

# THE STRUCTURE OF EUCLIDEAN ARTIN GROUPS

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ABSTRACT. The Coxeter groups that act geometrically on euclidean space have long been classified and presentations for the irreducible ones are encoded in the well-known extended Dynkin diagrams. The corresponding Artin groups are called euclidean Artin groups and, despite what one might naively expect, most of them have remained fundamentally mysterious for more than forty years. Recently, my coauthors and I have resolved several long-standing conjectures about these groups, proving for the first time that every irreducible euclidean Artin group is a torsion-free centerless group with a decidable word problem and a finite-dimensional classifying space. This article surveys our results and the techniques we use to prove them.

The reflection groups that act geometrically on spheres and euclidean spaces are all described by presentations of an exceptionally simple form and general Coxeter groups are defined by analogy. These spherical and euclidean Coxeter groups have long been classified and their presentations are encoded in the well-known Dynkin diagrams and extended Dynkin diagrams, respectively. Artin groups are defined by modified versions of these Coxeter presentations, and they were initially introduced to describe the fundamental group of a space constructed from the complement of the hyperplanes in a complexified version of the reflection arrangement for the corresponding spherical or euclidean Coxeter group. The most basic example of a Coxeter group is the symmetric group and the corresponding Artin group is the braid group, the fundamental group of a quotient of the complement of a complex hyperplane arrangement called the braid arrangement.

The spherical Artin groups, that is the Artin groups corresponding to the Coxeter groups acting geometrically on spheres, have been well understood ever since Artin groups themselves were introduced by Pierre Deligne [Del72] and by Brieskorn and Saito [BS72] in adjacent articles in the *Inventiones* in 1972. One might have expected the euclidean Artin groups to be the next class of Artin groups whose

structure was well-understood, but this was not to be. Despite the centrality of euclidean Coxeter groups in Coxeter theory and Lie theory more generally, euclidean Artin groups have remained fundamentally mysterious, with a few minor exceptions, for the past forty years.

In this survey, I describe recent significant progress in the study of these groups. In particular, my coauthors and I have succeeded in clarifying the structure of all euclidean Artin groups. We do this by showing that each of these groups is isomorphic to a subgroup of a new class of Garside groups that we believe to be of independent interest. The results discussed are contained in the following papers: “Factoring euclidean isometries” with Noel Brady [BM], “Dual euclidean Artin groups and the failure of the lattice property” [McC], and “Artin groups of euclidean type” with Robert Sulway [MS]. The first two are foundational in nature; the third establishes the main results. The structure of this survey follows that of the talks I gave in Durham. The first part corresponds to my first talk and the second part corresponds to my second talk.

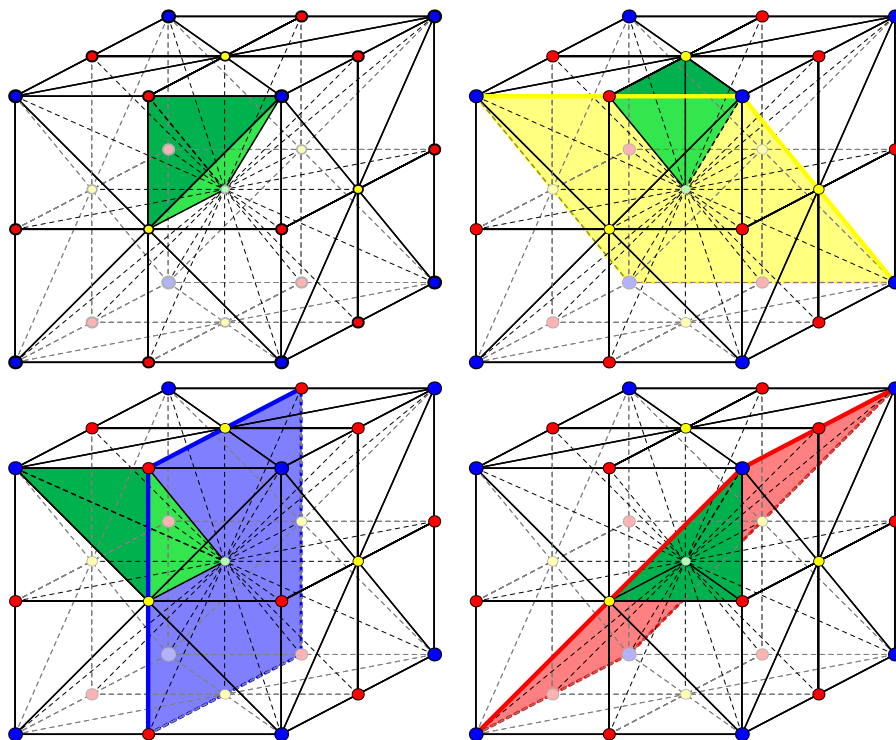
## Part 1. Factoring euclidean isometries

I begin with a brief sketch of some elementary facts and known results about Coxeter groups and Artin groups in order to establish a context for our results. The discussion then shifts to a seemingly unrelated topic: the structure of the poset of all minimum length reflection factorizations of an arbitrary euclidean isometry. The connection between these two disparate topics is rather indirect and its description is postponed until the second part of the article.

### 1. COXETER GROUPS

Recall that a group is said to act *geometrically* when it acts properly discontinuously and cocompactly by isometries, and an action on euclidean space is *irreducible* if there does not exist a nontrivial orthogonal decomposition of the underlying space so that the group is a product of subgroups acting on these subspaces.

**Definition 1.1** (Spherical Coxeter groups). The irreducible *spherical Coxeter groups* are those groups generated by reflections that act geometrically and irreducibly on a sphere in some euclidean space fixing its center. The classification of such groups is classical and their presentations are encoded in the well-known Dynkin diagrams. The *type* of a Dynkin diagram is its name in the Cartan-Killing classification and it is *crystallographic* or *non-crystallographic* depending on whether or not it

FIGURE 1. The spherical Coxeter group  $\text{COX}(B_3)$ .

extends to a euclidean Coxeter group. The crystallographic types consist of three infinite families ( $A_n$ ,  $B_n = C_n$ , and  $D_n$ ) and five sporadic examples ( $G_2$ ,  $F_4$ ,  $E_6$ ,  $E_7$ , and  $E_8$ ). The non-crystallographic types are  $H_3$ ,  $H_4$  and  $I_2(m)$  for  $m \neq 3, 4, 6$ . The subscript is the dimension of the euclidean space containing the sphere on which it acts.

**Example 1.2** (Simplices and cubes). The spherical Coxeter groups of types  $A$  and  $B$  are the best known and represent the symmetry groups of regular simplices and high-dimensional cubes, respectively. As groups they are the *symmetric groups* and extensions of symmetric groups by elementary 2-groups called a *signed symmetric groups*. For example, the group  $\text{COX}(A_3) \cong \text{SYM}_4$  is the symmetric group of a regular tetrahedron and the group  $\text{COX}(B_3) \cong (\mathbb{Z}_2)^3 \rtimes \text{SYM}_3$  and is the group of symmetries of the 3-cube shown in Figure 1.

**Definition 1.3** (Euclidean Coxeter groups). The irreducible *euclidean Coxeter groups* are the groups generated by reflections that act geometrically and irreducibly on euclidean space. The classification of such groups is also classical and their presentations are encoded in the

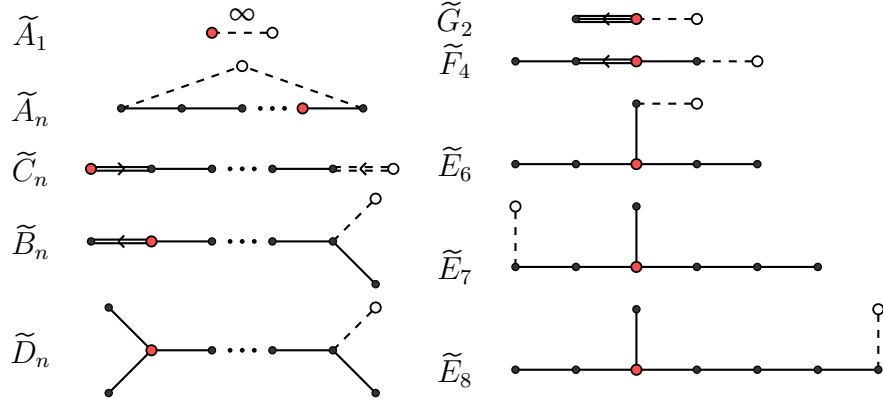


FIGURE 2. Four infinite families and five sporadic examples.

extended Dynkin diagrams shown in Figure 2. There are four infinite families ( $\tilde{A}_n$ ,  $\tilde{B}_n$ ,  $\tilde{C}_n$  and  $\tilde{D}_n$ ) and five sporadic examples ( $\tilde{G}_2$ ,  $\tilde{F}_4$ ,  $\tilde{E}_6$ ,  $\tilde{E}_7$ , and  $\tilde{E}_8$ ). The subscript is the dimension of the euclidean space on which it acts. Removing the white dot and the attached dashed edge or edges from the extended Dynkin diagram  $\tilde{X}_n$  produces the corresponding Dynkin diagram  $X_n$ .

These extended Dynkin diagrams index many different objects including the Artin groups that are our primary focus, but in the present context, it is more relevant that they index euclidean simplices with restricted dihedral angles.

**Definition 1.4** (Euclidean Coxeter simplices). Every extended Dynkin diagram encodes a simplex in euclidean space, unique up to rescaling, with the following properties: the vertices of the diagram are in bijection with the facets of the simplex, i.e. its codimension one faces, and vertices  $s$  and  $t$  in the diagram are connected with 0, 1, 2, or 3 edges iff the corresponding facets intersect with a dihedral angle of  $\frac{\pi}{2}$ ,  $\frac{\pi}{3}$ ,  $\frac{\pi}{4}$ , or  $\frac{\pi}{6}$ , respectively. These conventions are sufficient to describe the simplices associated to each diagram with one exception: the diagram  $\tilde{A}_1$  corresponds to a 1-simplex in  $\mathbb{R}^1$  whose facets are its endpoints. These do not intersect and this is indicated by the infinity label on its unique edge. The extended Dynkin diagrams form a complete list of those euclidean simplices where every dihedral angle is of the form  $\frac{\pi}{m}$  for some integer  $m > 1$ . We call these *euclidean Coxeter simplices*.

From these euclidean Coxeter simplices we can recover the corresponding euclidean Coxeter groups and an associated euclidean tiling.

**Definition 1.5** (Euclidean tilings). Let  $\tilde{X}_n$  be an extended Dynkin diagram and let  $\sigma$  be the corresponding euclidean  $n$ -simplex described above. The group generated by the collection of  $n + 1$  reflections which fix some facet of  $\sigma$  is the corresponding euclidean Coxeter group  $W = \text{COX}(\tilde{X}_n)$  and the images of  $\sigma$  under the action of  $W$  group tile euclidean  $n$ -space. As an illustration, consider the extended Dynkin diagram  $\tilde{G}_2$ . It represents a euclidean triangle with dihedral angles  $\frac{\pi}{3}$ ,  $\frac{\pi}{6}$  and  $\frac{\pi}{2}$  and the euclidean Coxeter group  $\text{COX}(\tilde{G}_2)$  generated by the reflections in its sides is associated with the tiling of  $\mathbb{R}^2$  by congruent 30-60-90 triangles shown in Figure 10.

**Remark 1.6** (Spherical analogues). For an ordinary Dynkin diagram of type  $X_n$ , one constructs spherical simplex  $\sigma$  with similarly restricted dihedral angles and recovers the spherical Coxeter group  $\text{COX}(X_n)$  as the group generated by the reflections in the facets of  $\sigma$ . The images of  $\sigma$  under this action yield a spherical tiling. This is illustrated in Figure 1 if one intersects the cell structure shown with a small sphere around the center of the cube. The cube in the upper left shades a tetrahedron which intersects with the small sphere to produce a spherical triangle with dihedral angles  $\frac{\pi}{3}$ ,  $\frac{\pi}{4}$  and  $\frac{\pi}{2}$ . The other three cubes illustrate its image under the action of the three reflections in its sides.

And finally a short remark about how the spherical and euclidean cases relate to the general theory.

**Remark 1.7** (General Coxeter groups). The general theory of Coxeter groups was pioneered by Jacques Tits in the early 1960s and the spherical and euclidean Coxeter groups are key examples that motivate their introduction. Coxeter groups are defined by simple presentations and in that first unpublished paper, Tits proved that every Coxeter group has a faithful linear representation preserving a symmetric bilinear form and thus has a solvable word problem. Irreducible Coxeter groups can be coarsely classified by the signature of the symmetric bilinear forms they preserve and the irreducible spherical and euclidean groups are those that preserve positive definite and positive semi-definite forms, respectively.

## 2. ARTIN GROUPS

As mentioned in the introduction, Artin groups first appear in print in 1972 in a pair of articles by Pierre Deligne [Del72] and by Brieskorn and Saito [BS72]. Both articles focus on spherical Artin groups as fundamental groups of spaces constructed from complements of complex hyperplane arrangements and successfully analyze their structure using

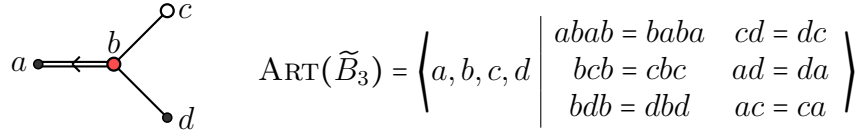


FIGURE 3. The  $\tilde{B}_3$  diagram and the presentation for the corresponding euclidean Artin group.

different techniques. The resulting presentations resemble Artin’s standard presentation for the braid groups, which is, of course, the most prominent example of a spherical Artin group. In the spherical and euclidean context these presentations are extremely easy to describe.

**Definition 2.1** (Euclidean Artin groups). Let  $\tilde{X}_n$  be an extended Dynkin diagram. The standard presentation for the Artin group of type  $\tilde{X}_n$  has a generator for each vertex and at most one relation for each pair of vertices. More precisely, if  $s$  and  $t$  are vertices connected by 0, 1, 2, or 3 edges, then the presentation contains the relation  $st = ts$ ,  $sts = tst$ ,  $stst = tsts$  or  $ststst = tststs$  respectively. And finally, in the case of  $\tilde{A}_1$ , the edge labeled  $\infty$  indicates that there is no relation corresponding to this pair of vertices. As an illustration, Figure 3 shows the extended Dynkin diagram of type  $\tilde{B}_3$  along with the explicit presentation for the corresponding euclidean Artin group  $\text{ART}(\tilde{B}_3)$ .

General Artin groups are defined by similarly simple presentations encoded in the same diagrams as general Coxeter groups and then coarsely classified in the same way. Given the centrality of euclidean Coxeter groups and the elegance of their structure, one might have expected euclidean Artin groups to be well understood shortly thereafter. It is now 40 years later and these groups are still revealing their secrets.

**Definition 2.2** (Four conjectures). In a recent survey article, Eddy Godelle and Luis Paris highlight how little we know about general Artin groups by highlighting four basic conjectures that remain open [GP]. Their four conjectures are:

- (A) All Artin groups are torsion-free.
- (B) Every non-spherical irreducible Artin group has a trivial center.
- (C) Every Artin group has a solvable word problem.
- (D) All Artin groups satisfy the  $K(\pi, 1)$  conjecture.

Godelle and Paris also remark that these conjectures remain open and are a “challenging question” even in the case of the euclidean Artin groups. These are precisely the conjectures that my collaborators and I set out to resolve.

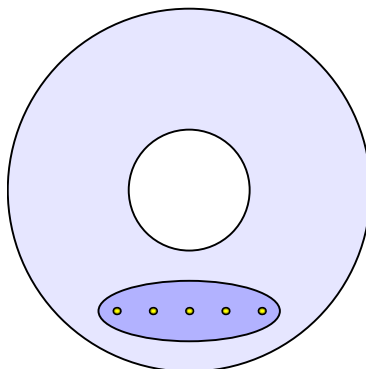


FIGURE 4. Five punctures in a disk inside an annulus.

There are a few euclidean Artin groups with a well-understood structure. The earliest results are by Craig Squier.

**Example 2.3** (planar Artin groups). In a 1987 article, Squier successfully analyzes the structure of the three irreducible euclidean Artin groups  $\text{ART}(\tilde{A}_2)$ ,  $\text{ART}(\tilde{C}_2)$ , and  $\text{ART}(\tilde{G}_2)$  that correspond to the three irreducible euclidean Coxeter groups which act geometrically on the euclidean plane [Squ87]. He works directly with the presentations and analyzes them as amalgamated products and HNN extensions of well-known groups. This technique does not appear to generalize to other euclidean Artin groups.

A second result is the consequence of an unusual embedding of a euclidean Artin group into a spherical Artin group.

**Example 2.4** (Annular braids). It has been repeatedly observed that the euclidean Artin group  $\text{ART}(\tilde{A}_n)$  embeds into the spherical Artin group  $\text{ART}(B_{n+1})$  and is, in fact, part of a short exact sequence

$$\text{ART}(\tilde{A}_n) \hookrightarrow \text{ART}(B_{n+1}) \twoheadrightarrow \mathbb{Z}$$

which greatly clarifies its structure [tD98, All02, KP02, CP03]. The group  $\text{ART}(B_n)$  is sometimes called the *annular braid group* because it can be interpreted as the braid group of the annulus [Bir74]. If one selects a disk in the annulus containing all the punctures, as shown in Figure 4, then the path traced by each puncture, viewed as a path that starts and ends in the disk, has a winding number. The sum of these individual winding numbers is a global winding number for each element of  $\text{ART}(B_{n+1})$ , and this assignment of a global winding number is a group homomorphism onto  $\mathbb{Z}$  with  $\text{ART}(\tilde{A}_n)$  as its kernel.

In other words, the group  $\text{ART}(\tilde{A}_n)$  is the subgroup of annular braids with global winding number 0.

And finally, there are two recent results due to François Digne.

**Example 2.5** (Garside structures). Digne showed that the groups  $\text{ART}(\tilde{A}_n)$  and  $\text{ART}(\tilde{C}_n)$  have infinite-type Garside structures [Dig06, Dig12]. In the first article Digne uses the embedding  $\text{ART}(\tilde{A}_n) \hookrightarrow \text{ART}(B_{n+1})$  to show that the euclidean Artin groups of type  $\tilde{A}_n$  have infinite-type Garside structures and in the second he uses a delicate analysis of the some maps relating type  $C$  and type  $A$  to show that the euclidean Artin groups of type  $\tilde{C}_n$  also has an infinite-type Garside structure. Our approach to arbitrary euclidean Artin groups is closely related to Digne’s work and the second part of the article contains a more detailed description of Garside structures and their uses.

To my knowledge, these euclidean Artin groups, i.e. the ones of type  $\tilde{A}_n$ ,  $\tilde{C}_n$ , and  $\tilde{G}_2$ , are the only ones whose structure was previously fully understood. In fact, one of the main frustrations in the area is the stark contrast between the utter simplicity of the presentations involved and the fact that we typically know very little about the groups they define.

For example, all four conjectures identified by Godelle and Paris were open for the group  $\text{ART}(\tilde{B}_3)$  shown in Figure 3 – including a solution to its word problem – until 2010 when my Ph.D. student Robert Sulway analyzed its structure as part of his dissertation [Sul10]. As an extension of Sulway’s work, he and I are now able to give positive solutions to Conjectures (A), (B) and (C) for all euclidean Artin groups and we also make some progress on Conjecture (D). We prove, in particular, that every irreducible euclidean Artin group  $\text{ART}(\tilde{X}_n)$  is a torsion-free centerless group with a solvable word problem and a finite-dimensional classifying space. Our proofs rely heavily on the structure of intervals in euclidean Coxeter groups and other euclidean groups generated by reflections, and so we now shift our attention to structural aspects of the set of all factorizations of a euclidean isometry into reflections.

### 3. ISOMETRIES

Every euclidean isometry can be built out of reflections and the Cayley graph of the euclidean isometry group with respect to this natural reflection generating set has bounded diameter. This follows from a fact that most mathematicians learn early on in their education: every isometry of  $n$ -dimensional euclidean space is a product of at most  $n + 1$  reflections. The goal of the next few sections is to describe in some detail the structure of the portion of this Cayley graph between the



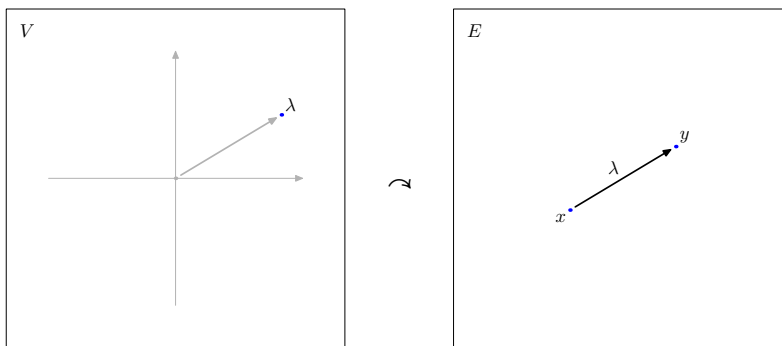


FIGURE 5. Vectors acting on points.

identity and a fixed euclidean isometry. We begin with a coarse classification of euclidean isometries and their basic invariants following the approach taken in [BM]. The first step is elementary but important for conceptual clarity: we make a sharp distinction between points and vectors.

**Definition 3.1** (Points and vectors). Let  $V$  be a vector space with a simple transitive action on a set  $E$  as shown in Figure 5. We call  $E$  an *affine space*, the elements of  $E$  are called *points* and the elements of  $V$  are called *vectors*. The main difference between  $V$  and  $E$  is that  $E$  is essentially a vector space with no distinguished point identified as its origin.

Both  $V$  and  $E$  have a natural collections of subspaces which are used to defines the basic invariants of euclidean isometries.

**Definition 3.2** (Subspaces). A subset of  $V$  is *linear* if it is closed under linear combination. A subset of  $V$  or  $E$  is *affine* if for every pair of elements in the subset, the line through these elements is also in the subset. Thus the vector space  $V$  has *linear subspaces* through the origin and other *affine subspaces* not through the origin. The affine space  $E$  only has affine subspaces. For any affine subspace  $B \subset E$ , vectors between points in  $B$  form a linear subspace  $\text{DIR}(B) \subset V$  called its *space of directions*.

Posets are obtained by ordering these natural subspaces by inclusion.

**Definition 3.3** (Poset structure). The linear subspaces of  $V$  ordered by inclusion define a poset  $\text{LIN}(V)$  which is a graded, bounded, self-dual lattice. The affine subspaces of  $E$  ordered by inclusion define a poset  $\text{AFF}(E)$  which is graded and bounded above, but not bounded below,

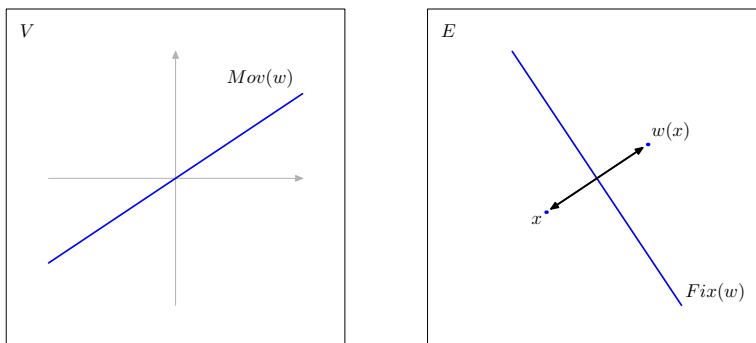


FIGURE 6. Basic invariants of a reflection.

not self-dual and not a lattice. Also note that there is a well-defined rank-preserving map  $\text{AFF}(E) \rightarrow \text{LIN}(V)$  that sends  $B$  to  $\text{DIR}(B)$ .

If one equips  $V$  with a positive definite inner product, this induces a euclidean metric on  $E$  and a corresponding set of euclidean isometries that preserve this metric.

**Definition 3.4** (Basic invariants). Let  $w$  be an isometry of the euclidean space  $E$ . Its *move-set* is the subset  $\text{MOV}(w) \subset V$  of all the motions that its points undergo. In symbols,

$$\text{MOV}(w) = \{w(x) - x \in V \mid x \in E\}$$

and it is easy to show that  $\text{MOV}(w)$  is an affine subspace of  $V$ . As an affine subspace,  $\text{MOV}(w)$  is a translation of a linear subspace. If  $U$  denotes the unique linear subspace of  $V$  which differs from  $\text{MOV}(w)$  by a translation and  $\mu$  is the unique vector in  $\text{MOV}(w)$  closest to the origin, then we call  $U + \mu$  the *standard form* of  $\text{MOV}(w)$ . The points in  $E$  that undergo the motion  $\mu$  are a subset  $\text{MIN}(w) \subset E$  called the *min-set* of  $w$  and it is also easy to show that  $\text{MIN}(w)$  is an affine subspace of  $E$ . We call these the *basic invariants* of  $w$ .

Euclidean isometries naturally divide into two types.

**Definition 3.5** (Elliptic and hyperbolic). Let  $w$  be a euclidean isometry. If its move-set  $\text{MOV}(w)$  includes the origin, then  $\mu$  is trivial, and its min-set  $\text{MIN}(w)$  is also its fix-set  $\text{FIX}(w)$ . Under these equivalent conditions  $w$  is called *elliptic*. Otherwise,  $w$  is called *hyperbolic*.

The simplest euclidean isometries are reflections and translations.

**Definition 3.6** (Translations). For each vector  $\lambda \in V$  there is a *translation* isometry  $t_\lambda$  whose min-set is all of  $E$  and whose move-set is the single point  $\{\lambda\}$ . So long as  $\lambda$  is nontrivial,  $t_\lambda$  is a hyperbolic isometry.

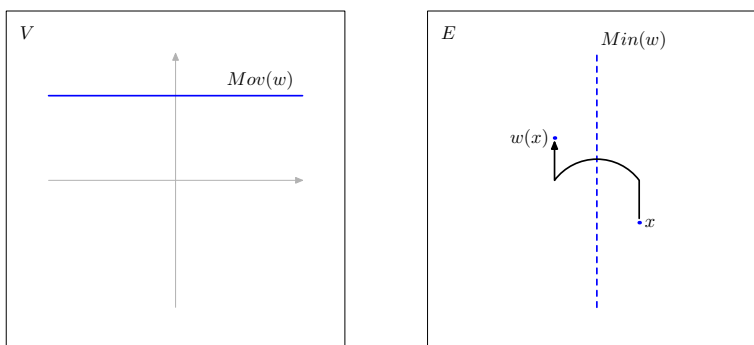


FIGURE 7. Basic invariants of a glide reflection.

**Definition 3.7** (Reflections). For each *hyperplane*  $H$  in  $E$  (an affine subspace of codimension 1) there is a unique nontrivial isometry  $r$  fixing  $H$  called a *reflection*. It is elliptic with fix-set  $H$  and its move-set is a line through the origin in  $V$ . We call any nontrivial vector  $\alpha$  in this line a *root* of  $r$ . The basic invariants of a typical reflection are shown in Figure 6.

A more interesting example which better illustrates these ideas is given by a glide reflection. The move-set of a glide reflection such as the one shown in Figure 7 is a non-linear affine line in  $V$ . It has a unique point  $\mu$  closest to the origin and the points in  $E$  which undergo the motion  $\mu$  are those on its min-set, also known as its glide axis.

#### 4. INTERVALS

Let  $V$  be an  $n$ -dimensional vector space over  $\mathbb{R}$ , let  $E$  be an  $n$ -dimensional euclidean space on which it acts, and let  $L = \text{ISOM}(E)$  be the Lie group of euclidean isometries of  $E$ . The structure I want to describe is the portion of the Cayley graph of  $L$  generated by its reflections between the identity element and a fixed euclidean isometry  $w$ . We begin by recalling that in any metric space there is a notion of “betweenness” which can be used to construct intervals that are posets.

**Definition 4.1** (Intervals in metric spaces). In any metric space a point  $z$  is said to be *between* points  $x$  and  $y$  when the triangle inequality becomes an equality, that is when  $d(x, z) + d(z, y) = d(x, y)$ . The set of all points between  $x$  and  $y$  form the *interval*  $[x, y]$  and this set can be given a partial ordering by defining  $z \leq w$  if and only if  $d(x, z) + d(z, w) + d(w, y) = d(x, y)$ .

As an illustration, consider the unit 2-sphere with standard angle metric. If  $x$  and  $y$  are not antipodal, the only points between  $x$  and

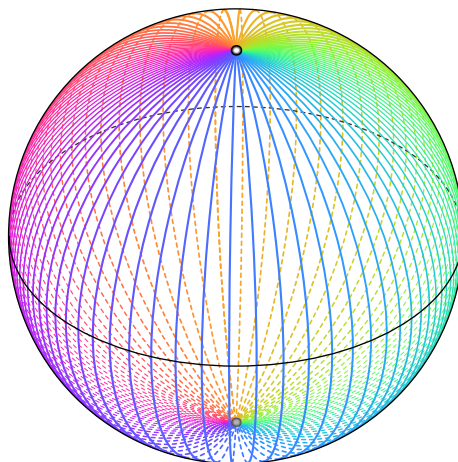


FIGURE 8. The interval between antipodal points on a 2-sphere.

$y$  are those along the unique geodesic connecting them with the usual ordering of an interval in  $\mathbb{R}$ . If, however,  $x$  and  $y$  are antipodal, say  $x$  is the south pole and  $y$  is the north pole, then all points on  $\mathbf{S}^2$  are between  $x$  and  $y$ , the interval  $[x, y]$  is all of  $\mathbf{S}^2$  and its ordering is that  $z < w$  iff they lie on a common longitude line with the latitude of  $z$  below that of  $w$ . See Figure 8. When Cayley graphs are viewed as metric spaces, they can be used to construct intervals.

**Definition 4.2** (Intervals in groups). Let  $G$  be a group with a fixed symmetric generating set  $S$ . If we assign the elements of  $S$  positive weights and the set of all possible weights is a discrete subset of the reals, then  $G$  can be viewed as a metric space where the distance  $d(g, h)$  is calculated as the minimum total length of a path in the Cayley graph from  $v_g$  to  $v_h$ . One convention is to assign every generator a weight of 1 and for finite generating sets the discreteness condition is always true, but for infinite generating sets with varying weights, some condition is needed so that the infimum of distances between two points is achieved by some actual path in the Cayley graph. Let  $[g, h]^G$  denote the portion of the Cayley graph between  $g$  and  $h$ , by which I mean the union of all the minimal length directed paths from  $v_g$  to  $v_h$ . This is an edge-labeled directed graph which also encodes the poset structure.

Since Cayley graphs are homogeneous,  $d(g, h) = d(1, g^{-1}h)$  and the interval  $[g, h]^G$  is isomorphic as an edge-labeled directed graph to the interval  $[1, g^{-1}h]^G$ . Thus it is sufficient to restrict our attention to distances from the identity and intervals of the form  $[1, g]^G$ . Note that

the formulas given above are for right Cayley graphs with their natural left group action; for left Cayley graphs  $[g, h]^G \cong [1, hg^{-1}]^G$ .

**Question 4.3** (Euclidean intervals). In this language our goal is to describe the poset structure of intervals in the Lie group  $L = \text{ISOM}(E)$  of all euclidean isometries generated by its full set of reflections with each reflection given unit weight. Questions one might ask include: What are the possible poset structures for these intervals  $[1, w]^L$ ? To what extent is this poset structure independent of  $w$ ? Are these posets lattices? i.e. do well-defined meets and joins always exist? A good test case for these questions is when  $w$  is a loxodromic ‘‘corkscrew’’ isometry of  $\mathbb{R}^3$ , to which we return at the end of Section 5.

In [BM] Noel Brady and I answer these questions by completely characterizing the poset structure of all euclidean intervals. Our motivation was to create a technical tool that could be used to construct dual presentations of euclidean Artin groups, a process described in Section 7. The first step is to understand how far an isometry  $w$  is from the identity in this Cayley graph, i.e. its *reflection length*  $\ell_R(w)$ , and this is the content of a classical result known as Scherk’s theorem [Sch50].

**Theorem 4.4** (Reflection length). *Let  $w$  be a euclidean isometry with a  $k$ -dimensional move-set. If  $w$  is elliptic, its reflection length is  $k$ . If  $w$  is hyperbolic, its reflection length is  $k + 2$ .*

From there we build up an understanding of how the basic invariants of a euclidean isometry change when it is multiplied by a reflection. The following lemma is one of the results we establish.

**Lemma 4.5.** *Suppose  $w$  is hyperbolic with  $\ell_R(w) = k$  and  $\text{MOV}(w) = U + \mu$  in standard form,  $r$  is a reflection with root  $\alpha$  and let  $U_\alpha$  denote the span of  $U \cup \{\alpha\}$ .*

- *If  $\alpha \in U$  then  $rw$  is hyperbolic with  $\ell_R(rw) = k - 1$ .*
- *If  $\alpha \notin U$  and  $\mu \in U_\alpha$  then  $rw$  is elliptic with  $\ell_R(rw) = k - 1$ .*
- *If  $\alpha \notin U$  and  $\mu \notin U_\alpha$  then  $rw$  is hyperbolic with  $\ell_R(rw) = k + 1$ .*

What I would like the reader to notice is that the geometric relationships between the basic invariants of  $w$  and the basic invariants of  $r$  combine to determine key properties of the basic invariants of  $rw$ . This type of detailed information makes it possible to prove results such as the following:

**Proposition 4.6** (Elliptic intervals). *Let  $w$  be an elliptic isometry with  $\text{MOV}(w) = U \subset V$ . The map  $u \mapsto \text{MOV}(u)$  creates a poset isomorphism  $[1, w]^L \cong \text{LIN}(U)$ . In particular,  $[1, w]^L$  is a lattice.*

Alternatively the map  $u \mapsto \text{FIX}(u)$  gives a poset isomorphism with the affine subspaces containing  $\text{FIX}(w)$  under reverse inclusion. Proofs of this result can be found [Sch50], [BW02b] or [BM]. The most remarkable aspect of this proposition is that the structure of the interval  $[1, w]^L$  only depends on the fact that  $w$  is elliptic and the dimension of its move-set (or equivalently the codimension of its fix-set); it is otherwise independent of  $w$  itself. In other words, the fix-set of  $w$  completely determines the order structure of the interval.

## 5. MODELS

The main new result established in [BM] is an analysis of the structure of euclidean intervals for hyperbolic isometries. To describe these intervals, we first define an abstract poset which mimics the basic invariants of euclidean isometries.

**Definition 5.1** (Global Poset). Let  $E$  be an  $n$ -dimensional euclidean space and let  $V$  be the  $n$ -dimensional vector space that acts on it. We construct a poset  $P$  called the *global poset* with two types of elements: it has an element we call  $h^M$  for each nonlinear affine subspace  $M \subset V$  and an element we call  $e^B$  for each affine subspace  $B \subset E$ . The ordering of these elements is defined as follows:

$$\begin{aligned} h^M &\geq h^{M'} \text{ iff } M \supset M' \\ e^B &\geq e^{B'} \text{ iff } B \subset B' \\ h^M &> e^B \text{ iff } M^\perp \subset \text{DIR}(B) \\ \text{no } e^B &\text{ is ever above } h^M \end{aligned}$$

Next, we define a map from the Lie group  $L = \text{ISOM}(E)$  to the global poset  $P$ .

**Definition 5.2** (Invariant map). For each euclidean isometry  $w$ , the *invariant map* assigns an element of  $P$  based on its type and its basic invariants. More precisely, the invariant map  $\text{INV} : L \rightarrow P$  is defined by setting  $\text{INV}(u) = h^{\text{MOV}(u)}$  when  $u$  is hyperbolic and  $\text{INV}(u) = e^{\text{FIX}(u)}$  when  $u$  is elliptic.

One reason to introduce the poset  $P$  and the map  $\text{INV}$  is that there is a way to use distance from the identity to turn the Lie group  $L$  into a poset and under this ordering the invariant map is a rank-preserving poset map. It is, however, far from injective as can be seen from the fact that all rotations which fix the same subspace are sent to the same element of  $P$ . Because  $\text{INV} : L \rightarrow P$  is a well-defined map between posets, it sends the elements below  $w$  to the elements below  $\text{INV}(w)$ . The former are the intervals  $[1, w]^L$ . The latter are what we called model posets.

**Definition 5.3** (Model posets). For each affine subspace  $B \subset E$ , let  $P^B$  denote the poset of elements below  $e^B$  in the global poset  $P$ . Similarly, for each nonlinear affine subspace  $M \subset V$ , let  $P^M$  denote the poset of elements below  $h^M$  in global poset  $P$ . We call these our *model posets*. Finally, let  $P(w)$  be the model poset of elements below  $\text{INV}(w)$  in  $P$ .

As noted above, the invariant map sends elements in the interval  $[1, w]^L$  to elements in the model poset  $P(w)$ . In fact, one of the main results in [BM] is that these restrictions of the invariant map are poset isomorphisms.

**Theorem 5.4** (Models for euclidean intervals). *For each isometry  $w \in L$ , the map  $u \mapsto \text{INV}(u)$  is a poset isomorphism between the interval  $[1, w]^L$  and the model poset  $P(w)$ .*

When  $w$  is elliptic, this reduces to the previously known Proposition 4.6, but when  $w$  is hyperbolic this result is new. Theorem 5.4 allows attention to shift away from the isometries themselves and to focus instead on these model posets defined purely in terms of the affine subspaces of  $V$  and  $E$ . In particular, we are able to understand the structure of euclidean intervals well enough that we can determine when meets and joins exist.

**Corollary 5.5** (Lattice failure). *Let  $w \in L$  be a euclidean. The interval  $[1, w]^L$  is not a lattice iff  $w$  is a hyperbolic isometry and its move-set has dimension at least 2. All other intervals are lattices.*

In [BM] we give an explicit characterization of where these failures occur. For the application to euclidean Artin groups it is sufficient to describe these failure when  $w$  is a hyperbolic isometry of maximal reflection length. For such a  $w$ , its min-set is a line in  $E$  and its move-set is a nonlinear affine hyperplane (i.e. an affine subspace of codimension 1) in  $V$ . We call the direction of its min-set *vertical* and all of the orthogonal directions *horizontal*. More generally we call any motion with a non-trivial vertical component *vertical*. One consequence of Theorem 5.4 is that there is exactly one elliptic in  $[1, w]^L$  for each affine subspace  $M \subset E$  and exactly one hyperbolic for each affine subspace of  $\text{MOV}(w) \subset V$ . Using the model poset structure as a guide we coarsely partition the elements in the interval  $[1, w]^L$  into a grid with three rows.

**Definition 5.6** (Coarse structure). Let  $w \in L$  be a hyperbolic euclidean isometry of maximal reflection length. For every  $u \in [1, w]^L$  there is a unique  $v$  such that  $uv = w$  and we coarsely partition the elements of  $[1, w]^L$  into 3 rows based on the types of  $u$  and  $v$  and into

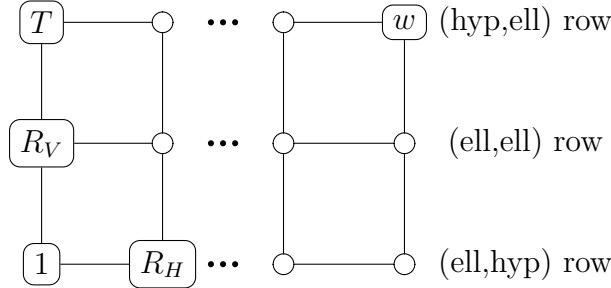


FIGURE 9. Coarse structure for a maximal hyperbolic isometry.

columns based on the dimensions of their basic invariants. See Figure 9. When  $u$  or  $v$  is hyperbolic, it turns out that the other must be an elliptic isometry where every point undergoes a motion that is purely horizontal. In particular, it is an elliptic whose fix-set is invariant under vertical translation, i.e translation in the direction of the line which is the min-set of  $w$ . When both  $u$  and  $v$  are elliptic, it turns out that both motions must have non-trivial vertical components and thus neither of their fix-sets is invariant under vertical translation. Within each row we grade based on the dimensions of the basic invariants. In the bottom row, the dimension of the fix-set of  $u$  decreases and the dimension of the move-set of  $v$  increases as we move from left to right. In the middle row, the dimension of the fix-set of  $u$  decreases and the dimension of the fix-set of  $v$  increases as we move from left to right. And in the top row, the dimension of the move-set of  $u$  increases and the dimension of the fix-set of  $v$  increases as we move from left to right.

The only element in the lower left-hand box is the identity element corresponding to the factorization  $1 \cdot w = w$  and the only element in the upper right-hand box is the element  $w$  corresponding to the factorization  $w \cdot 1 = w$ . All other boxes in this grid contain infinitely many elements. Descending in the poset order involves moving to elements in boxes down and/or to the left and covering relations involve elements in boxes that are adjacent either vertically or horizontally. As a consequence, the box an element is placed in determines its reflection length: its length equals the number of steps its box is from the lower left-hand corner.

**Definition 5.7** (Three special boxes). There are three particular boxes in this grid that merit additional description. The elements placed in the upper left-hand corner are hyperbolic isometries in  $[1, w]^L$  of reflection length 2 which means that they are produced by multiplying a pair of reflections fixing parallel hyperplanes. In other words they



are pure translations  $t_\lambda$ , and by construction of the model poset  $P(w)$ , the pure translations  $T$  which occur in the interval are precisely those where the translation vector  $\lambda$  is a element of the non-linear affine subspace  $\text{MOV}(w) \subset V$ . The elements in the second box in the bottom row have reflection length 1, i.e. they are themselves reflections and since they are in the bottom row, they have fixed hyperplanes invariant under vertical translation. We call this set  $R_H$  the *horizontal reflections* since they move points in a horizontal direction. All such reflections occur in the interval  $[1, w]^L$ . And finally the first box in the middle row contains reflections whose fixed hyperplane is not invariant under vertical translation. We call this set  $R_V$  the *vertical reflections* because the motions they produce contain a nontrivial vertical component. All such reflections also occur in the interval  $[1, w]^L$ .

We conclude our discussion of intervals in the full euclidean isometry group by describing where in the grid meets and joins fail to exist.

**Example 5.8** (Lattice failure). The simplest euclidean isometry whose interval fails to be a lattice is a loxodromic “corkscrew” motion  $w$  in  $\mathbb{R}^3$ . This isometry has reflection length 4 and it has a coarse structure with three rows and three columns. If we consider any pair of hyperbolic isometries from the middle box of the top row whose min-sets are parallel vertically invariant planes and a pair of horizontal reflections in the middle box of the bottom row whose fixed planes are parallel to each other and to the min-sets of the chosen hyperbolic isometries, then it is straight-forward to check that these hyperbolic isometries are distinct minimal upper bounds for this pair of elliptic isometries and these elliptic isometries are distinct minimal lower bounds for these hyperbolic isometries. In [BM] we call this situation a *bowtie*.

## Part 2. Crystallographic Garside groups

In this second part of the article I describe how knowing the structure of intervals in the full euclidean isometry group leads to an understanding of similar intervals inside a euclidean Coxeter group, and how these Coxeter intervals provide the technical foundation at the heart of our successful attempt to understand euclidean Artin groups using infinite-type Garside structures.

### 6. COXETER ELEMENTS

Let  $W = \text{COX}(\tilde{X}_n)$  be an irreducible euclidean Coxeter group acting geometrically on an  $n$ -dimensional euclidean space  $E$ . The Coxeter group  $W$  is discrete subgroup of the Lie group  $L = \text{ISOM}(E)$  and if we

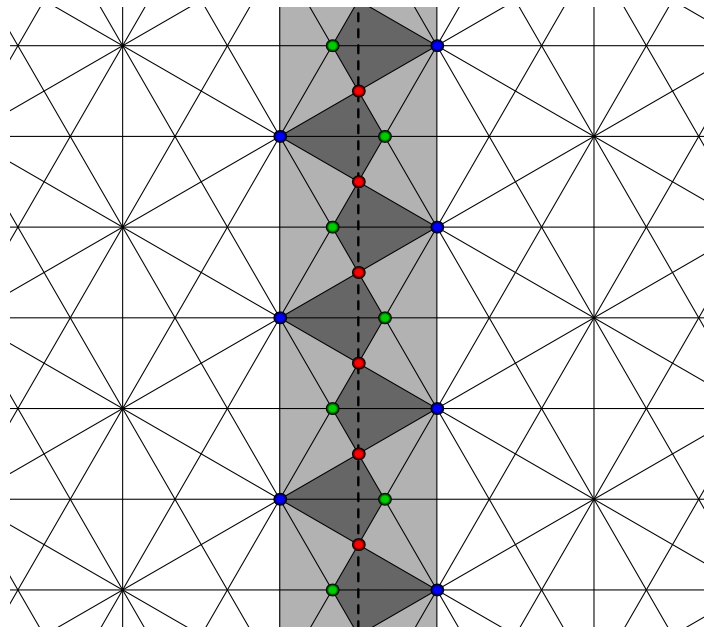


FIGURE 10. The euclidean Coxeter Group  $\text{COX}(\tilde{G}_2)$ .

continue to view  $L$  as a group generated by all reflections and we view  $W$  as the subgroup generated by those reflections which occur in  $W$ , then one might naturally expect there to be a close relationship between the interval  $[1, w]^W$  and the interval  $[1, w]^L$  for each  $w \in W$ . For generic elements the connection is not as close as one might hope. In fact, even the distance to the origin might be different in the two contexts, which makes the sets of minimal length reflection factorizations completely disjoint, as Kyle Petersen and I explored in [MP11]. There is, however, a close connection when  $w$  is a Coxeter element of  $W$ .

**Definition 6.1** (Coxeter elements). Let  $W = \text{COX}(\tilde{X}_n)$  be an irreducible euclidean Coxeter group with Coxeter generating set  $S$ . A *Coxeter element*  $w \in W$  is obtained by multiplying the elements of  $S$  in some order. This produces many different Coxeter elements depending on the order in which these elements are multiplied, but so long as the diagram  $\tilde{X}_n$  is a tree, as it is in all cases except for  $\tilde{A}_n$ , all Coxeter elements in  $W$  belong to the same conjugacy class and act on the corresponding euclidean tiling in the exact same way [McC, Proposition 7.5]. Thus it makes sense to talk about *the* Coxeter element in most irreducible euclidean contexts.

Coxeter elements of irreducible euclidean Coxeter groups are hyperbolic euclidean isometries whose geometric invariants play a large role in our understanding of the structure of the interval  $[1, w]^W$ .

**Definition 6.2** (Axial features). Let  $w$  be a Coxeter element for an irreducible euclidean Coxeter group  $W = \text{COX}(\tilde{X}_n)$ . It is a hyperbolic isometry whose reflection length is  $n + 1$  when measured in either  $W$  or  $L$ . In  $L$  this reflection length is the maximum possible, its min-set  $\text{MIN}(w)$  is a line in  $E$  called its *axis* and its move-set  $\text{MOV}(w)$  is a non-linear affine hyperplane in  $V$ . The top-dimensional simplices whose interior nontrivially intersects the axis are called *axial simplices* and the vertices of these simplices are *axial vertices*.

**Example 6.3** ( $\text{COX}(\tilde{G}_2)$ ). Figure 10 illustrates these ideas for the Coxeter group  $\text{COX}(\tilde{G}_2)$ . Its Coxeter element is a glide reflection whose glide axis, i.e. its min-set, is shown as a dashed line. The corresponding axial simplices are heavily shaded their axial vertices are shown as enlarged dots.

For Coxeter elements, the interval  $[1, w]^W$  is a restriction of edge-labeled subposet  $[1, w]^L$  to the union of minimal length paths in the Cayley graph of  $L$  from  $v_1$  to  $v_w$  where every edge is labeled by a reflection in  $W = \text{COX}(\tilde{X}_n)$ . The original product of elements in the Coxeter generating set  $S$  which produces  $w$  as a Coxeter element is one such minimal length path. As such, the elements of this *Coxeter interval*  $[1, w]^W$  have a coarse structure as described in Definition 5.6. The first difference we find is that whereas every reflection in  $L$  labels some edge in the interval  $[1, w]^L$ , in the Coxeter interval  $[1, w]^W$  only a proper subset of the reflections in  $W$  actually label edges in the interval. For the unused reflections do not occur in a minimal length factorization of  $w$  where every reflection must belong to  $W$ . In [McC, Theorem 9.6] the reflections that do occur as edge labels in the interval are precisely characterized as follows.

**Theorem 6.4** (Reflection generators). *Let  $w$  be a Coxeter element of an irreducible euclidean Coxeter group  $W = \text{COX}(\tilde{X}_n)$ . A reflection labels an edge in the interval  $[1, w]^W$  iff its fixed hyperplane contains an axial vertex.*

If we separate the reflections labeling edges in  $[1, w]^W$  into those which are horizontal and those which are vertical, in the sense defined in the previous section, then there are infinitely many vertical reflections (all those whose hyperplanes cross the Coxeter axis) and a finite number of horizontal reflections (those whose hyperplanes bound the convex

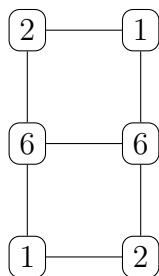


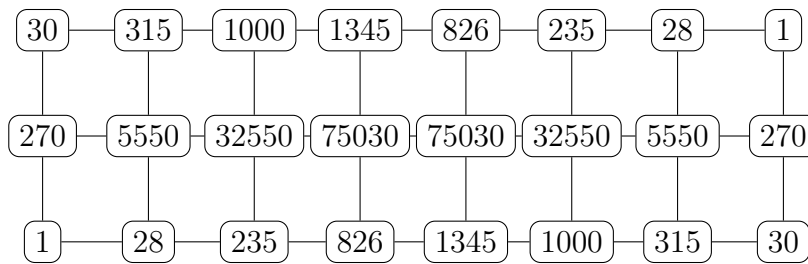
FIGURE 11. Coarse structure for the  $\tilde{G}_2$  interval.

hull of the axial simplices). More generally, the coarse structure of the interval  $[1, w]^W$  will have only finitely many elements in each box along the top row and finitely many elements in each box along the bottom row. The boxes in the middle row, on the other hand, have infinitely many elements in each. Nevertheless, there is a periodicity to the convex hull of the axial simplices and this means that the infinitely many elements in each box in the middle row falls into a finite number of infinitely repeating patterns. We illustrate this with the  $\tilde{G}_2$  Coxeter group where one can view the entire euclidean tiling.

**Example 6.5** (Coarse structure of the  $\tilde{G}_2$  interval). As can be seen in Figure 10, the euclidean Coxeter group  $W = \text{COX}(\tilde{G}_2)$  has exactly 2 horizontal reflections, the ones with vertical fixed lines which bound the lightly shaded region and this is indicated by the 2 in the second box in the bottom row of the coarse structure for the interval  $[1, w]^W$  shown schematically in Figure 11. On the other hand, there are 6 essentially different ways that a fixed line can cross the glide axis and the corresponding 6 infinite families of vertical reflections below  $w$  are indicated by the 6 in the first box of the middle row. Similarly, there are 6 infinite families of rotations fixing an axial vertex represented by the 6 in the second box of the middle row and exactly two pure translations below  $w$  indicated by the 2 in the first box of the top row. Finally, although it is not immediately obvious, it turns out that this bounded, graded, self-dual poset with finite height and an infinite number of elements is a lattice.

For a more interesting example, consider the largest sporadic irreducible euclidean Coxeter group  $\text{COX}(\tilde{E}_8)$ .

**Example 6.6** (Coarse structure of the  $\tilde{E}_8$  interval). Let  $w$  be a Coxeter element for the Coxeter group  $W = \text{COX}(\tilde{E}_8)$ . The coarse structure of the interval  $[1, w]^W$  is shown schematically in Figure 12. As in the

FIGURE 12. Coarse structure for the  $\tilde{E}_8$  interval.

case of  $\text{COX}(\tilde{G}_2)$ , the numbers listed in the top row and in bottom row indicate the actual number of elements in each box, but the numbers in the middle row only indicate the number of infinite families of such elements. The equivalence relation used is that two middle row elements below  $w$  belong to the same family iff they differ by (conjugation by) a translation of the tiling in the direction of the Coxeter axis. Thus, from the coarse structure we see that the interval  $[1, w]^W$  has 28 horizontal reflections (second box in the bottom row), 30 translations (first box in the top row), 270 infinite families of vertical reflections (first box in the middle row) and 5550 infinite families of elements that rotation around a 6-dimensional fix-set that is not invariant under vertical translation (second box in the middle row), and so on. Representatives of each family were computed using a program I wrote called `euclid.sage` that is available from my webpage. The software also checks whether this bounded, graded, self-dual poset of finite height with an infinite number of elements is a lattice and in this case the answer is “No”.

## 7. DUAL ARTIN GROUPS

Now that the technical foundations are in place, it is time to shift our attention to the irreducible euclidean Artin groups themselves, to finally explain why intervals in euclidean Coxeter groups are relevant and why we are interested in whether or not these intervals are lattices. The answers are relatively straightforward. First, intervals in irreducible euclidean Coxeter groups can be used to give alternative, so called “dual” presentations for irreducible euclidean Artin groups as I show in [McC]. Next, when these Coxeter intervals are lattices, the dual Artin group has an infinite-type Garside structure and groups with Garside structures have good computational properties. This “grand scheme”, closely related to the approach taken by François Digne in [Dig06] and [Dig12] was the initial strategy by which my coauthors and we hoped to use it to understand arbitrary euclidean Artin groups.

Unfortunately, a detailed examination of the groups themselves caused this scheme to fail because the Coxeter intervals turned out to be more poorly behaved than expected. Nonetheless, a modified grand scheme, described in the later sections, does eventually succeed. The first step is to understand how intervals lead to presentations of new groups.

**Definition 7.1** (Interval groups). Let  $G$  be a group with a fixed symmetric discretely weighted generating set and let  $[1, g]^G$  be an interval in  $G$ , viewed as an edge-labeled directed graph sitting inside the Cayley graph of  $G$ . The *interval group*  $G_g$  is a new group generated by the labels of edges in the interval subject to the set of all relations that are visible in the interval. Since there is a natural function from the generators of  $G_g$  to  $G$  and since the set of relations used to define  $G_g$  is a subset of the relations which hold in  $G$ , this function extends to a group homomorphism  $G_g \rightarrow G$ . If, moreover, the labels on the edges in the interval  $[1, g]^G$  include a generating set for  $G$  then this natural map is onto.

To see how this works in practice, consider the following example.

**Example 7.2** (Noncrossing partitions). Let  $G = \text{SYM}_n$  be the symmetric group on  $n$  elements, let  $g$  be the  $n$ -cycle  $(1, 2, \dots, n)$  and fix the full set of transpositions as its generating set. It turns out in this case that the poset structure of the interval  $[1, g]^G$  is a well-known combinatorial object called the *noncrossing partition lattice* defined as follows. Start with a convex regular  $n$ -gon in the plane whose vertices are labeled 1 through  $n$  in a clockwise fashion. A partitioning of its vertex set  $[n] = \{1, 2, \dots, n\}$  is called *noncrossing* if the convex hulls of the blocks of the partition are pairwise disjoint. For example, the partition  $\{\{1, 3\}, \{2, 4\}\}$  of  $[4]$  is not a noncrossing partition because the convex hulls are two line segments that intersect. Noncrossing partitions can be ordered by declaring  $\sigma < \tau$  when every block of  $\sigma$  is a subset of a block of  $\tau$  and a noncrossing partition can be converted into a permutation by clockwise permuting the vertices in the boundary of the convex hull of each block. This function defines a poset isomorphism between the noncrossing partition lattice  $NC_n$  and the Coxeter interval  $[1, g]^G$ . See [McC06] for an elementary discussion of these ideas. When  $n = 3$ , there are only 3 transpositions, the noncrossing partition lattice is particularly simple and the presentation for the corresponding interval group  $G_g$  is  $\langle a, b, c \mid ab = bc = ca \rangle$ , which is an alternate presentation for the 3-string braid group.

This example, and its generalization to all spherical Coxeter groups described below, leads to the following general definition.

**Definition 7.3** (Dual Artin groups). Let  $W = \text{COX}(\Gamma)$  be an arbitrary Coxeter group viewed as a group generated by its full set of reflections and let  $w$  be one of its Coxeter elements. The interval group  $W_w$  defined by the interval  $[1, w]^W$  is called a *dual Artin group* and denoted  $\text{ART}^*(\Gamma, w)$ . The notation is meant to highlight the fact that in general Coxeter groups there are geometrically distinct Coxeter elements and thus there are distinct dual presentations which heavily depend on the choice of Coxeter element.

The study of dual presentations in general is motivated by the work of Davis Bessis [Bes03] and Tom Brady and Colum Watt [BW02a] on spherical Artin groups. Here are their main results translated into this terminology.

**Theorem 7.4** (Dual spherical Artin groups). *If  $W = \text{COX}(X_n)$  is a spherical Coxeter group generated by its reflections, and  $w$  is a Coxeter element of  $W$ , then the Coxeter interval  $[1, w]^W$  is isomorphic to the  $W$ -noncrossing partition lattice and the interval group  $W_w = \text{ART}^*(X_n, w)$  is isomorphic to the corresponding Artin group  $\text{ART}(X_n)$ .*

The terminology “dual Artin group” was introduced in [McC] because in general it is not known whether or not Artin groups and dual Artin groups are isomorphic. Fortunately, I was able to establish that they are isomorphic in the euclidean case [McC].

**Theorem 7.5** (Dual euclidean Artin groups). *If  $W = \text{COX}(\tilde{X}_n)$  is an irreducible euclidean Coxeter group generated by its reflections, and  $w$  is a Coxeter element, then the dual Artin group  $W_w = \text{ART}^*(\tilde{X}_n, w)$  is naturally isomorphic to  $\text{ART}(\tilde{X}_n)$ .*

The proof uses a result from quiver representation theory to greatly simplify the dual presentations for dual euclidean Artin groups and it is then relatively straight-forward to establish that these simplified presentations define the same groups as do the standard euclidean Artin presentations [McC]. In other words, the interval  $[1, w]^W$  gives a new presentation (with infinitely many generators and infinitely many relations) of the corresponding Artin group in the irreducible euclidean case. The interest in whether or not these Coxeter intervals are lattices has to do with the following result which is essentially due to David Bessis [Bes03]. See [McC] for a more detailed discussion.

**Theorem 7.6** (Sufficient conditions). *Let  $G$  be a group with a fixed symmetric discretely weighted generating set closed under conjugation. For each  $g \in G$ , if the interval  $[1, g]^G$  is a lattice, then  $G_g$  is a Garside group, possibly of infinite-type.*

In this article, Garside structures are treated as a black box. For a book-length discussion of Garside structures see [DDG<sup>+</sup>]. The main idea is that if there is a portion of the Cayley graph which contains a generating set and has well-defined meets and joins (plus a few more technical conditions) then this local lattice structure can be used to systematically construct normal forms for all group elements, thereby solving the word problem for the group and allowing one to construct a finite dimensional classifying space.

**Theorem 7.7** (Garside consequences). *If an interval group  $G_g$  is a Garside group, possibly of infinite-type, then  $G_g$  is a torsion-free group with a solvable word problem and a finite dimensional classifying space.*

**Remark 7.8** (Artin groups with Garside structures). By the early years of the new millenium, many dual Artin groups were known to have Garside structures. We have already mentioned that the dual Artin group  $\text{ART}(\Gamma)$  is Garside when  $\text{COX}(\Gamma)$  is spherical (due to Bessis [Bes03] and Brady-Watt [BW02a] independently) and when  $\Gamma$  is an extended Dynkin diagram of type  $\tilde{A}_n$  or  $\tilde{C}_n$  (due to Digne [Dig06, Dig12]). David Bessis also proved this for the free group, thought of as the Artin group where every  $m_{ij} = \infty$  [Bes06]. In addition, there are unpublished results due to myself and John Crisp which show this to be true for all Artin groups with at most 3 standard generators and also for all Artin groups defined by a diagram in which every  $m_{ij}$  is at least 6. Note that the 3 standard generator result means that the three planar Artin groups investigated by Craig Squier in [Squ87] have dual presentations which are Garside structures (as do all of the 3-generator Artin groups whose Coxeter groups naturally act on the hyperbolic plane). In particular,  $\text{ART}(\tilde{G}_2)$  has an infinite-type Garside structure.

This list of results naturally lead one to conjecture that Coxeter intervals using Coxeter elements are always lattices and that all the corresponding dual Artin groups are Garside groups. Unfortunately, as we have already seen in Example 6.6, this natural conjecture turns out to be too optimistic and false even for some of the irreducible euclidean Coxeter groups such as the group  $W = \text{COX}(\tilde{E}_8)$ .

## 8. HORIZONTAL ROOTS

Several years ago John Crisp and I systematically investigated whether every irreducible euclidean Coxeter group has a Coxeter interval which is a lattice, and what we found was not what we expected to find. The only irreducible euclidean Coxeter groups whose Coxeter intervals are lattices are those of type  $\tilde{A}_n$ ,  $\tilde{C}_n$ , and  $\tilde{G}_2$ . In other words, the only



Type	Horizontal root system
$A_n$	$\Phi_{A_{p-1}} \cup \Phi_{A_{q-1}}$
$C_n$	$\Phi_{A_{n-1}}$
$B_n$	$\Phi_{A_1} \cup \Phi_{A_{n-2}}$
$D_n$	$\Phi_{A_1} \cup \Phi_{A_1} \cup \Phi_{A_{n-3}}$
$G_2$	$\Phi_{A_1}$
$F_4$	$\Phi_{A_1} \cup \Phi_{A_2}$
$E_6$	$\Phi_{A_1} \cup \Phi_{A_2} \cup \Phi_{A_2}$
$E_7$	$\Phi_{A_1} \cup \Phi_{A_2} \cup \Phi_{A_3}$
$E_8$	$\Phi_{A_1} \cup \Phi_{A_2} \cup \Phi_{A_4}$

TABLE 1. Horizontal root systems by type.

dual Artin groups with Garside structures are those where the group structure was already well understood by Craig Squier and/or François Digne. Further investigation revealed that the reason why a euclidean Coxeter group might have a Coxeter interval which failed to be a lattice is closely related to the structure of what we call its horizontal root system.

**Definition 8.1** (Horizontal root system). Let  $W = \text{COX}(\tilde{X}_n)$  be an irreducible euclidean Coxeter group and let  $w$  be a Coxeter element of  $W$ . The horizontal reflections in the interval  $[1, w]^W$  are those whose roots are orthogonal to the direction of the Coxeter axis of  $w$ . The set of all such roots form a subroot system of the full root system of  $W$  that we call the *horizontal root system*.

It turns out that horizontal root system is easy to describe as a subdiagram of the original extended Dynkin diagram.

**Remark 8.2** (Finding horizontal roots). Let  $W = \text{COX}(\tilde{X}_n)$  be an irreducible euclidean Coxeter group with Coxeter element  $w$ . The horizontal root system with respect to  $w$  is itself a root system for a spherical Coxeter group whose Dynkin diagram is obtained by removing two dots from the extended Dynkin diagram  $\tilde{X}_n$  or one dot from Dynkin diagram  $X_n$ . In Figure 2 the dots to be removed are slightly larger than the others. Removing the large white dot produces the Dynkin diagram of type  $X_n$ . Also removing the large shaded dot produces the diagram for the horizontal root system. In all cases, the shaded dot is the long end of a multiple bond or the branch point if either exists in  $X_n$ . The only case where neither exists is in type  $\tilde{A}_n$ , where the shaded

dot might be any of the remaining dots and different choices arise from the geometrically different choices for the Coxeter element in this case.

The types of the horizontal root systems are listed in Table 1. The key property turns out to be whether or not the remaining diagram is connected, or equivalently, whether or not the horizontal root system is reducible. The horizontal root systems for types  $C$  and  $G$  are irreducible, the horizontal root systems for types  $B$ ,  $D$ ,  $E$  and  $F$  are reducible and for type  $A$  it depends on the choice of Coxeter element. The following theorem is a restatement of and explanation for the computational result originally discovered in collaboration with John Crisp. It is proved in [McC], an article which is, morally speaking, the result of a collaboration with John Crisp even though it was only written up after he left research mathematics.

**Theorem 8.3** (Failure of the lattice property). *The interval  $[1, w]^W$  is a lattice iff the horizontal root system is irreducible. In particular, types  $C$  and  $G$  are lattices, types  $B$ ,  $D$ ,  $E$  and  $F$  are not, and for type  $A$  the answer depends on the choice of Coxeter element.*

As a consequence of this theorem, it is clear which irreducible euclidean Artin groups have dual presentations that lead to Garside structures.

**Corollary 8.4.** *The dual euclidean Artin group  $\text{ART}^*(\tilde{X}_n, w)$  is Garside when  $X$  is  $C$  or  $G$  and it is not Garside when  $X$  is  $B$ ,  $D$ ,  $E$  or  $F$ . When the group has type  $A$  there are distinct dual presentations and the one investigated by Digne is the only one that is Garside.*

At this point, the grand scheme has failed and no additional euclidean Artin groups have been understood. There is, however, a ray of hope. The explicit nature of the euclidean model posets enables an explicit examination of the pairs of elements which fail to have well-defined meets or well-defined joins. It turns out that pairs of elements with no well-defined join must occur in the bottom row of the coarse structure and pairs of elements with no well-defined meet must occur in the top row of the coarse structure. Since the top and bottom rows only contain finitely many elements, this means that out of the infinitely many pairs of elements in the infinite interval  $[1, w]^W$ , only finitely many fail to be well-behaved. This leaves open the possibility that one can systematically fix these finitely many failures.

## 9. NEW GROUPS

Let  $W = \text{COX}(\tilde{X}_n)$  be an irreducible euclidean Coxeter group with Coxeter element  $w$ . As remarked above, the finitely many pairs of elements

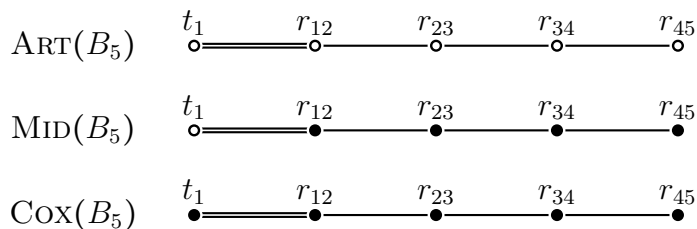


FIGURE 13. Presentation diagrams for the Artin, middle and Coxeter groups of type  $B_5$ .

in the Coxeter interval  $[1, w]^W$  which cause the lattice property to fail all occur in the top and bottom rows of its coarse structure. Thus it makes sense to focus on the subgroup corresponding to this portion of the interval. Its structure is closely related to an elementary group which does not appear to have a standard name in the literature. In fact, we have not found any references to it in the literature so far.

**Definition 9.1** (Middle groups). Consider the group of isometries of  $\mathbb{Z}^n$  in  $\mathbb{R}^n$  generated by coordinate permutations and integral translations. We call this group the *middle group* and denote it  $\text{MID}(B_n)$ . It is generated by the reflections  $r_{ij}$  that switch coordinates  $i$  and  $j$  and the translations  $t_i$  that adds 1 to the  $i$ -th coordinate and it is a semidirect product  $\mathbb{Z}^n \rtimes \text{SYM}_n$  with the translations  $t_i$  generating the normal free abelian subgroup. A standard minimal generating set for  $\text{MID}(B_n)$  is the set  $\{t_1\} \cup \{r_{12}, r_{23}, \dots, r_{n-1n}\}$  and it has a presentation similar to  $\text{ART}(B_n)$  and  $\text{COX}(B_n)$  [MS]. See Figure 13. A solid dot means the corresponding generator has order 2 and an empty dot means the corresponding generator has infinite order.

If we consider  $\text{MID}(B_n)$  as a group generated by the full set of translations and reflections, then the factorizations of  $w = t_1 r_{12} r_{23} \cdots r_{n-1n}$  form an interval isomorphic to the type  $B$  noncrossing partition lattice, exactly the same poset as the Coxeter interval in the spherical Coxeter group  $\text{COX}(B_n)$ . This explains the use of  $B_n$  in the notation. The name “middle group” is suggested by its location in the center of a diagram that shows its relation to several closely related Coxeter groups and Artin groups. See Figure 14. The top row is the short exact sequence that is often used to understand  $\text{ART}(\tilde{A}_{n-1})$ . Geometrically middle groups are easy to recognize as a symmetric group generated by reflections and a translation with a component out of this subspace. Middle groups are introduced in [MS] in order to succinctly describe

$$\begin{array}{ccccc}
\text{ART}(\tilde{A}_{n-1}) & \hookrightarrow & \text{ART}(B_n) & \twoheadrightarrow & \mathbb{Z} \\
\downarrow & & \downarrow & & \downarrow \\
\text{COX}(\tilde{A}_{n-1}) & \hookrightarrow & \text{MID}(B_n) & \twoheadrightarrow & \mathbb{Z} \\
& & \downarrow & & \\
& & \text{COX}(B_n) & & 
\end{array}$$

FIGURE 14. Relatives of middle groups.

the structure of the diagonal subgroup build from the top and bottom rows of the coarse structure of a euclidean Coxeter interval.

**Definition 9.2** (Diagonal subgroup). Let  $W = \text{COX}(\tilde{X}_n)$  be an irreducible euclidean Coxeter group with Coxeter element  $w$  and let  $R_H$  and  $T$  denote the set of horizontal reflections and translations contained in the Coxeter interval  $[1, w]^W$ . In addition let  $D$  be the subgroup of  $W$  generated by the set  $R_H \cup T$ . If we assign a weight of 1 to each horizontal reflection and a weight of 2 to each translation, then distances in the Cayley graph of  $D$  match distances in the Cayley graph of  $W$  and the interval  $[1, w]^D$  is an induced subposet of the Coxeter interval  $[1, w]^W$  consisting of only the top and bottom rows. We write  $D_w$  to denote the interval group defined by this restricted interval.

The interval  $[1, w]^D$  is almost a direct product of posets and the group  $D$  is almost a direct product of middle groups. More precisely, the poset  $[1, w]^D$  is almost a direct product of type  $B$  noncrossing partitions lattices and the missing elements are added if we factor the translations of  $D$ .

**Definition 9.3** (Factored translations). Each pure translation  $t$  in  $[1, w]^D$  projects nontrivially onto the Coxeter axis and onto each of the irreducible components of the horizontal root system of the corresponding Coxeter group  $W = \text{COX}(\tilde{X}_n)$ . Let  $t^{(i)}$  be the translation which agrees with  $t$  on the  $i$ -th component and contains  $1/k$  of the translation in the Coxeter direction where  $k$  is the number of irreducible components of the horizontal root system. We call each  $t^{(i)}$  a *factored translation* and let  $T_F$  denote the set of all such factored translations.

We use the factored translations to introduce a slightly larger group.

**Definition 9.4** (Factorable groups). The *factorable group*  $F$  is defined as the group of euclidean isometries generated by  $R_H \cup T_F$ . It is crystallographic in the sense that it acts geometrically on euclidean space but it is not a Coxeter group in general because it is not generated by

Name	Symbol	Generating set
Horizontal	$H$	$R_H$
Diagonal	$D$	$R_H \cup T$
Coxeter	$W$	$R_H \cup R_V (\cup T)$
Factorable	$F$	$R_H \cup T_F (\cup T)$
Crystallographic	$C$	$R_H \cup R_V \cup T_F (\cup T)$

TABLE 2. Five generating sets.

reflections. If we assign a weight of  $2/k$  to each factored translation then distances in the Cayley graph of  $F$  agree with those in  $D$  and the interval  $[1, w]^D$  is an induced subinterval of  $[1, w]^F$ . The main advantage of the interval  $[1, w]^F$  is that it factors as a direct product of type  $B$  noncrossing partition lattices with one factor for each irreducible component of the horizontal root system. The edge labels in the  $i$ -th factor poset correspond to the factored translations which project nontrivially onto  $i$ -th component of the horizontal root system together with the horizontal reflections whose roots belong to this component. Moreover, the isometries that occur as labels in any one factor generator a group isomorphic to a middle group acting on the subspace whose directions are spanned by a component of the horizontal root system and the direction of the Coxeter axis. The structure of the factorable group  $F$  is not quite as clean since each of the component middle groups contain central elements which are pure translations in the direction of the Coxeter axis. Thus  $F$  is merely a central product of the associated middle groups.

In addition to the groups  $W$ ,  $D$ , and  $F$ , we introduce several other groups that help to clarify the structure of the corresponding euclidean Artin group.

**Definition 9.5** (Ten groups). Let  $W = \text{COX}(\tilde{X}_n)$  be an irreducible euclidean Coxeter group with Coxeter element  $w$  and let  $D$  and  $F$  be the diagonal and factorable groups acting on  $n$ -dimensional euclidean space as defined above. There are two other euclidean isometry groups we need to define. Let  $H$  denote the subgroup of  $W$  generated by horizontal reflections  $R_H$  that label edges in the interval  $[1, w]^W$  and let  $C$  be the group generated by the set  $R_H \cup R_V \cup T_F (\cup T)$  of all generating isometries considered so far. The set  $T$  of translations is in parentheses because its elements are products of other generators. Note that  $C$ , like  $F$ , is crystallographic in that it acts geometrically on euclidean space but it is not in general a Coxeter group since we have added translation

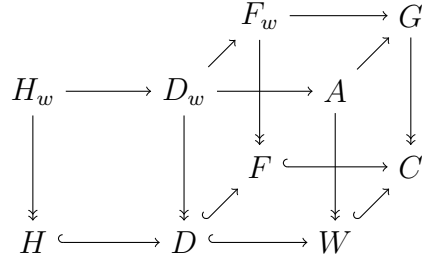


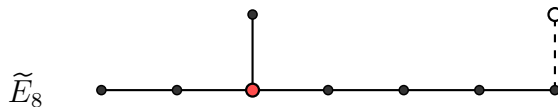
FIGURE 15. Ten groups.

generators which are not products of reflections contained in  $W$ . The five generating sets introduced are listed in Table 2 along with the euclidean isometry groups they generate. Of these five groups  $H$  and  $W$  are Coxeter groups, while  $D$ ,  $F$  and  $C$  are merely crystallographic.

Next we construct five groups defined by presentations. Let  $D_w$ ,  $F_w$ ,  $W_w$  and  $C_w$  denote the interval groups defined by the interval  $[1, w]$  in each of the four contexts, but note that we write  $A = W_w$  and  $G = C_w$  since these turn out to be the corresponding Artin group and a previously unstudied Garside group, respectively. A final group  $H_w$  is defined by a presentation with  $R_H$  as its generators and subject to the relations among these generators which are visible in the interval  $[1, w]^W$ . This is not quite an interval group since there is no interval of the form  $[1, w]^H$ . This is because the element  $w$  is not in the subgroup  $H$  as it requires a vertical motion in order to be constructed. Some of the maps between these ten groups are shown in Figure 15. The maps in the bottom level are the natural inclusions among these five euclidean isometry groups, the vertical arrows are the projections from the five groups defined by presentations to the groups from which they were constructed, and the maps in the top level are the natural homomorphisms that extend the inclusions on their generating sets.

A review of some of the various posets and groups associated with the euclidean Coxeter group of type  $\tilde{E}_8$  might help to clarify these definitions.

**Example 9.6** (Groups of type  $\tilde{E}_8$ ). When  $W$  is the irreducible euclidean Coxeter group of type  $\tilde{E}_8$ , its horizontal root system decomposes as  $\Phi_{A_1} \cup \Phi_{A_2} \cup \Phi_{A_4}$ . See Figure 16. The factorable group  $F$  is a central product of middle groups  $\text{MID}(B_2)$ ,  $\text{MID}(B_3)$  and  $\text{MID}(B_5)$ . The interval  $[1, w]^F$  is isomorphic to the direct product  $NC_{B_2} \times NC_{B_3} \times NC_{B_5}$  of type  $B$  noncrossing partition lattices, and the interval  $[1, w]^F$  defines an interval group  $F_w$  which is a direct product  $\text{ART}(B_2) \times \text{ART}(B_3) \times$

FIGURE 16. The  $\tilde{E}_8$  diagram.

$\text{ART}(B_5)$  of type  $B$  spherical Artin groups. The horizontal reflections in the interval  $[1, w]^W$  generate a group  $H$  isomorphic to a direct product  $\text{COX}(\tilde{A}_1) \times \text{COX}(\tilde{A}_2) \times \text{COX}(\tilde{A}_4)$  of type  $A$  euclidean Coxeter groups, and the relations among these reflections visible in the interval  $[1, w]^W$  define a group  $H_w$  which is isomorphic to a direct product  $\text{ART}(\tilde{A}_1) \times \text{ART}(\tilde{A}_2) \times \text{ART}(\tilde{A}_4)$  of type  $A$  euclidean Artin groups.

## 10. STRUCTURAL RESULTS

The addition of the factored translations as extra generators completely solves the lattice problem and makes it possible to prove the three main structural results established in by myself and Rob Sulway in [MS]. The first establishes the existence of a new class of Garside groups based on intervals in the crystallographic groups  $C$  introduced in the previous section.

**Theorem A** (Crystallographic Garside groups). *If  $C = \text{CRYST}(\tilde{X}_n, w)$  is the crystallographic group obtained by adding the factored translations to the generating set of the irreducible euclidean Coxeter group  $W = \text{COX}(\tilde{X}_n)$ , then the interval  $[1, w]^C$  in the Cayley graph of  $C$  is a lattice. As a consequence, this interval defines an group  $G = C_w$  with an infinite-type Garside structure.*

As is typical, the most difficult step in the entire article is the proof that these augmented intervals are lattices. Using the program `euclid.sage`, we verify that this is the case for all irreducible euclidean Coxeter groups up to rank 9, which includes all of the sporadic examples. Then special properties of the infinite families are used to complete the proof. It would, of course, be more desirable to have case-free proof of the lattice property, that project has not yet been completed.

The second main result of [MS] establishes that the crystallographic Garside group  $G$  has the structure of an amalgamated free product, and as a consequence, the natural map from corresponding euclidean Artin group  $A$  to the crystallographic Garside group  $G$  is injective.

**Theorem B** (Artin groups are subgroups). *For each irreducible euclidean Coxeter group  $W = \text{COX}(\tilde{X}_n)$  and for each choice of Coxeter element  $w$ , the crystallographic Garside group  $G = \text{GAR}(\tilde{X}_n, w)$  is an amalgamated free product of explicit groups with the euclidean Artin group  $A = \text{ART}(\tilde{X}_n)$  as one of its factors. In particular, the euclidean Artin group  $A$  injects into the crystallographic Garside group  $G$ .*

In terms of the groups defined in the previous section the crystallographic Garside group  $G$  is an amalgamated product of  $F_w$  and  $A$  over  $D_w$ . This also means that all of the group homomorphisms on the top level of Figure 15 are injective. Injectivity is a consequence of our structural analysis and not something that was immediately obvious from the definitions of the maps. The final result of [MS] uses these embeddings of euclidean Artin groups into crystallographic Garside groups to elucidate their structure.

**Theorem C** (Structure of euclidean Artin groups). *Every irreducible euclidean Artin group  $A = \text{ART}(\tilde{X}_n)$  is a torsion-free centerless group with a solvable word problem and a finite-dimensional classifying space.*

Most of these structural results follow immediately from Theorem B. The only aspect that required a bit more work is the center. The Garside structure on  $G$ , the product structure on  $F_w$ , and the fact that we are amalgamating over  $D_w$  are all used in the proof that shows the center of  $A$  is trivial. See [MS] for details.

To conclude this survey, I would like to highlight some of the questions that these results raise. Now that we understand the word problem for the euclidean types, can we devise an Artin group intrinsic solution that avoids the introduction of the crystallographic Garside groups? Or perhaps the crystallographic Garside groups we define are merely the first instance of a natural geometric completion process? What about hyperbolic Artin groups and beyond? Are there similar procedures that work in these more general contexts?

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