ARTIN GROUPS OF EUCLIDEAN TYPE

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Abstract. This article resolves several long-standing conjectures about Artin groups of euclidean type. We prove, in particular, that every irreducible euclidean Artin group is a torsion-free centerless group with a decidable word problem and a finite-dimensional classifying space. We do this by showing that it is isomorphic to a subgroup of a Garside group in the expanded sense of Digne. The Garside groups involved are introduced here for the first time. They are constructed by applying semi-standard procedures to crystallographic groups that contain euclidean Coxeter groups but which need not be generated by the reflections they contain.

Arbitrary Coxeter groups are groups defined by a particularly simple type of presentation, but the central motivating examples that lead to the general theory are the groups generated by reflections that act properly discontinuously and cocompactly by isometries on spheres and euclidean spaces. Presentations for these spherical and euclidean Coxeter groups are encoded in the well-known Dynkin diagrams and extended Dynkin diagrams, respectively. Arbitrary Artin groups are groups defined by a modified version of these simple presentations, a definition designed to describe the fundamental group of a space constructed from the complement of the hyperplanes in a complexified version of the reflection arrangement for the corresponding Coxeter group. The spherical Artin groups, i.e. the Artin groups corresponding to the spherical Coxeter groups, have been well understood ever since Artin groups themselves were introduced in 1972 by Deligne [11] and by Brieskorn and Saito [6]. Given the centrality of euclidean Coxeter groups in Coxeter theory and Lie theory more generally, it has been somewhat surprising that the structure of most euclidean Artin groups has remained mysterious for the past forty years. In this article we clarify the structure of all euclidean Artin groups by showing that they are isomorphic to subgroups of a new class of Garside groups that we believe to be of independent interest. More specifically we prove three main results. The first establishes the existence of a new class

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$$\begin{array}{ccccc} \textsc{Art}(\widetilde{X}_n) & = & \textsc{Art}^*(\widetilde{X}_n,w) & \hookrightarrow & \textsc{Gar}(\widetilde{X}_n,w) \\ \downarrow & & \downarrow & & \downarrow \\ \textsc{Cox}(\widetilde{X}_n) & = & \textsc{Cox}^*(\widetilde{X}_n,w) & \hookrightarrow & \textsc{Cryst}(\widetilde{X}_n,w) \end{array}$$

Figure 1. For each Coxeter element w in an irreducible euclidean Coxeter group of type $\widetilde{X}_n$ we define several related groups.

of Garside groups based on intervals in crystallographic groups closely related to the irreducible euclidean Coxeter groups.

Theorem A (Crystallographic Garside groups). *Let $W = \textsc{Cox}(\widetilde{X}_n)$ be an irreducible euclidean Coxeter group and let R be its set of reflections. For each Coxeter element $w \in W$ there exists a crystallographic group $\textsc{Cryst}(\widetilde{X}_n, w)$ containing W with a generating set $T \supset R$ so that the weighted factorizations of w over T form a balanced lattice. As a consequence, this collection of factorizations define a group $\textsc{Gar}(\widetilde{X}_n, w)$ with a Garside structure in the expanded sense of Digne.*

The second shows that irreducible euclidean Artin groups are isomorphic to subgroups of these crystallographic Garside groups.

Theorem B (Artin groups as subgroups). *For each irreducible euclidean Coxeter group $W = \textsc{Cox}(\widetilde{X}_n)$ and for each choice of Coxeter element $w \in W$, the Artin group $\textsc{Art}(\widetilde{X}_n)$ is isomorphic to a subgroup of the Garside group $\textsc{Gar}(\widetilde{X}_n, w)$.*

And the third uses the Garside structure of the crystallographic Garside supergroup to derive consequences for the euclidean Artin group.

Theorem C (Euclidean Artin groups). *Every irreducible euclidean Artin group $\textsc{Art}(\widetilde{X}_n)$ is a torsion-free centerless group with a solvable word problem and a finite-dimensional classifying space.*

The relations among these groups are shown in Figure 1. The notations in the middle column refer to the Coxeter group and the Artin group as defined by their dual presentations and it is these presentations that facilitate the connection between the Coxeter group $\textsc{Cox}(\widetilde{X}_n)$ and the crystallographic group $\textsc{Cryst}(\widetilde{X}_n, w)$ and between the Artin group $\textsc{Art}(\widetilde{X}_n)$ and the crystallographic Garside group $\textsc{Gar}(\widetilde{X}_n, w)$.

The article is structured as follows. We begin by reviewing the necessary background material from [3] and [16] followed by some remarks on the structure of a group we call the annular symmetric group. Next, we define ten different groups associated each irreducible euclidean type