the *Hurwitz action*, of the k -strand braid group on the set of all length- k reflection factorizations of a given word w .

1.2. Spherical Coxeter groups. In this section, we discuss the spherical Coxeter groups and their connection to root systems.

Definition 1.6 (Spherical Coxeter groups). A *euclidean vector space* V is an n -dimensional real vector space equipped with a positive definite inner product $\langle \cdot, \cdot \rangle$. A *crystallographic root system* Φ is a finite collection of vectors that span a real euclidean vector space V satisfying a few elementary properties; see [Hum90] for a precise definition. (While there are noncrystallographic root systems as well, it is only the crystallographic root systems that arise in the study of affine Coxeter groups.) The elements of Φ are called *roots*. Each crystallographic root system corresponds to a finite (or *spherical*) Coxeter group W_0 as follows: for each α in Φ , H_α is the hyperplane through the origin in V orthogonal to α , and the unique nontrivial isometry of V that fixes H_α pointwise is a *reflection* that we call r_α . The collection $R = \{r_\alpha \mid \alpha \in \Phi\}$ generates the spherical Coxeter group W_0 , and R is its set of reflections in the sense of Definition 1.1. In Figure 1 we see the hyperplanes for each irreducible crystallographic root system of rank two.

In a spherical Coxeter group constructed as above, each element is associated to two fundamental subspaces: its fixed space and its move-set.

Definition 1.7 (Fixed space and move-set). Given an orthogonal transformation w of V , its *fixed space* $\text{FIX}(w)$ is the set of vectors $\lambda \in V$ such that $w(\lambda) = \lambda$. In other words, it is the kernel $\text{KER}(w - 1)$. Its *move-set* $\text{MOV}(w)$ is the set of vectors μ in V for which there exists some λ in V such that $w(\lambda) = \lambda + \mu$. In other words, it is the image $\text{IM}(w - 1)$.

Remark 1.8 (Orthogonal transformations). Every element in a spherical Coxeter group constructed from a root system as above is an orthogonal transformation, and so the move-set and fixed space are orthogonal complements with respect to V .

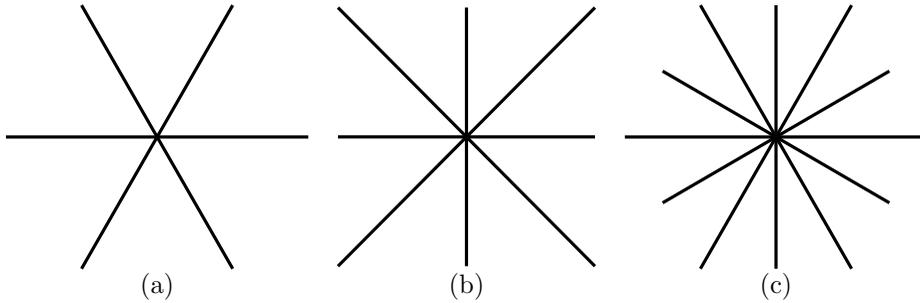


FIGURE 1. Reflecting hyperplanes associated to the root systems of types (a) A_2 , (b) B_2 and C_2 , and (c) G_2 .

1.3. Points, vectors, and affine Coxeter groups. In the same way that doing linear algebra using fixed coordinate systems can obscure an underlying geometric elegance, working in affine Coxeter groups with a predetermined origin can have an obfuscating effect. One way to avoid making such a choice is to distinguish between *points* and *vectors*, as in [ST89] and [BM15].

Definition 1.9 (Points and vectors). Let V be a euclidean vector space. A *euclidean space* is a set E with a uniquely transitive V -action; i.e., for every ordered pair of points $x, y \in E$ there exists a unique vector $\lambda \in V$ such that λ sends x to y . When this happens we write $x + \lambda = y$. The preceding sentences illustrate two conventions that we adhere to throughout the paper: the elements of E are *points* and are denoted by roman letters, such as x and y , while the elements of V are *vectors* and are denoted by Greek letters, such as λ and μ .

The main difference between E and V is that V has a well-defined origin, but E does not. If we select a point $x \in E$ to serve as the origin, then V and E can be identified by sending each vector λ to the point $x + \lambda$. We use this identification in the construction of the affine Coxeter groups.

Definition 1.10 (Affine Coxeter groups). Let E be a euclidean space whose associated vector space V contains the crystallographic root system Φ . An affine Coxeter group W can be constructed from Φ , as follows. Fix a point x in E to temporarily identify V and E . For each $\alpha \in \Phi$ and $j \in \mathbb{Z}$, let $H_{\alpha,j}$ denote the (affine) *hyperplane* in E of solutions to the equation $\langle v, \alpha \rangle = j$, where the brackets denote the standard inner product (treating x as the origin). The unique nontrivial isometry of E that fixes $H_{\alpha,j}$ pointwise is a *reflection* that we call $r_{\alpha,j}$. The collection $R = \{r_{\alpha,j} \mid \alpha \in \Phi, j \in \mathbb{Z}\}$ generates the affine Coxeter group W , and R is its set of reflections in the sense of Definition 1.1. A standard minimal generating set S can be obtained by restricting to those reflections that reflect across the facets of a certain polytope in E . The irreducible affine hyperplane arrangements of rank two are shown in Figure 2.

The affine Coxeter group W associated to a finite crystallographic root system Φ is closely related to the spherical Coxeter group W_0 .

Definition 1.11 (Subgroups and quotients). Let W be an affine Coxeter group constructed as in Definition 1.10. The map sending $r_{\alpha,j}$ in W to r_α in W_0 extends to a group homomorphism $p: W \rightarrow W_0$. The kernel of p is a normal abelian subgroup

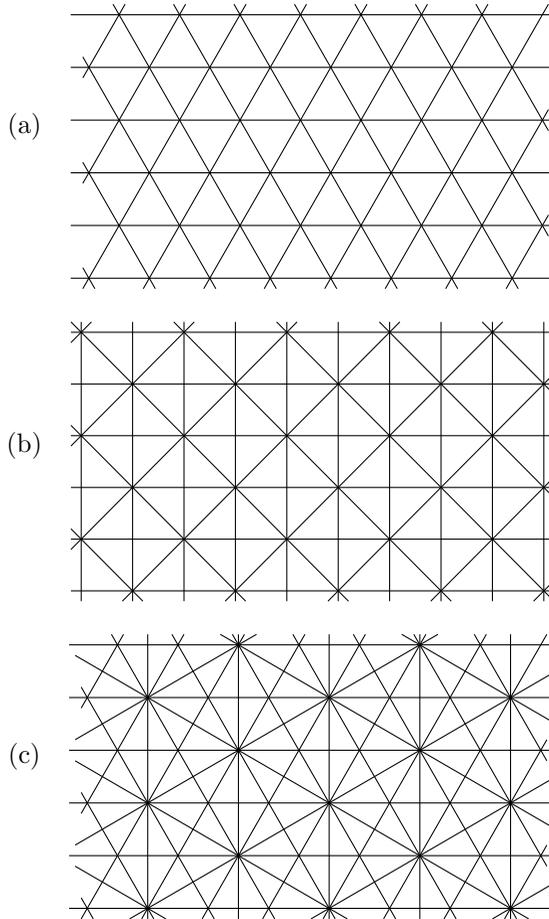


FIGURE 2. Affine hyperplanes for the root system of type (a) A_2 , (b) B_2 , and (c) G_2 .

T , isomorphic to \mathbb{Z}^n , whose elements are called *translations*, and $W_0 \cong W/T$. When x is the point used in the construction of W , the map $i: W_0 \hookrightarrow W$ sending r_α to $r_{\alpha,0}$ is a section of the projection p , identifying W_0 with the subgroup of all elements of W that fix x . (Note also the composition $i \circ p$ sends reflection $r_{\alpha,j}$ to r_α to $r_{\alpha,0}$.) Thus W may be identified as a semidirect product $W \cong T \rtimes W_0$.

Of course, the identification of W_0 with a subgroup of W is not unique: conjugation by elements of T gives an infinite family of such subgroups.

Remark 1.12 (Geometric interpretation of W , W_0 , and p). Given W , an *alcove* is (the closure of) a maximal connected component in the complement of the reflection hyperplanes $H_{\alpha,j}$ in the euclidean space E . These are the small triangles in Figures 2 and 3. Each of these triangles can play the role of a fundamental domain of the action of W on E . The choice of a point x as origin and a fundamental alcove c_x having x as a vertex determines a bijection between elements of W and alcoves. Here the fundamental alcove corresponds to the identity, and every element $w \in W$ corresponds to the image of c_x under w .

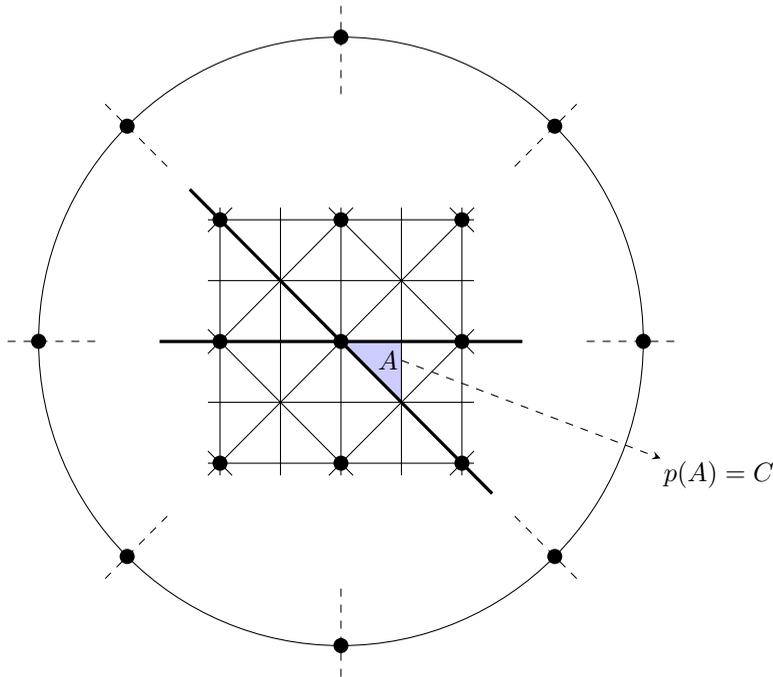


FIGURE 3. Elements of W can be put in bijection with alcoves in the tessellated plane, and the orbit of x under translations in W is illustrated with the fat vertices of these alcoves. Elements of W_0 are in bijection with chambers on the boundary sphere. The projection map p maps an alcove A to the chamber C at infinity that points “in the same direction”. For details see Remark 1.12.

The boundary of E is a sphere S whose points are the parallelism classes of geodesic rays. Its simplicial structure is induced by the tessellation of the plane. A parallelism class of hyperplanes in E corresponds to a reflection hyperplane, i.e., an equator, in the sphere S . We call the maximal connected components in the complement of the hyperplanes in S *chambers*. Each chamber is also the parallelism class of simplicial cones in E . Namely, for any one of the black vertices in Figure 3, say y , the complement of all the hyperplanes going through y decomposes into simplicial cones. We call them *Weyl cones* based at y . The parallelism classes of Weyl cones are in bijection with the chambers in S . Moreover, the chambers in S are in a natural bijection with elements of W_0 , as follows: Each element $w \in W$ is an affine motion of E that obviously maps parallel rays to parallel rays. Hence it induces an isometry of (the tessellation of) S where translations on E induce the identity on S . If we want the bijection between W_0 and the chambers in S to be compatible with these induced isometries of W on S , we need to map the identity in W_0 to the parallelism class of the unique Weyl cone based at x that contains the alcove c_x .

From this perspective, the projection map $p : W \twoheadrightarrow W_0$ can be understood geometrically as the map that sends an element $w \in W$ to the induced isometry on S . In geometric terms this means that an alcove A with vertex y is mapped to

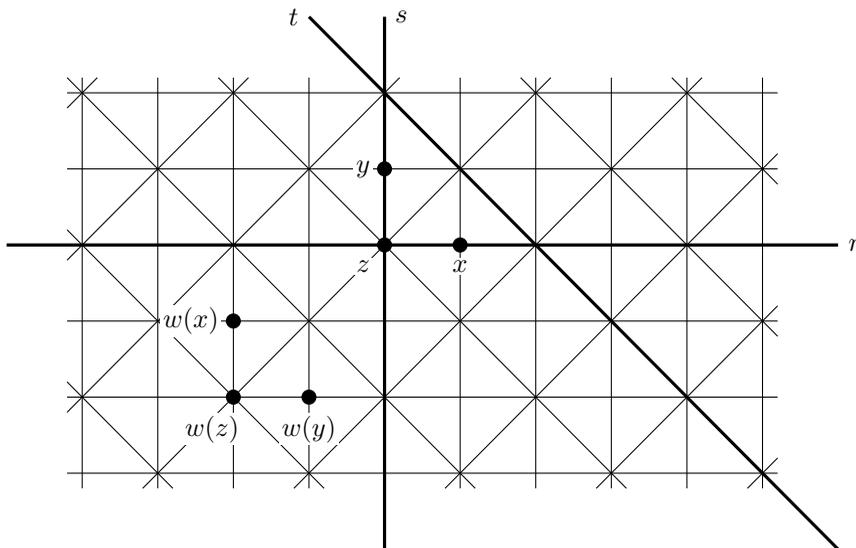


FIGURE 4. The action of $w = rst$ in the group of affine type B_2 on the points x, y, z in the euclidean plane, as in Example 1.15.

the chamber C at infinity that is the parallelism class of the cone based at y that contains A . This is indicated by the dotted arrow in Figure 3. One can think of this as “walking to infinity in the direction of A at y ”. More about the interplay between faces of alcoves and faces of chambers is studied in work of Marcelo Aguiar and the third author [AP15].

The notions of fixed space and move-set extend easily to affine Coxeter groups.

Definition 1.13 (Move-sets). The *motion* of a point $x \in E$ under a euclidean isometry w is the vector $\lambda \in V$ such that $w(x) = x + \lambda$. The *move-set* of w is the collection of all motions of the points in E ; by [BM15, Proposition 3.2], it is an affine subspace of V . In symbols, $\text{MOV}(w) = \{\lambda \mid w(x) = x + \lambda \text{ for some } x \in E\} \subset V$.

Definition 1.14 (Fixed space). The *fixed space* $\text{FIX}(w)$ of an isometry w is the subset of points $x \in E$ such that $w(x) = x$. Equivalently, $\text{FIX}(w)$ consists of all points whose motion under w is the vector 0. When $\text{FIX}(w)$ is nonempty, it is an (affine) subspace of E .

The complementarity between $\text{MOV}(w)$ and $\text{FIX}(w)$ mentioned in Remark 1.8 does not hold for isometries of euclidean space. It can, however, be recovered if the fixed space is replaced with the min-set of points that are moved a minimal distance under w . Then the space of directions for the move-set and the space of directions for the min-set give an orthogonal decomposition of V ; see [BM15, Lemma 3.6].

Example 1.15 (The move-set of an element in affine B_2). To make the notion of a move-set concrete, we describe the move-set of an element in affine B_2 . Suppose $w = rst$, where r, s , and t are reflections across the lines pictured in Figure 4, and suppose x, y , and z are the points labeled. We have also labeled the locations of $w(x)$, $w(y)$, and $w(z)$.

The standard basis vectors ε_1 and ε_2 in \mathbb{R}^2 are the vectors that send z to x and z to y , respectively. By abuse of notation, we will write $\varepsilon_1 = x - z$ and $\varepsilon_2 = y - z$. Having made this identification, we can express any point u in $E \cong \mathbb{R}^2$ as

$$u = z + a(x - z) + b(y - z)$$

for some $(a, b) \in \mathbb{R}^2$. Using this coordinate system, we have

$$\begin{aligned} w(x) &= z - 2(x - z) - (y - z) = x - 3(x - z) - (y - z), \\ w(y) &= z - (x - z) - 2(y - z) = y - (x - z) - 3(y - z), \quad \text{and} \\ w(z) &= z - 2(x - z) - 2(y - z). \end{aligned}$$

We see that the motion of x is the vector $(-3, -1)$, the motion of y is $(-1, -3)$, and the motion of z is $(-2, -2)$.

In general, we use linearity to compute

$$\begin{aligned} w(u) &= w(z) + aw(x - z) + bw(y - z) \\ &= w(z) + a(w(x) - w(z)) + b(w(y) - w(z)) \\ &= z - 2(x - z) - 2(y - z) + a(y - z) + b(x - z) \\ &= u + (b - a - 2)(x - z) + (a - b - 2)(y - z). \end{aligned}$$

Thus, a generic motion is $\lambda = (-2, -2) + (b - a) \cdot (1, -1)$, and the move-set of this element w is an affine line:

$$\text{MOV}(w) = (-2, -2) + \mathbb{R}(1, -1).$$

Move-sets for general euclidean plane isometries are given in Example 1.35.

1.4. Elliptics and translations. In this section, we record some basic facts about special kinds of elements in an affine Coxeter group.

Definition 1.16 (Roots and reflections). For a reflection r whose fixed space is the hyperplane H , the motion of any point under r is in a direction orthogonal to H , and $\text{MOV}(r)$ is the line through the origin in V in this direction. A vector $\alpha \in V$ is called a *root* of r if $\text{MOV}(r) = \text{SPAN}(\alpha) = \mathbb{R}\alpha$. For affine Coxeter groups, the fixed spaces of the reflections come in finitely many parallel families. Within each family, the fixed hyperplanes are equally spaced, and the length of their common root α can be chosen to encode additional information such as the distance between adjacent parallel fixed hyperplanes. Once the roots are normalized in this way, the result is the usual crystallographic root system Φ inside V . The \mathbb{Z} -span of Φ in V is called the *root lattice*. It is an abelian group (under vector addition) isomorphic to \mathbb{Z}^n where $n = \dim(V)$.

Definition 1.17 (Elliptic elements and elliptic part). An element w in an affine Coxeter group W is called *elliptic* if its fixed space is nonempty. Equivalently, these are exactly the elements of W of finite order. Given an arbitrary element $w \in W$, its *elliptic part* $w_e = p(w)$ is its image under the projection $p : W \twoheadrightarrow W_0$. (In particular, the elliptic part is an element of W_0 , acting naturally on V rather than on E .)

One can characterize elliptic elements in a variety of ways.

Lemma 1.18 (Elliptic elements). *For a euclidean isometry w , the following are equivalent: (1) w is elliptic, (2) $\text{MOV}(w) \subset V$ is a linear subspace, and (3) $\text{MOV}(w)$ contains the origin.*

Proof. [BM15, Proposition 3.2 and Definition 3.3]. \square

Definition 1.19 (Coroots). Let $\Phi \subset V$ be a crystallographic root system. For each root $\alpha \in \Phi$, its *coroot* is the vector $\alpha^\vee = \frac{2}{\langle \alpha, \alpha \rangle} \alpha$. The collection of these coroots is another crystallographic root system $\Phi^\vee = \{\alpha^\vee \mid \alpha \in \Phi\}$. The \mathbb{Z} -span $L(\Phi^\vee)$ of Φ^\vee in V is the *coroot lattice*; as an abelian group, it is also isomorphic to \mathbb{Z}^n where $n = \dim(V)$.

Definition 1.20 (Translations). For every vector $\lambda \in V$ there is a euclidean isometry t_λ of E called a *translation* that sends each point $x \in E$ to $x + \lambda$. Let W be an affine Coxeter group acting on E with root system $\Phi \subset V$. An element of W is a translation in this sense if and only if it is in the kernel T of the projection $p: W \rightarrow W_0$. Moreover, the set of vectors in V that define the translations in T is identical to the set of vectors in the coroot lattice $L(\Phi^\vee)$.

The next result follows immediately from the definitions.

Proposition 1.21 (Translated move-sets). *For any euclidean isometry w and any translation t_λ , $\text{MOV}(t_\lambda w) = \lambda + \text{MOV}(w)$.*

Definition 1.22 (Translation-elliptic factorizations). Let W be an affine Coxeter group. There are many ways to write an element $w \in W$ as a product of a translation $t_\lambda \in W$ and an elliptic $u \in W$. We call any such factorization $w = t_\lambda u$ a *translation-elliptic factorization* of w . In any such factorization we call t_λ the *translation part* and u the *elliptic part*.

Definition 1.23 (Normal forms). Let W be an affine Coxeter group acting co-compactly on a euclidean space E . An identification of W as a semidirect product $T \rtimes W_0$ corresponds to a choice of an inclusion map $i: W_0 \hookrightarrow W$ that is a section of the projection map $p: W \rightarrow W_0$. The unique point $x \in E$ fixed by the subgroup $i(W_0)$ serves as our origin, and every element has a unique factorization $w = t_\lambda u$ where t_λ is a translation in T and u is an elliptic element in $i(W_0)$. In particular, $u = i(w_e)$ is the image under i of the elliptic part of w (Definition 1.17). For a fixed choice of a section i , we call this unique factorization $w = t_\lambda u$ the *normal form* of w under this identification.

If $w = t_\lambda u$ is a translation-elliptic factorization, then $u_e = w_e \in W_0$ and t_λ is in the kernel of p . Some translation-elliptic factorizations come from an identification of W and $T \rtimes W_0$ as in Definition 1.23, but not all of them do; see Example 2.4.

Proposition 1.24 (Recognizing elliptics). *Let W be an affine Coxeter group and let $w = t_\lambda u$ be a translation-elliptic factorization of an element $w \in W$. The following are equivalent: (1) w is elliptic, (2) $\lambda \in \text{MOV}(u)$, (3) $\text{MOV}(w) = \text{MOV}(u)$, and (4) $\text{MOV}(w) = \text{MOV}(w_e)$.*

Proof. By Proposition 1.21, $\text{MOV}(w) = \lambda + \text{MOV}(u)$. Since u is elliptic, $\text{MOV}(u)$ is a linear subspace of V (Lemma 1.18). Thus, $\text{MOV}(w)$ is a linear subspace of V if and only if $\lambda \in \text{MOV}(u)$, so (1) and (2) are equivalent. Moreover, $\lambda \in \text{MOV}(u)$ if and only if $\text{MOV}(u) = \lambda + \text{MOV}(u) = \text{MOV}(w)$, so (2) and (3) are equivalent. Finally, for any elliptic element w , choose a point x fixed by w ; then $w(x + \lambda) = x + w_e(\lambda)$ for all vectors λ . Thus $w(x + \lambda) = (x + \lambda) + (w_e(\lambda) - \lambda)$, so μ belongs to $\text{MOV}(w)$ if and only if μ is of the form $w_e(\lambda) - \lambda$, i.e., if μ belongs to $\text{MOV}(w_e)$. Thus (1) and (4) are equivalent. \square

When isometries are composed, the new move-set is contained in the vector sum of the affine subspaces that are the individual move-sets.

Proposition 1.25 (Move-set addition). *If w_1 and w_2 are elements of an affine Coxeter group W , then $\text{MOV}(w_1 \cdot w_2) \subset \text{MOV}(w_1) + \text{MOV}(w_2)$. Moreover, if r is a reflection and its root α is not in $\text{MOV}(w_e)$, then $\text{MOV}(wr) = \text{MOV}(rw) = \text{MOV}(r) + \text{MOV}(w)$.*

Proof. The motion vector of a point x under the product w_1w_2 is the motion vector of x under w_2 plus the motion vector of $w_2(x)$ under w_1 . The second assertion is part of [BM15, Proposition 6.2]. □

This quickly leads to Carter’s elegant geometric characterization of the reflection length in spherical Coxeter groups [Car72] and its extension to reflection length for elliptic elements in affine Coxeter groups.

Lemma 1.26 (Factoring elliptics). *Let $w = r_1r_2 \cdots r_k$ be a product of reflections in an affine Coxeter group, where reflection r_i is through an affine hyperplane H_i orthogonal to the root α_i . If w is elliptic and $\ell_R(w) = k$, then the roots α_i are linearly independent. Conversely, if the roots α_i are linearly independent, then w is elliptic, $\ell_R(w) = k$, $\text{FIX}(w) = H_1 \cap \cdots \cap H_k$, and $\text{MOV}(w) = \text{SPAN}(\{\alpha_1, \alpha_2, \dots, \alpha_k\})$.*

Proof. In [BM15, Lemmas 3.6 and 6.4], these facts are proved in the full isometry group of the euclidean space E generated by all possible reflections, but the statements and their proofs easily restrict to the case where every reflection is in the set $R \subset W$ and reflection length is computed with respect to this collection of reflections. □

Remark 1.27 (Maximal elliptics). Let W be an affine Coxeter group acting cocompactly on an n -dimensional euclidean space E . When u is an elliptic element of reflection length $n = \dim(E)$ (such as a Coxeter element of a maximal parabolic subgroup of W), its move-set $\text{MOV}(u)$ is all of V (Lemma 1.26). By the equivalence of parts (1) and (2) in Proposition 1.24, the elements $t_\lambda u$ are elliptic for all choices of translation $t_\lambda \in T$ and, in particular, $\ell_R(t_\lambda u) = \ell_R(u) = n$ for every $t_\lambda \in T$.

1.5. Dimension of an element. This section shows how to assign a dimension to each element in a spherical or affine Coxeter group. It is based on the relationship between move-sets and root spaces.

Definition 1.28 (Root spaces). Let V be a euclidean vector space with root system Φ . A subset $U \subset V$ is called a *root space* if it is the span of the roots it contains. In symbols, U is a root space when $U = \text{SPAN}(U \cap \Phi)$. Equivalently, U is a root space when U is a linear subspace of V that is spanned by a collection of roots or when U has a basis consisting of roots. Since Φ is a finite set, there are only finitely many root spaces; the collection of all root spaces in V is called the *root space arrangement* $\text{ARR}(\Phi) = \{U \subset V \mid U = \text{SPAN}(U \cap \Phi)\}$.

Definition 1.29 (Root dimension). For any subset $A \subset V$, we define its *root dimension* $\dim_\Phi(A)$ to be the minimal dimension of a root space in $\text{ARR}(\Phi)$ that contains A . Since V itself is a root space, $\dim_\Phi(A)$ is defined for every subset A in V .

Definition 1.30 (Dimension of an element). When w is an element of a spherical or affine Coxeter group, its move-set is contained in a euclidean vector space V

that also contains the corresponding root system Φ . The *dimension* $\dim(w)$ of such an element is defined to be the root dimension of its move-set. In symbols, $\dim(w) = \dim_{\Phi}(\text{MOV}(w))$. Let W be an affine Coxeter group acting on a euclidean space E and let $p: W \rightarrow W_0$ be its projection map. For each element $w \in W$ we can compute the dimension of w and the dimension of its elliptic part $w_e = p(w) \in W_0$. We call $e = e(w) = \dim(w_e)$ the *elliptic dimension* of w . Instead of focusing on the dimension of w itself, we focus on the number $d = d(w) = \dim(w) - \dim(w_e)$, which we call the *differential dimension* of w . Note that $\dim(w) = d + e$.

The statistics $d(w)$ and $e(w)$ carry geometric meaning.

Proposition 1.31 (Statistics and geometry). *Let W be an affine Coxeter group. An element $w \in W$ is a translation if and only if $e(w) = 0$, and w is elliptic if and only if $d(w) = 0$.*

Proof. If w is a translation, then it is in the kernel of the projection p , w_e is the identity, and $e(w) = 0$. Conversely, if $e(w) = 0$, then $\text{MOV}(w_e) = \{0\}$, w_e is the identity, w is in the kernel of p and thus is a translation.

If w is elliptic, then by Proposition 1.24 $\text{MOV}(w) = \text{MOV}(w_e)$, so $\dim(w) = \dim(w_e)$ and $d(w) = 0$. Conversely, if $d(w) = 0$, then $\dim(w) = \dim(w_e)$. By Lemma 1.26, $\text{MOV}(w_e)$ is itself a root subspace, so we must have $\text{MOV}(w) = \text{MOV}(w_e)$. Then w is elliptic by Proposition 1.24. \square

Thus one way to interpret these numbers is that, roughly speaking, $d(w)$ measures how far w is from being an elliptic element, and $e(w)$ measures how far w is from being a translation. We record one more elementary fact for later use.

Lemma 1.32 (Separation). *Let M and U be linear subspaces of a vector space V and let λ be a vector. The space M contains $\lambda + U$ if and only if M contains both U and λ .*

Proof. If M contains $\lambda + U$, then it contains λ (since U contains the origin) and it contains $-\lambda$ (since M is closed under negation). Thus it contains $(\lambda + U) + (-\lambda) = U$. The other direction is immediate. \square

The next two propositions record the basic relationships between these dimensions and reflection length.

Proposition 1.33 (Inequalities). *Let W be an affine Coxeter group. For every element $w \in W$, $\ell_R(w) \geq \dim(w) \geq \dim(w_e) = \dim(U)$ where $U = \text{MOV}(w_e)$ is the move-set of the elliptic part of w .*

Proof. If w can be written as a product of k reflections, then $\text{MOV}(w)$ is contained in the (linear) span of their roots (Proposition 1.25). Thus $k \geq \dim(w)$, and choosing a minimum-length reflection factorization for w shows that $\ell_R(w) \geq \dim(w)$. Next, if $w = t_\lambda u$ is a translation-elliptic factorization of w , then $\text{MOV}(w) = \lambda + U$, where $U = \text{MOV}(u) = \text{MOV}(w_e)$ (Propositions 1.21 and 1.24). By Lemma 1.32, any root space M that contains $\text{MOV}(w) = \lambda + U$ also contains $\text{MOV}(w_e) = U$, which proves $\dim(w) \geq \dim(w_e)$. Finally, since U is itself a root subspace by Lemma 1.26, we have $\dim(w_e) = \dim(U)$. \square

Proposition 1.34 (Elliptic equalities). *Let W be an affine Coxeter group. When w is elliptic, $\ell_R(w) = \dim(w) = \dim(w_e) = \dim(U)$ where $U = \text{MOV}(w) = \text{MOV}(w_e)$.*

TABLE 1. Basic invariants for the 5 types of elements in an affine Coxeter group acting on the euclidean plane.

w	$\text{MOV}(w)$	d	e	ℓ_R
identity	the origin	0	0	0
reflection	a root line	0	1	1
rotation	the plane	0	2	2
translation	an affine point	1 or 2	0	2 or 4
glide reflection	an affine line	1	1	3

Proof. By Proposition 1.24, $\text{MOV}(w) = \text{MOV}(w_e) = U$. By Lemma 1.26, U is a root space and $\ell_R(w) = \dim(U)$. Proposition 1.33 completes the proof. \square

Example 1.35 (Euclidean plane). The affine Coxeter groups that act on the euclidean plane have five different types of move-sets (see Table 1). Among the elliptic elements, the move-set of the identity is the point at the origin, the move-set of a reflection r with root $\alpha \in \Phi$ is the root line $\mathbb{R}\alpha$, and the move-set of any nontrivial rotation is all of $V = \mathbb{R}^2$. For these elements, $d(w) = 0$ and $\ell_R(w) = \dim(w) = \dim(w_e) = e(w)$ where this common value is 0, 1, or 2, respectively. The move-set of a nontrivial translation t_λ is the single nonzero vector $\{\lambda\}$. Its elliptic dimension is 0 and its differential dimension is either 1 (when λ is contained in a root line $\mathbb{R}\alpha$) or 2. By [MP11, Proposition 4.3], $\ell_R(t_\lambda)$ is twice its dimension. Finally, when w is a glide reflection, $\text{MOV}(w)$ is a line not through the origin, so $e = \dim(w_e) = 1$, $\dim(w) = 2$, $d = 2 - 1 = 1$, and $\ell_R(w) = 3$.

We finish this section with a pair of remarks about computing e and d in general. Let W be an affine Coxeter group with a fixed identification of W with $T \rtimes W_0$ and let $w \in W$ be an element that is given in its semidirect product normal form $w = t_\lambda u$ for some vector λ in the coroot lattice $L(\Phi^\vee)$ and some elliptic element u . Computing the elliptic dimension $e(w)$ is straightforward.

Remark 1.36 (Computing elliptic dimension). To compute the elliptic dimension $e(w)$ it is sufficient to simply compute the dimension of the move-set of the elliptic part u of its normal form. Indeed, by Definition 1.30, $e(w) = \dim(w_e)$, but since $\text{MOV}(w_e)$ is itself a root subspace by Lemma 1.26, $\dim_\Phi(\text{MOV}(w_e)) = \dim(\text{MOV}(w_e))$. Finally, since $p(u) = p(w) = w_e$, $\text{MOV}(w_e) = \text{MOV}(u)$ by Proposition 1.34, and so $\dim(\text{MOV}(w_e)) = \dim(\text{MOV}(u))$.

Computing the differential dimension $d(w)$ is more complicated, but it can be reduced to computing the dimension of a point in a simpler arrangement of subspaces in a lower dimensional space. How much lower depends on the elliptic dimension $e(w)$.

Remark 1.37 (Computing differential dimension). By Definition 1.30, to compute the differential dimension $d(w)$, we need to find the minimal dimension of a root subspace containing $\text{MOV}(w)$ and then subtract $e(w)$ from this value. Since $w = t_\lambda u$, $\text{MOV}(w) = \lambda + U$ where $U = \text{MOV}(u) = \text{MOV}(w_e)$. Moreover, by Lemma 1.32 we only need to consider root spaces that contain λ and U or, equivalently, root spaces that contain $\lambda + U$ and U . Let $q: V \twoheadrightarrow V/U$ be the natural quotient linear transformation whose kernel is U . Under the map q , the coset $\lambda + U$ is sent to a

point in V/U that we call λ/U , and the subspaces in $\text{ARR}(\Phi)$ containing U are sent to a collection of subspaces in V/U that we call $\text{ARR}(\Phi/U)$. Let $\dim_{\Phi/U}(\lambda/U)$ be the minimal dimension of a subspace in $\text{ARR}(\Phi/U)$ that contains the point λ/U . Since the dimensions involved have all been diminished by $e(w) = \dim(U)$, we have that $\dim_{\Phi/U}(\lambda/U) = d(w)$ is the differential dimension of w .

2. PROOFS OF MAIN THEOREMS

In the following subsections we prove our main results.

2.1. Proof of Theorem A. In this subsection, we prove Theorem A by showing the claimed value is both a lower bound and an upper bound for the reflection length of w . Let W be an affine Coxeter group with root system Φ and projection map $p: W \rightarrow W_0$. For each element $w \in W$ we write $w_e = p(w)$ for the elliptic part of w , $e = e(w) = \dim(w_e)$ for its elliptic dimension, and $d = d(w) = \dim(w) - \dim(w_e)$ for its differential dimension.

Proposition 2.1 (Lower bound). *For every $w \in W$, $\ell_R(w) \geq 2d + e$.*

Proof. Let $w = r_1 r_2 r_3 \cdots r_k$ be a reflection factorization of w . For each reflection r_i , let α_i be one of its roots. Let M be the span of this set of vectors α_i and let $m = \dim(M)$. By Proposition 1.25, $\text{MOV}(w) \subset M$ and thus $d + e = \dim(w) \leq m$.

Next, pick a basis for M from among the α_i . By Lemma 1.4, we may assume that the reflections corresponding to our chosen basis are the reflections r_1, \dots, r_m . Define $u = r_1 r_2 \cdots r_m$ and $v = r_{m+1} \cdots r_{k-1} r_k$, so that $w = uv$ and $u = wv^{-1}$.

By Lemma 1.26, u is elliptic and $\text{MOV}(u) = \text{MOV}(u_e) = M$, and so $\dim(u) = \dim(u_e) = m$ (Proposition 1.34). By Proposition 1.33, $k - m \geq \dim(v) \geq \dim(v_e) = \dim(v_e^{-1})$ because v is a product of $k - m$ reflections. Projecting $u = wv^{-1}$ gives $u_e = w_e v_e^{-1}$ and thus $\text{MOV}(u_e) \subset \text{MOV}(w_e) + \text{MOV}(v_e^{-1})$ by Proposition 1.25. Taking dimensions shows that $m \leq e + (k - m)$. Thus, $k \geq 2m - e \geq 2(d + e) - e = 2d + e$, as claimed. □

Proposition 2.2 (Upper bound). *For every $w \in W$, $\ell_R(w) \leq 2d + e$.*

Proof. Let $w = t_\lambda u$ be the normal form of w in W for some identification of W and $T \rtimes W_0$ and let $U = \text{MOV}(u)$. By Propositions 1.21 and 1.24, $\text{MOV}(w) = \lambda + U$, $\text{MOV}(w_e) = U$, and $e = \dim(U)$. Let $m = \dim(w) = d + e$. By definition, there exists an m -dimensional root subspace M containing the affine subspace $\text{MOV}(w)$. By Lemma 1.32, M must contain both λ and U . Since M has a basis of roots, there exists a relative root basis $\{\alpha_1, \alpha_2, \dots, \alpha_d\}$ for M over U , i.e., $M = \text{SPAN}(\{\alpha_1, \dots, \alpha_d\}) \oplus U$ (and, in particular, $\text{SPAN}(\{\alpha_1, \dots, \alpha_d\}) \cap U = \{0\}$).

For each i , let r_i be any reflection with root α_i . Consider the element $v = w r_1 r_2 \cdots r_d$. Iteratively applying Proposition 1.25 one reflection at a time shows that $\text{MOV}(v) = \text{SPAN}(\{\alpha_1, \dots, \alpha_d\}) + \text{MOV}(w)$. Since $\text{MOV}(w) = \lambda + U$, it follows that $\text{MOV}(v) = \text{SPAN}(\{\alpha_1, \dots, \alpha_d\}) + \lambda + U = \lambda + M = M$. Since M is a linear subspace of V , v is elliptic and $\ell_R(v) = \dim(M) = d + e$. Since $w = v r_d \cdots r_2 r_1$, $\ell_R(w) \leq \ell_R(v) + d = (d + e) + d = 2d + e$, as claimed. □

Together Propositions 2.1 and 2.2 prove Theorem A.

2.2. Proof of Theorem B. In this subsection, we prove Theorem B using a recent technical result about inefficient factorizations in spherical Coxeter groups due to the first author and Vic Reiner [LR16, Corollary 1.4].

Proposition 2.3 (Inefficient factorizations). *Let W_0 be a spherical Coxeter group and let w be an element of W_0 . If $\ell_R(w) = \ell$, then every factorization of w into k reflections lies in the Hurwitz orbit of some factorization $w = r_1 r_2 \cdots r_k$ where $r_1 = r_2$, $r_3 = r_4$, \dots , $r_{k-\ell-1} = r_{k-\ell}$, and $r_{k-\ell+1} \cdots r_k$ is a minimum-length reflection factorization of w .*

We use this proposition about inefficient factorizations of an element in a spherical Coxeter group to find a way to rewrite an efficient factorization of an element in an affine Coxeter group into a particular form. This is our second main result.

Theorem B (Factorization). *Let W be an affine Coxeter group. For any element $w \in W$ there is a translation-elliptic factorization $w = t_\lambda u$ such that $\ell_R(t_\lambda) = 2d(w)$ and $\ell_R(u) = e(w)$. In particular, $\ell_R(w) = \ell_R(t_\lambda) + \ell_R(u)$ for this factorization of w .*

Proof. Let $k = 2d + e = \ell_R(w)$ and let $w = r'_1 r'_2 \cdots r'_k$ be a minimum-length reflection factorization of w . The projection $p: W \rightarrow W_0$ sends this efficient factorization of $w \in W$ to a (not necessarily efficient) factorization $w_e = p(w) = p(r'_1) p(r'_2) \cdots p(r'_k)$ of $w_e \in W_0$. By Carter’s lemma (see Remark 1.2) and Proposition 1.24, minimum-length reflection factorizations of w_e all have length $e = e(w)$.

By Proposition 2.3, there is a sequence of Hurwitz moves on the given W_0 -factorization that produces a factorization of a special form. The exact sequence of Hurwitz moves applied to the factorization in W_0 can be mimicked on the original factorization in W . Since the Hurwitz action is easily seen to be compatible with p , the result is an R -factorization $w = r_1 r_2 \cdots r_k$ of w such that $p(r_1) = p(r_2)$, $p(r_3) = p(r_4)$, \dots , $p(r_{2d-1}) = p(r_{2d})$, and $p(r_{2d+1}) \cdots p(r_k)$ is a minimum-length reflection factorization of $p(w)$. This means that r_1 and r_2 are reflections through parallel hyperplanes, and so $t_{\lambda_1} = r_1 r_2$ is a (possibly trivial) translation. Similarly $t_{\lambda_i} = r_{2i-1} r_{2i}$ is a translation for $i = 1, \dots, d$. Thus the product of these first $2d$ reflections is a translation $t_\lambda = t_{\lambda_1} t_{\lambda_2} \cdots t_{\lambda_d}$ and $\ell_R(t_\lambda) \leq 2d$.

Let $u = r_{2d+1} \cdots r_k$ be the product of the remaining e reflections, so that $w = t_\lambda u$. Because the given factorization of u projects to a minimum-length reflection factorization of $u_e = w_e$ in W_0 , the roots of $p(r_{2d+1}), \dots, p(r_k)$ must be linearly independent (Lemma 1.26), which means that the same is true for r_{2d+1}, \dots, r_k in W . By Lemma 1.26, u is elliptic and $\ell_R(u) = e$. Finally, by Theorem A and the triangle inequality (Remark 1.3), we must have $\ell_R(t_\lambda) = 2d$. □

It is important to note that this theorem does *not* assert that there exists an identification of W with $T \rtimes W_0$ such that the corresponding normal form $w = t_\lambda u$ satisfies the conclusion of the theorem. In fact this stronger assertion is demonstrably false.

Example 2.4 (Normal forms are insufficient). In all irreducible affine Coxeter groups other than the affine symmetric groups, there is a maximal parabolic subgroup W' that is not isomorphic to W_0 . Let w be a Coxeter element for one of these alternative maximal parabolic subgroups.

Being a Coxeter element of a spherical Coxeter group, w has a unique fixed point; denote it by x' . Reflections across the hyperplanes passing through x' generate W' .

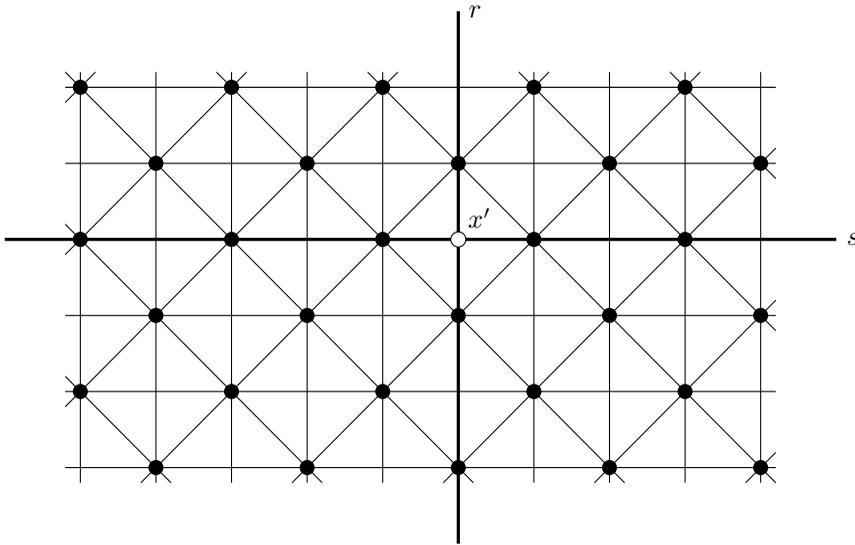


FIGURE 5. The element $w = rs$ has $\ell_R(w) = \ell_R(w_e) = 2$. The origin can only be identified with the black nodes, whereas x' is the unique fixed point of w . Thus every normal form for w has a nontrivial translation.

On the other hand, reflections across the hyperplanes passing through any choice of origin used in the construction of W (i.e., the $r_{\alpha,0}$ reflecting across the $H_{\alpha,0}$ as in Definitions 1.10 and 1.11) must generate a group isomorphic to W_0 . As W' is not isomorphic to W_0 , this means x' cannot be the origin for any such identification of W with $T \rtimes W_0$.

Now fix an identification of W and $T \rtimes W_0$ having origin $x \neq x'$, and consider the corresponding normal form $w = t_\lambda u$. By Remark 1.27, $\ell_R(w) = \ell_R(u) = n$. However, since $w(x) \neq x = u(x)$, we have $t_\lambda \neq 1$; i.e., the normal form for w includes a nontrivial translation. Thus, for this element w , $\ell_R(w) < \ell_R(t_\lambda) + \ell_R(u)$.

Figure 5 provides an illustration. Here we see a portion of the affine line arrangement for B_2 . The nodes in black correspond to all possible identifications of an origin to form the semidirect product.

Let r and s be the reflections across the lines indicated in bold, and let $w = rs$. Let x' denote the point of intersection for these lines. (The reflections r and s generate the maximal parabolic subgroup of type $A_1 \times A_1$.) We have $\ell_R(w) = \ell_R(w_e) = 2$. However, if we choose an origin $x \neq x'$ and write $w = t_\lambda u$ with respect to this origin, then the motion u will be a π -rotation about x which requires at least 2 reflections on its own, i.e., $\ell_R(u) = 2$. But since x and x' are not equal and w does not fix x , t_λ cannot be trivial. Thus $\ell_R(t_\lambda) > 0$ and $\ell_R(w) < \ell_R(t_\lambda) + \ell_R(u)$.

In an affine symmetric group, all maximal parabolic subgroups are isomorphic, and every vertex (maximal intersection of hyperplanes) in the hyperplane arrangement can play the role of the origin in an identification with a semidirect product. This immediately establishes the following result.

Corollary 2.5 (Affine symmetric normal form). *Let $W = \tilde{\mathfrak{S}}_n$ be the affine symmetric group. For each element $w \in W$ there is an identification of W and $T \rtimes W_0$ so that w has normal form $w = t_\lambda u$ and $\ell_R(w) = \ell_R(t_\lambda) + \ell_R(u)$.*

Proof. Let $w = t_\lambda u$ be the factorization from Theorem B. Choose an identification of W and $T \rtimes W_0$ so that the role of the origin is played by one of the vertices that is fixed by u . □

3. LOCAL STATISTICS

In a spherical Coxeter group $W = W_0$, Shephard and Todd [ST54, Theorem 5.3] showed that the generating function

$$f_0(t) = \sum_{u \in W_0} t^{\ell_R(t)}$$

has a particularly nice form,

$$(1) \quad f_0(t) = \prod_{i=1}^n (1 + e_i t),$$

where the numbers e_i are positive integers called the *exponents* of W_0 . In earlier work [MP11], the second and third authors asked whether there are similarly nice generating functions associated to an affine Coxeter group. In this section, we explore this question.

For an affine Coxeter group W , reflection length is bounded and $|W|$ is infinite, so the naive generating function is not defined. A natural fix is to consider only a finite piece of W .

Definition 3.1 (Local generating function). Given an element λ of the coroot lattice $L(\Phi^\vee)$, define the bivariate generating function

$$f_\lambda(s, t) = \sum_{u \in W_0} s^{d(t_\lambda u)} \cdot t^{e(t_\lambda u)} = \sum_{u \in W_0} s^{\dim(t_\lambda u)} \cdot (t/s)^{\dim(u)}$$

that tracks the statistics of differential and elliptic dimension. By Theorem A, we have

$$f_\lambda(t^2, t) = \sum_{u \in W_0} t^{\ell_R(t_\lambda \cdot u)}.$$

By mild abuse of notation, we let $f_\lambda(t) = f_\lambda(t^2, t)$ denote this local reflection length generating function.

The term “local” here makes sense geometrically, once the elements of W have been identified with alcoves in the reflecting hyperplane arrangement for W as in Remark 1.12. The alcoves neighboring the origin (each of which is identified with a unique element of W_0) form a W_0 -invariant polytope P . The set $\{t_\lambda u \mid u \in W_0\}$ corresponds to the set of alcoves in $t_\lambda \cdot P$, i.e., those alcoves neighboring λ . In Figure 6 we have shaded the alcoves according to reflection length. In each example, the identity element is identified with the black alcove, and lighter colored cells have greater reflection length. The coroots are highlighted in white.

We collect here some easy facts about the local generating function.

Proposition 3.2 (Properties of local generating functions). *Let λ be a vector in the coroot lattice $L(\Phi^\vee)$.*

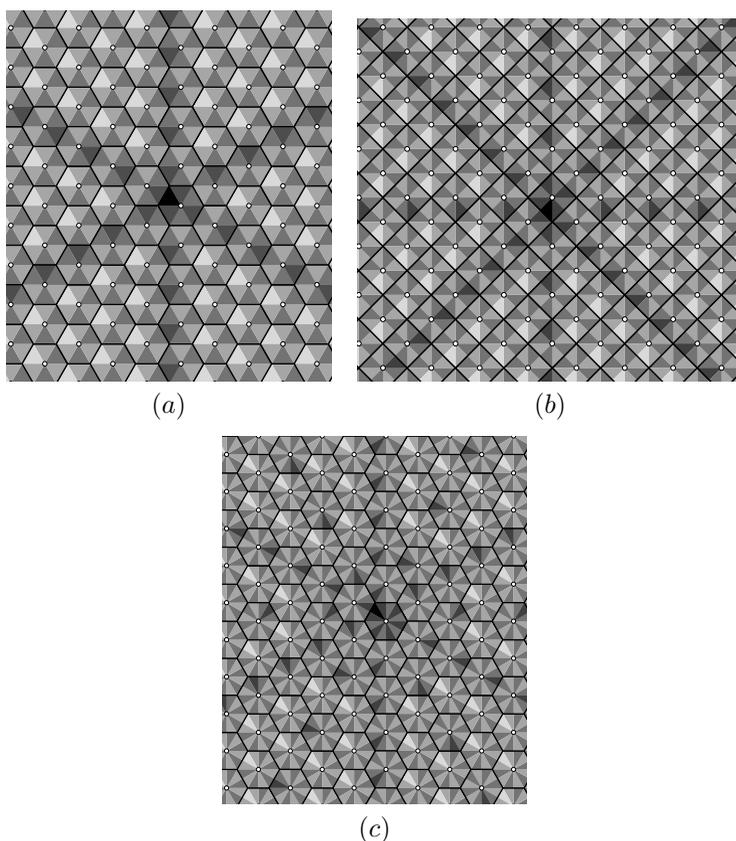


FIGURE 6. Reflection length in the affine Coxeter groups of (a) type A_2 , (b) type B_2 , and (c) type G_2 . Lighter colors correspond to greater reflection length. White nodes correspond to translations.

- (i) (*The origin*) If $\lambda = 0$, then $f_\lambda(s, t) = f_0(t)$ is the generating function for reflection length in W_0 .
- (ii) (*Generic*) If λ is generic, i.e., if it is not a member of any proper root subspace of V , then $f_\lambda(s, t) = s^n f_0(t/s) = \prod_{i=1}^n (s + e_i t)$, where the e_i are the exponents of W_0 .
- (iii) (*Permutations*) If λ and λ' belong to the same W_0 -orbit (that is, there is some $w \in W_0$ such that $\lambda' = w(\lambda)$), then $f_\lambda(s, t) = f_{\lambda'}(s, t)$.

Proof. If $\lambda = 0$, then t_λ is the identity, and so (i) follows immediately from the definitions.

For (ii), since u is elliptic we have $\lambda \in \text{MOV}(t_\lambda u)$. Thus, since λ is generic, $\text{MOV}(t_\lambda u)$ is not contained in any proper root subspace of V , and so $\dim(t_\lambda u) = n$. Substituting this into the definition of f_λ gives the first equality. The second equality follows from Shephard and Todd's result.

For (iii), suppose that $\lambda' = w(\lambda)$. Then $t_{\lambda'} = wt_\lambda w^{-1}$ and so for every u in W_0 we have $t_{\lambda'} u = w(t_\lambda(w^{-1}uw))w^{-1}$. Conjugation of group elements by w acts on

TABLE 2. Local generating functions for affine A_2 , B_2 , and G_2 .

A_2		B_2	
λ	$f_\lambda(s, t)$	λ	$f_\lambda(s, t)$
● 0	$(1+t)(1+2t)$	● 0	$(1+t)(1+3t)$
● α^\vee	$(s+t)(1+2t)$	● α^\vee	$(s+t)(1+3t)$
○ generic	$(s+t)(s+2t)$	○ generic	$(s+t)(s+3t)$

G_2	
λ	$f_\lambda(s, t)$
● 0	$(1+t)(1+5t)$
● α^\vee	$(s+t)(1+5t)$
○ generic	$(s+t)(s+5t)$

TABLE 3. Local generating functions for affine A_3 . Here α_1 and α_3 are any two orthogonal roots, while α_2 and α_1 are not orthogonal.

A_3	
λ	$f_\lambda(s, t)$
● 0	$(1+t)(1+2t)(1+3t)$
● α^\vee	$(s+t)(1+2t)(1+3t)$
○ $\alpha_1^\vee + \alpha_3^\vee$	$2t^2 + 6t^3 + 4st + 9st^2 + s^2 + 2s^2t$
● generic span of $\alpha_1^\vee, \alpha_2^\vee$	$(s+t)(s+2t)(1+3t)$
● generic span of $\alpha_1^\vee, \alpha_3^\vee$	$(s+t)(t+6t^2+s+4st)$
● generic	$(s+t)(s+2t)(s+3t)$

move-sets as multiplication by w . Since $w \in W_0$, this preserves root dimensions, and so $\dim(t_\lambda u) = \dim(t_\lambda(w^{-1}uw))$. Finally, as u runs over the spherical group W_0 , $w^{-1}uw$ does as well, and $\dim(w^{-1}uw) = \dim(u)$, so the joint distribution of dimensions is the same over both sets. \square

To get a feel for the local generating functions, Table 2 lists the different local generating functions for types A_2 , B_2 , and G_2 . In Figure 7 we revisit the alcove picture, but instead of individual alcoves, we have colored the translates $t_\lambda \cdot P$ according to $f_\lambda(t)$. In Figure 8 we see affine A_3 with the translates $t_\lambda \cdot P$ colored according to $f_\lambda(t)$. In this case there are six different local generating functions, as listed in Table 3. (Only five are visible in Figure 8 as the origin is hidden.) The important thing to know about the notation in Table 3 is that α_1 and α_3 are orthogonal to each other, whereas neither is orthogonal to α_2 .

From the pictures it appears that the local generating functions line up along faces of a hyperplane arrangement. The faces of the arrangement are intersections of maximal root subspaces, and if two coroots lie in the same face, they have the same local generating function. However, note that while some of these intersections are themselves root subspaces, not all of them are. For example, the white balls in Figure 8 lie along lines that are not of the form $\mathbb{R}\alpha$ for any root α .

We make the phenomenon precise here.

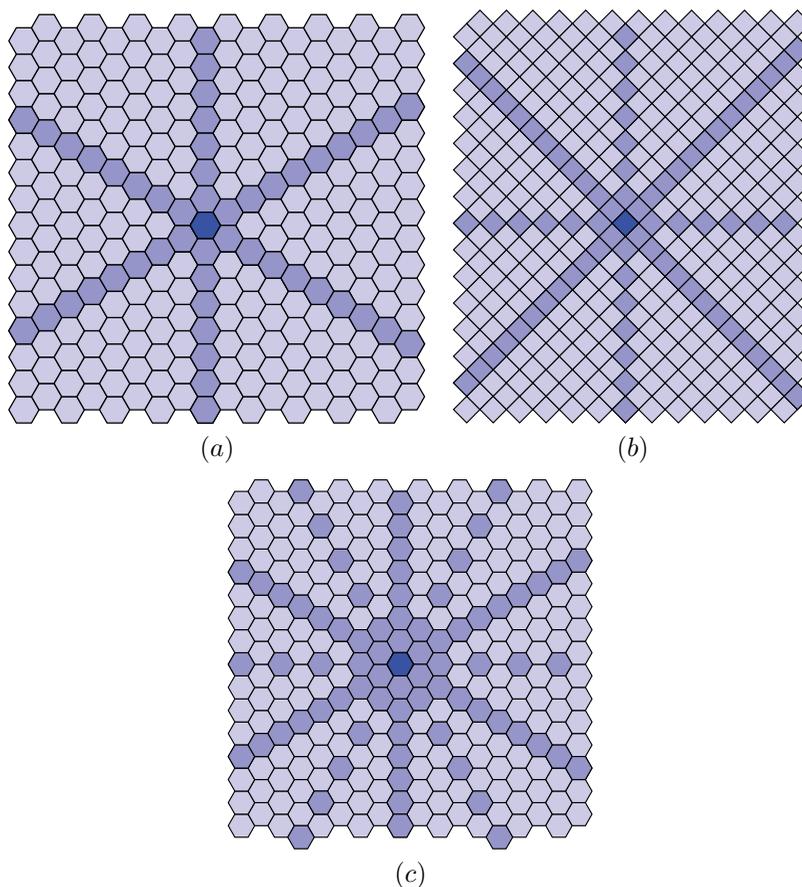


FIGURE 7. The translates $t_\lambda \cdot P$ in (a) type A_2 , (b) type B_2 , and (c) type C_2 , colored according to the local distribution of reflection length.

Theorem 3.3 (Equality of local generating functions). *Suppose that λ and μ are two elements of the coroot lattice L . Suppose furthermore that λ and μ belong to the same collection of root subspaces. Then $f_\lambda(s, t) = f_\mu(s, t)$.*

Proof. In fact, something stronger is true: if λ and μ are as described, then $\dim(t_\lambda u) = \dim(t_\mu u)$ for every u in W_0 . Indeed, by Lemma 1.32, for $u \in W_0$, a root subspace U contains $\text{MOV}(t_\lambda u)$ if and only if it contains λ and $\text{MOV}(u)$. But by hypothesis, every such U contains μ and $\text{MOV}(u)$, and so contains $\text{MOV}(t_\mu u)$. Then the result follows immediately from the definition of f_λ . \square

Unfortunately, while Theorem 3.3 and Proposition 3.2 imply bounds on the number of local generating functions in terms of the number of W_0 -orbits of intersections of root subspaces, it is probably intractable to compute all $f_\lambda(s, t)$, or even all $f_\lambda(t)$, in general. We show in Appendix A that computing $d(t_\lambda)$ for an element λ of the type A_n coroot lattice is essentially equivalent to the NP-complete problem **SubsetSum**.

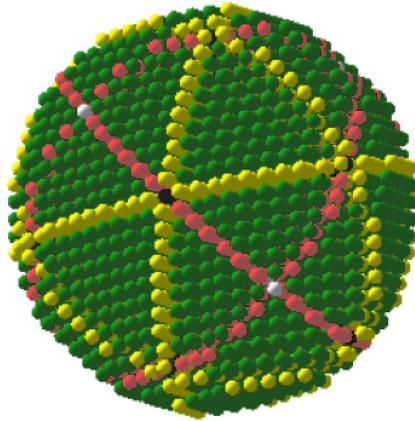


FIGURE 8. The translates $t_\lambda \cdot P$ in type A_3 colored according to the local distribution of reflection length.

A different approach to understanding the distribution of reflection length would be to introduce a new statistic that grows with λ and take a bivariate generating function, either over the whole group W or over the elements with fixed elliptic part (that is, over a coset of the translations). Thus, for a given element $u \in W_0$ one could consider the generating function

$$g_u(q, t) = \sum_{\lambda \in L} t^{d(t_\lambda u)} q^{\text{stat}(\lambda, u)}$$

for some statistic “stat”, e.g., usually the Coxeter length of t_λ . By Theorem A, the corresponding generating function for reflection length is $t^{\dim(u)} g_u(q, t^2)$.

If u has maximal reflection length in W_0 , e.g., if u is a Coxeter element, then by Remark 1.27 we have $d(t_\lambda u) = 0$, and this generating function simplifies to $\sum_{\lambda \in L} q^{\text{stat}(\lambda, u)}$. However, at this point we have no good candidate for $\text{stat}(\lambda, u)$ and have made little progress in this direction.

4. AFFINE SYMMETRIC GROUPS

In this section, we restrict our attention to the affine symmetric groups and give simple combinatorial descriptions of the statistics $d(w)$ and $e(w)$ used to define reflection length. This provides an affine analog of the formula for the symmetric group given by Dénes [Dén59].

Throughout this part of the article, we can safely relax our distinction between “points” in euclidean space and “vectors” in a vector space. To this end, fix a euclidean vector space \mathbb{R}^n with a fixed ordered orthonormal basis e_1, e_2, \dots, e_n .

4.1. Permutations and affine permutations. In this section, we describe the symmetric group and affine symmetric group from both the geometric and combinatorial perspectives.

Definition 4.1 (Permutations). A *permutation* of the set $[n]$ is a bijection $\pi: [n] \rightarrow [n]$, and the set of all permutations under composition is the *symmetric group* \mathfrak{S}_n . A permutation π may be represented in *one line notation* as the sequence $[\pi(1), \dots, \pi(n)]$ of its values.

Definition 4.2 (Root system). For every permutation $\pi \in \mathfrak{S}_n$, there is an isometry $u_\pi: \mathbb{R}^n \rightarrow \mathbb{R}^n$ that sends (v_1, \dots, v_n) to $(v_{\pi(1)}, \dots, v_{\pi(n)})$. All of these isometries fix the line $\mathbb{R}\mathbf{1}$ in direction $\mathbf{1} = (1, \dots, 1)$. Let V be the orthogonal complement of this line, consisting of all vectors whose coordinate sum is 0. Then \mathfrak{S}_n is a spherical Coxeter group acting on V , as in Definition 1.6. The reflections are exactly the *transpositions* that interchange two values $i, j \in [n]$ while fixing all others. For the reflection that switches the i -th and j -th coordinates we select the vectors $\pm(e_i - e_j)$ as its roots, and we denote by Φ_n the full root system $\Phi_n = \{e_i - e_j \mid i, j \in [n], i \neq j\}$.

We record a few elementary properties of Φ_n without proof.

Proposition 4.3 (Coroots). *Let Φ_n be the root system for \mathfrak{S}_n . Roots are the same as coroots, the root system spans V , and the lattice of (co)roots is the set of vectors in V with integer coordinates. In symbols, $\Phi_n = (\Phi_n)^\vee$, $\text{SPAN}(\Phi_n) = V$, and $L(\Phi_n) = V \cap \mathbb{Z}^n$.*

Following Definition 1.10, one can use the root system Φ_n to construct the *affine symmetric group* $\tilde{\mathfrak{S}}_n$ as a group of isometries of V . By Definition 1.11, every *affine permutation* $w \in \tilde{\mathfrak{S}}_n$ can be written as $w = t_\lambda u_\pi$ where u_π is the elliptic isometry of \mathbb{R}^n indexed by a permutation $\pi \in \mathfrak{S}_n$ and t_λ is a translation by a vector λ in the \mathbb{Z} -span of Φ_n , i.e., a vector with integer coordinates and coordinate sum 0. There is also an alternative description of affine permutations, due originally to Lusztig [Lus83], that is common in the combinatorics literature (see, e.g., [EE98]).

Definition 4.4 (Affine permutations as bijections). Let $w: \mathbb{Z} \rightarrow \mathbb{Z}$ be a bijection from the integers to the integers. We call the map w an *affine permutation* of order n if it satisfies three conditions:

- (1) $w(i + n) = n + w(i)$ for all $i \in \mathbb{Z}$,
- (2) $w(i) = w(j) \pmod n$ if and only if $i = j \pmod n$, and
- (3) $w(1) + w(2) + \dots + w(n) = 1 + 2 + \dots + n$.

By the first condition, such periodic maps can be described in *one line notation* by listing the n values $[w(1), w(2), \dots, w(n)] \in \mathbb{Z}^n$. It is easy to check whether such a list of n integers satisfies the second and third conditions.

Remark 4.5 (Isomorphisms). The isomorphisms between the combinatorial and geometric descriptions of $\tilde{\mathfrak{S}}_n$ go as follows. Since every integer can be uniquely written as a number in $[n]$ plus a multiple of n , the one line notation $[w(1), w(2), \dots, w(n)]$ can be uniquely written as $u + t$, where u has all entries in $[n]$ and $t = n \cdot \lambda$ for some vector $\lambda \in \mathbb{Z}^n$. By the second condition, u is the one line notation of a permutation π in \mathfrak{S}_n , while by the third condition, λ has coordinate sum 0. The bijection w with this one line notation is sent by the isomorphism to the element $t_\lambda u_\pi \in \tilde{\mathfrak{S}}_n$.

4.2. Partitions, cycles, and root arrangements. In this section, we describe the move-sets and fixed spaces of permutations in terms of their cycle structure. For this purpose, it is helpful to work with the language of set partitions.

Definition 4.6 (Partitions). A *partition* P of a set A is a collection of pairwise disjoint nonempty subsets whose union is A . The subsets in P are called *blocks*, and the *size* $|P|$ of P is the number of blocks it contains. Concretely, a partition of size k is a collection $P = \{B_1, B_2, \dots, B_k\}$ with $\emptyset \neq B_i \subset A$ for all i , $B_i \cap B_j = \emptyset$ for $i \neq j$, and $B_1 \cup \dots \cup B_k = A$.

Definition 4.7 (Cycles and partitions). For every permutation $\pi \in \mathfrak{S}_n$ there is a partition of $[n]$, $P(\pi)$, with i and j in the same block of $P(\pi)$ if and only if i and j are in the same orbit under the action of π . Concretely, i and j are in the same block of $P(\pi)$ if and only if there is an integer ℓ so that $\pi^\ell(i) = j$. When π is written in *cycle notation*, each block of $P(\pi)$ corresponds to the set of numbers contained in one cycle of π . For example, the permutation π in \mathfrak{S}_6 with one line notation $[4, 5, 1, 3, 2, 6]$ has cycle notation $\pi = (1, 4, 3)(2, 5)(6)$ and associated partition $P(\pi) = \{\{1, 3, 4\}, \{2, 5\}, \{6\}\}$. Let $\text{CYC}(\pi)$ be the set of cycles in π , so that $|\text{CYC}(\pi)| = |P(\pi)|$ is the number of cycles.

Definition 4.8 (Partition lattice). Let P and Q be two partitions of the same set A . We say that P *refines* Q and write $P \leq Q$ if every block of P is contained in some block of Q . We denote by Π_A the set of all partitions of a fixed finite set A , ordered by refinement. This poset is a bounded, graded lattice, and two partitions are at the same height if and only if they have the same size. The partition $P_1 = \{A\}$ with only one block is the unique maximum element. The partition P_0 with $|A|$ blocks, each containing a single element, is the unique minimum element. When $A = [n]$, we write Π_n instead of $\Pi_{[n]}$.

The next few definitions describe the spaces that turn out to be equal to the fixed spaces and move-sets of permutations.

Definition 4.9 (Special vectors). Let \mathbb{R}^n be a euclidean vector space and let e_1, e_2, \dots, e_n be a fixed ordered orthonormal basis. For each subset $\emptyset \neq B \subset [n]$, let $\mathbf{1}_B$ be the sum of the basis vectors indexed by the numbers in B , i.e., $\mathbf{1}_B = \sum_{i \in B} e_i$. We call $\mathbf{1}_B$ a *special vector*. When $B = \{i\}$ contains only a single element, $\mathbf{1}_{\{i\}} = e_i$ is a basis vector. At the other extreme, $\mathbf{1}_{[n]} = \mathbf{1} = (1, \dots, 1)$ is the all-1's vector.

Definition 4.10 (F -spaces). For each partition $P \in \Pi_n$, we define the F -space $F_P \subset \mathbb{R}^n$ to be the set of all vectors whose coordinates are constant on the blocks of P . In other words, a vector $v = (v_1, \dots, v_n)$ is in F_P if and only if $v_i = v_j$ for all i, j belonging to the same block $B \in P$.

Remark 4.11 (F -spaces). When $P = \{B_1, B_2, \dots, B_k\}$, we have that F_P is equal to the span of $\mathbf{1}_{B_1}, \mathbf{1}_{B_2}, \dots, \mathbf{1}_{B_k}$, and these special vectors form a basis for F_P . In particular, $\dim(F_P) = |P|$. At the extremes, $F_{P_0} = \text{SPAN}(\{e_1, e_2, \dots, e_n\}) = \mathbb{R}^n$ is the entire space, and $F_{P_1} = \text{SPAN}(\{\mathbf{1}\}) = \mathbb{R}\mathbf{1}$ is a line. Also note that $F_P \subset F_Q$ if and only if $Q \leq P$ in Π_n . Thus every F -space contains the line $F_{P_1} = \mathbb{R}\mathbf{1}$.

Definition 4.12 (M -spaces). For each partition $P \in \Pi_n$, we define the M -space $M_P \subset \mathbb{R}^n$ to be the orthogonal complement of the corresponding F -space F_P . In symbols, $M_P = (F_P)^\perp$ and $F_P \oplus M_P = \mathbb{R}^n$. In terms of coordinates, M_P is the set of vectors v such that, for each block B of P , the sum of the coordinates of v indexed by elements of B is equal to 0.

Remark 4.13 (M -spaces). Since $\dim(F_P) = |P|$, we have $\dim(M_P) = n - |P|$. Since $F_{P_0} = \mathbb{R}^n$, $M_{P_0} = \{0\}$ is the trivial subspace. At the other extreme, since $F_{P_1} = \mathbb{R}\mathbf{1}$ is a line, $M_{P_1} = (F_{P_1})^\perp$ is the codimension-1 subspace V from Definition 4.2. Taking orthogonal complements reverses containment, so $M_P \subset M_Q$ in \mathbb{R}^n if and only if $P \leq Q$ in Π_n . Because every F -space contains the line $\mathbb{R}\mathbf{1} = F_{P_1}$, every M -space is contained in the subspace $V = M_{P_1}$.

Proposition 4.14 (Permutations and partitions). *For each $\pi \in \mathfrak{S}_n$ with partition $P(\pi) \in \Pi_n$, the elliptic isometry u_π acting on \mathbb{R}^n has $\text{FIX}(u_\pi) = F_{P(\pi)}$ and $\text{MOV}(u_\pi) = M_{P(\pi)}$.*

Proof. That a point v is fixed under the action of u_π if and only if v is in $F_{P(\pi)}$ is clear from the definition of the action, which proves the assertion about the fixed space. Since move-sets and fixed spaces of elliptic isometries are orthogonal complements [BM15, Lemma 3.6], as are $F_{P(\pi)}$ and $M_{P(\pi)}$ (Definition 4.12), the assertion about move-sets follows from the assertion about the fixed space. \square

Remark 4.15 (Root arrangement). Proposition 4.14 shows that the move-set of every permutation $\pi \in \mathfrak{S}_n$ is an M -space. Moreover, it is easy to see that every partition P of $[n]$ is induced as $P = P(\pi)$ for some permutation π . Thus, the move-sets of permutations are exactly the M -spaces. It follows from Lemma 1.26 that these spaces are also exactly the set of root spaces for the root system Φ_n , and so the arrangement of M -spaces is exactly the root space arrangement (Definition 1.28).

The root arrangement for Φ_n should not be confused with the better known braid arrangement.

Remark 4.16 (Braid arrangement). The *braid arrangement* in \mathbb{R}^n is the collection of $\binom{n}{2}$ hyperplanes that are orthogonal to a root in Φ_n . Concretely, there is a hyperplane H_{ij} for all distinct $i, j \in [n]$ that contains the vectors v with $v_i = v_j$. The result of taking all intersections of hyperplanes in the braid arrangement is a subspace arrangement that is equal to the collection of F -spaces $\{F_P \mid P \in \Pi_n\}$. By Remark 4.15, the spaces in the root arrangement $\text{ARR}(\Phi_n)$ are the M -spaces, i.e., the orthogonal complements of the F -spaces. The hyperplanes in $\text{ARR}(\Phi_n)$ are the M -spaces M_P where P is a partition of size 2, and there are exactly $(2^n - 2)/2 = 2^{n-1} - 1$ of these. Unlike the F -spaces, the M -spaces are *not* closed under intersection. For example, in \mathbb{R}^4 , the intersection $M_{\{1,2\},\{3,4\}} \cap M_{\{1,4\},\{2,3\}}$ is the line consisting of vectors of the form $(a, -a, a, -a)$ for $a \in \mathbb{R}$, and this is not an M -space.

4.3. Null partitions. In this section, we assemble the final combinatorial and linear-algebraic objects necessary to reinterpret reflection length.

Definition 4.17 (L -maps). Let v be a vector in \mathbb{R}^n . For each nonempty subset $B \subset [n]$, we define the real number v_B to be the sum of the coordinates of v in the positions indexed by the set B . Equivalently, $v_B = \langle v, \mathbf{1}_B \rangle$, where $\mathbf{1}_B$ is the special vector defined in Definition 4.9. For each partition $P \in \Pi_n$ with $|P| = k$, we define a linear transformation $L_P: \mathbb{R}^n \rightarrow \mathbb{R}^k$ as follows. When $P = \{B_1, \dots, B_k\}$ we assign the names e_{B_i} to an orthonormal basis of \mathbb{R}^k and we define $L_P(v) = \sum_{i=1}^k v_{B_i} e_{B_i}$. We call the maps of this form *L-maps*.

The identification $e_{B_i} \mapsto e_i$ in \mathbb{R}^k corresponds to fixing an order on the blocks of P , but note that L_P is a well-defined map independent of such an ordering. In coordinates, $L_P(v) = (v_{B_1}, v_{B_2}, \dots, v_{B_k})$. For example, if $v = (-5, 2, 1, 0, 4, -2)$ and $P = \{\{1, 3\}, \{2, 4, 6\}, \{5\}\}$ with the blocks in that order, then $L_P(v) = (-4, 0, 4)$.

Proposition 4.18 (M -space membership). *Let $P \in \Pi_n$ be a partition of $[n]$. Then $\ker L_P = M_P$.*

Proof. Let $P = \{B_1, \dots, B_k\}$. The vector v is in the space M_P if and only if it is orthogonal to every vector in F_P . Since $\{\mathbf{1}_{B_1}, \dots, \mathbf{1}_{B_k}\}$ is a basis of F_P , it is

sufficient to test whether v is orthogonal to each $\mathbf{1}_{B_i}$, and this is equivalent to being in the kernel of map L_P . \square

Definition 4.19 (Null partitions). Let v be a vector in $V = M_{\mathbf{1}}$. A nonempty subset $B \subset [n]$ is called a *null block* of v if the sum v_B (Definition 4.17) of the entries of v indexed by B is equal to 0 or equivalently if v is orthogonal to the special vector $\mathbf{1}_B$ (Definition 4.9). A partition $P \in \Pi_n$ is called a *null partition* of v if every block of P is a null block of v . Equivalently, P is a null partition of v if and only if v is in the kernel of L_P (Definition 4.17) and if and only if $v \in M_P$ (Proposition 4.18). We write $\Pi^0(v)$ for the collection of all null partitions of v . This collection is not empty since $v \in V$ implies $\langle v, \mathbf{1} \rangle = 0$, which means $P_{\mathbf{1}} \in \Pi^0(v)$. The *nullity* of v is the maximal size of a null partition of v . In symbols, $\nu(v) = \max\{|P| \mid P \in \Pi^0(v)\}$.

Example 4.20 (Null partitions). Consider $v = (-5, 2, 1, 0, 4, -2)$ in $V = M_{\mathbf{1}} \subset \mathbb{R}^6$. The partition $P = \{\{1, 3, 5\}, \{4\}, \{2, 6\}\}$ is a null partition of v , and its blocks are null blocks of v . The set $\{1, 2, 3\}$ is not a null block of v because $\langle v, \mathbf{1}_{\{1,2,3\}} \rangle = -2 \neq 0$.

Proposition 4.21 (Nullity and dimension). *For each $v \in V$, $\nu(v) + \dim_{\Phi_n}(\{v\}) = n$.*

Proof. By Definition 1.29, $\dim_{\Phi_n}(\{v\})$ is the minimal dimension of a root space in $\text{ARR}(\Phi_n)$ that contains v . By Remark 4.15, root spaces are M -spaces. For each $P \in \Pi_n$, $P \in \Pi^0(v)$ if and only if $v \in M_P$ (Definition 4.19) and $|P| + \dim(M_P) = n$ (Remark 4.13). Thus, among the null partitions for v , the ones that maximize $|P|$ simultaneously minimize $\dim(M_P)$. For a partition P with these properties, the equation $|P| + \dim(M_P) = n$ becomes $\nu(v) + \dim_{\Phi_n}(\{v\}) = n$, as claimed. \square

4.4. Combinatorial formulas for statistics. In this section, we complete the work of Section 4. For any affine permutation w , we give combinatorial expressions for the elliptic dimension $e(w)$ and the differential dimension $d(w)$. Thus by Theorem A, we obtain a combinatorial expression for the reflection length $\ell_R(w)$.

First, we consider the elliptic dimension. Our result generalizes Dénes’s result that the reflection length of a permutation π is $n - |\text{CYC}(\pi)|$.

Proposition 4.22 (Elliptic dimension). *Let $w \in \tilde{\mathfrak{S}}_n$ be an affine permutation, with normal form $w = t_\lambda u_\pi$ (so that u_π is the elliptic part of w). Then $e(w) = n - |\text{CYC}(\pi)|$.*

Proof. By Remark 1.36, $e(w) = \dim(\text{MOV}(u_\pi))$. By Proposition 4.14, this is equal to $\dim(M_{P(\pi)})$. As noted in Remark 4.13, this is equal to $n - |P(\pi)| = n - |\text{CYC}(\pi)|$, as claimed. \square

For the differential dimension we need a relative notion of nullity.

Definition 4.23 (Relative nullity). Let $w = t_\lambda u_\pi$ be an affine permutation in $\tilde{\mathfrak{S}}_n$ with $\pi \in \mathfrak{S}_n$ and λ in the \mathbb{Z} -span of Φ_n and let $k = |P(\pi)| = \text{CYC}(\pi)$ be the number of cycles of the permutation π (Definition 4.1). We call the nullity of the point $L_{P(\pi)}(\lambda)$ in the arrangement $\text{ARR}(\Phi_k)$ the *relative nullity of λ modulo π* and we denote this number by $\nu(\lambda/\pi)$. In combinatorial language, the relative nullity is the maximum size of a null partition P of $\lambda = w(0)$ such that every cycle of $\pi = p(w)$ is contained in a single part of P .

Proposition 4.24 (Differential dimension). *An affine permutation $w \in \tilde{\mathfrak{S}}_n$ with normal form $w = t_\lambda u_\pi$ has $d(w) = |\text{CYC}(\pi)| - \nu(\lambda/\pi)$.*

Proof. Let $U = \text{MOV}(u_\pi)$ be the move-set of the elliptic part u_π of w , and let $k = |P(\pi)| = |\text{CYC}\pi|$. By Remark 1.37, the differential dimension $d(w)$ is equal to the dimension of the point λ/U in the subspace arrangement $\text{ARR}(\Phi_n/U)$. By Proposition 4.14, the move-set U is equal to the M -space $M_{P(\pi)}$, and by Proposition 4.18, quotienting by the M -space $M_{P(\pi)}$ is achieved by applying the L -map $L_{P(\pi)}$. In particular, the point λ/U is the point $L_{P(\pi)}(\lambda)$ in \mathbb{R}^k .

Let \bar{i} denote the block of $P(\pi)$ containing i . The image of the root $e_i - e_j$ in Φ_n under $L_{P(\pi)}$ is $e_{\bar{i}} - e_{\bar{j}}$, which is either equal to 0 (if i and j belong to the same cycle of π) or is a root in Φ_k . Moreover, for each root in Φ_k , we can find a preimage in Φ_n by picking representatives. It follows that the quotient arrangement $\text{ARR}(\Phi_n/U)$ is equal to the root arrangement $\text{ARR}(\Phi_k)$. Thus $d(w)$ is the dimension of the point $L_{P(\pi)}(\lambda)$ in the arrangement $\text{ARR}(\Phi_k)$. By Proposition 4.21 and Definition 4.23, we have

$$\nu(\lambda/\pi) + \dim_{\Phi_k}(\lambda/U) = k.$$

As $k = |\text{CYC}(\pi)|$, we can rewrite this as

$$d(w) = |\text{CYC}(\pi)| - \nu(\lambda/\pi),$$

as claimed. \square

Combining these propositions gives us a combinatorial formula for the reflection length of an element in an affine symmetric group.

Theorem 4.25 (Formula). *An affine permutation $w \in \tilde{\mathfrak{S}}_n$ with normal form $w = t_\lambda u_\pi$ has $\ell_R(w) = n - 2 \cdot \nu(\lambda/\pi) + |\text{CYC}(\pi)|$.*

Proof. By Theorem A and Propositions 4.22 and 4.24,

$$\ell_R(w) = 2 \cdot (|\text{CYC}(\pi)| - \nu(\lambda/\pi)) + (n - |\text{CYC}(\pi)|),$$

which simplifies to the expression in the statement of the theorem. \square

5. FUTURE WORK

5.1. Non-euclidean Coxeter groups. Is there a natural extension of Theorem A to other infinite Coxeter groups? Because reflection length is unbounded for all irreducible non-euclidean Coxeter groups of infinite type [Dus12], there cannot be a simple formula with a bounded number of summands representing bounded dimensions. Thus any extension must necessarily have a different flavor.

5.2. Identifying reduced factorizations. It is easy to characterize when reflection is minimal in spherical Coxeter groups: the product $r_1 \cdots r_k$ of reflections in a spherical Coxeter group has reflection length k if and only if the associated roots are linearly independent. Is there a similar criterion for affine Coxeter groups? Because of the delicate behavior of some of our examples in the affine symmetric group, the criterion is likely to be complicated.

APPENDIX A. NULL PARTITION COMPUTATIONS

Theorem 4.25 reduces the computation of reflection length in the affine symmetric group (which involves minimizing something over an infinite search space) to the finite problem of computing two combinatorial quantities: the number of cycles

$|\text{CYC}(\pi)|$ and the relative nullity $\nu(\lambda/\pi)$. Counting the number of cycles in a permutation is an old problem. In the rest of this section, we focus on the computation of the nullity of a vector.

A first observation is that this task is computationally intractable in general: the NP-complete problem **SubsetSum** asks, “given a (multi)set of integers, does it have a subset with sum 0?” Given an instance S of **SubsetSum**, we may reduce it to a problem of relative nullity, as follows: create the vector λ whose entries consist of the entries of S in some order (with correct multiplicities) followed by the entry $-\sum_{s \in S} s$ and compute the nullity $\nu(\lambda/1)$. We have that the largest partition size is 1 if and only if S has no subset summing to 0, answering a **SubsetSum** instance.

Nevertheless, it is possible in practice to compute nullity by hand in examples of reasonable size by first identifying the null blocks, then the minimal null blocks, then maximal null partitions, and finally the nullity. Throughout this section, $v \in L(\Phi_n)$ is always an integer vector with coordinate sum 0, and we illustrate our procedures using the specific vector $v_0 = (-3, -2, -2, -1, 1, 2, 5) \in \mathbb{R}^7$. To compute the null blocks of a vector we divide its coordinates according to their sign.

Definition A.1 (Positive and negative weights). Let $X = \{i \mid v_i > 0\}$, $Y = \{i \mid v_i < 0\}$, and $Z = \{i \mid v_i = 0\}$. Of course $X \cup Y \cup Z = [n]$. Recall that for each $B \subset [n]$, $v_B = \langle v, \mathbf{1}_B \rangle$ is the sum of the coordinates indexed by the numbers in B and that saying B is a null block means $v_B = 0$. For each subset $B \subset [n]$, let $B_+ = B \cap X$, $B_- = B \cap Y$, and $B_0 = B \cap Z$. We call v_{B_+} the *positive weight* of B and v_{B_-} the *negative weight* of B and note that B is a null block if and only if $v_{B_+} + v_{B_-} = 0$.

To find the null partitions of maximal size, it suffices to restrict our attention to those constructed out of null blocks that are minimal under inclusion.

Lemma A.2 (Maximal null partitions). *Let P be a null partition of $v \in V$. If some block $B \in P$ is not minimal among the null blocks of v , then P is not maximal among the null partitions of v . In particular, maximal null partitions are constructed out of minimal null blocks.*

Proof. Let $B' \subset B$ be a proper nonempty subset of B where both are null blocks of v . Since the sum of the coordinates indexed by B and by B' add to 0, this means that the coordinates in $B'' = B \setminus B'$ also add to 0. Thus B'' is a null block of v , and the refined partition that replaces the block B with the pair of blocks B' and B'' is a new null partition of v of strictly larger size. \square

We now describe a handy way to identify minimal null blocks.

Definition A.3 (Profiles). We create two lists, $\text{POS}(v)$ and $\text{NEG}(v)$. Each list has length equal to $v_X = -v_Y$, and each entry is initially the empty set. For each nonempty subset $B_+ \subset X$ we add the set B_+ to the collection of sets in the i -th position of the first list where $i = v_{B_+}$ is the (positive) weight of B_+ . The result is $\text{POS}(v)$, the *positive profile* of v . Similarly, for each nonempty subset of $B_- \subset Y$ we add the set B_- to the collection of sets in the i -th position in the second list, where $i = -v_{B_-}$ is the absolute value of the negative weight of B_- . The result is $\text{NEG}(v)$, the *negative profile* of v . Let $\text{POS}(v, i)$ be the i -th entry in the list $\text{POS}(v)$ and let $\text{NEG}(v, i)$ be the i -th entry in the list $\text{NEG}(v)$.

Remark A.4 (Null blocks). If B is a null block of positive weight i , then B_+ is in $\text{POS}(v, i)$, B_- is in $\text{NEG}(v, i)$, and $B_0 \subset Z$. Conversely, for every choice of

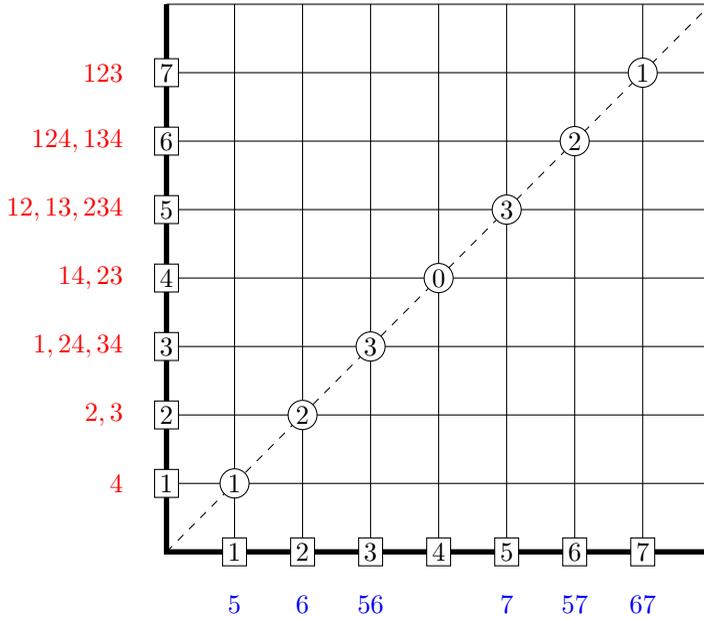


FIGURE 9. A graphical representation of the positive and negative profiles and the basic null blocks of v_0 .

$B_+ \in \text{POS}(v, i)$, $B_- \in \text{NEG}(v, i)$, and $B_0 \subset Z$, the set $B = B_+ \cup B_- \cup B_0$ is a null block of positive weight i . Since we are ultimately interested in minimal null blocks, we ignore the subset of Z and call a null block with $B \cap Z = \emptyset$ a *basic null block*. To create a list of all basic null blocks for v we simply take the “dot product” of the positive and negative profiles of v . Concretely $\text{BASIC}(v, i) = \{B_+ \cup B_- \mid B_+ \in \text{POS}(v, i), B_- \in \text{NEG}(v, i)\}$ is the collection of all *basic null blocks of positive weight i* , and if k is the positive weight of $[n]$, then $\text{BASIC}(v) = (\text{BASIC}(v, 1), \dots, \text{BASIC}(v, k))$ is a list of all basic null blocks, filtered by their positive weight.

Example A.5 (Profiles). For the vector $v_0 = (-3, -2, -2, -1, 1, 2, 5)$, $X = \{5, 6, 7\}$, $Y = \{1, 2, 3, 4\}$, $Z = \{\}$, and its positive weight is 8. The positive profile $\text{POS}(v_0)$, in simplified notation, is shown beneath the x -axis in Figure 9. Each proper nonempty subset of X is shown without internal commas and surrounding braces and placed beneath the x -axis at a location indicating its (positive) weight. The full set X is included as a set in $\text{POS}(v_0, 8)$, but it is not displayed. Similarly, the simplified negative profile of v_0 is shown to the left of the y -axis. The number on the diagonal at location (i, i) is the product $|\text{POS}(v_0, i)| \cdot |\text{NEG}(v_0, i)| = |\text{BASIC}(v_0, i)|$, the number of basic null blocks of positive weight i . The evident symmetry is not accidental since the complement of a proper null block is a proper null block. From this data we find that v_0 has twelve proper null blocks: $\{4, 5\}$, $\{2, 6\}$, $\{3, 6\}$, $\{1, 5, 6\}$, $\{2, 4, 5, 6\}$, $\{3, 4, 5, 6\}$, $\{1, 2, 7\}$, $\{1, 3, 7\}$, $\{2, 3, 4, 7\}$, $\{1, 2, 4, 5, 7\}$, $\{1, 3, 4, 5, 7\}$, and $\{1, 2, 3, 6, 7\}$.

Remark A.6 (Finding minimal null blocks). If B is a basic null block of v that is not minimal, then it must contain a basic null block of strictly smaller positive weight.

$$\begin{aligned}
 & | (45)(26, 36)(156, 2456, 3456)(127, 137, 2347)(12457, 13457)(12367) \\
 & \rightsquigarrow (45) | (26, 36)(156)(127, 137, 2347)(12367) \\
 & \rightsquigarrow (45)(26, 36) | (156)(127, 137, 2347) \\
 & \rightsquigarrow (45)(26, 36)(156) | (127, 137, 2347) \\
 & \rightsquigarrow (45)(26, 36)(156)(127, 137, 2347) |
 \end{aligned}$$

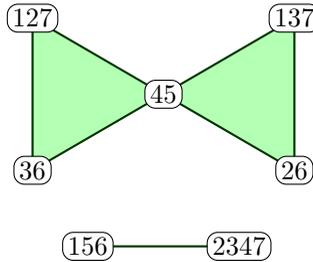
FIGURE 10. Minimal null block algorithm applied to v_0 .

This leads to an algorithm that quickly identifies the minimal null blocks from the filtered list $\text{BASIC}(v)$. Let L be a copy of $\text{BASIC}(v)$ and we proceed to modify L working from left to right. The basic null blocks, if any, in the first entry of L are necessarily minimal null blocks, so we count them as confirmed and remove from L any basic null blocks (to the right) that contain one of these as a subset. At this point, the basic null blocks remaining in the second entry are necessarily minimal, so we count them as confirmed and remove basic null blocks that contain one of them as a subset. And so on. In the end the only blocks that remain are minimal null blocks.

Example A.7 (Finding minimal null blocks). Figure 10 shows the progress of the minimum null block algorithm described in Remark A.6 when applied to v_0 , presented in a simplified notation. As in Example A.5 blocks are listed without commas and braces; commas separate distinct blocks. Parentheses are used to indicate blocks with the same positive weight, and these sets are listed from left to right according to their common positive weight. The vertical bar separates the confirmed minimal null blocks to the left from the unprocessed basic null blocks to the right. At the first step the null block $\{4, 5\}$ is seen to be minimal, and we remove $\{2, 4, 5, 6\}$, $\{3, 4, 5, 6\}$, $\{1, 2, 4, 5, 7\}$, and $\{1, 3, 4, 5, 7\}$ from the list. In the second step both $\{2, 6\}$ and $\{3, 6\}$ are confirmed as minimal, and $\{1, 2, 3, 6, 7\}$ is removed from the list. The third step confirms $\{1, 5, 6\}$ is minimal, and the fourth step confirms $\{1, 2, 7\}$, $\{1, 3, 7\}$, and $\{2, 3, 4, 7\}$ are minimal. Thus v_0 has exactly 7 minimal null blocks.

Once the minimal null blocks have been identified, it remains to identify the maximal null partitions, and one way to do this is to build a flag simplicial complex.

Definition A.8 (Null partition complex). Let Γ be a graph with vertices indexed by the minimal null blocks of v and with an edge connecting two vertices if and only if their minimal null blocks are disjoint. Every graph can be turned into a flag simplicial complex by adding a simplex spanning a subset of vertices if and only if the 1-skeleton of that simplex is already visible in the graph. In other words, there is a simplex in the complex for every complete subgraph of the graph. Let $\text{NULLCPLX}(v)$ be the flag simplicial complex built from Γ . We call $\text{NULLCPLX}(v)$ the *null partition complex of v* . Since complements of null blocks are null blocks and every null block is a union of pairwise disjoint minimal null blocks, a null partition of v is maximal if and only if it corresponds to a maximal simplex in $\text{NULLCPLX}(v)$. This direct correspondence means that the nullity of a vector is one more than the dimension of its null partition complex. In symbols, $\nu(v) = \dim(\text{NULLCPLX}(v)) + 1$.

FIGURE 11. Null partition complex for v_0 .

Example A.9 (Null partition complex). The null partition complex of v_0 is shown in Figure 11. It has 7 vertices, 7 edges, and 2 triangles. There are three maximal simplices: two triangles and an edge, and these correspond to maximal null partitions $\{\{1, 2, 7\}, \{3, 6\}, \{4, 5\}\}$, $\{\{1, 3, 7\}, \{2, 6\}, \{4, 5\}\}$, and $\{\{1, 5, 6\}, \{2, 3, 4, 7\}\}$, respectively. The dimension of $\text{NULLCPLX}(v_0)$ is 2 and its nullity is 3.

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