Braid groups and Curvature
Talk 3: The Proof

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Consequences of being a CAT(0) group

For those not familiar with CAT(0) groups:
When a group G has a nice action on a CAT(0) space there are many algebraic consequences for G. Here is a list taken from Chapter III.Γ in the book by Bridson and Haeflieger.

Remark (CAT(0) groups)

If G has a geometric action on a CAT(0) space X, then:

1. G is finitely presented.
2. G has finitely many conjugacy classes of finite subgroups.
3. Every solvable subgroup of G is virtually abelian.
4. Every abelian subgroup of G is finitely generated.
5. If G is torsion-free, then G is the fundamental group of a compact cell complex with contractible universal cover.

Closure Properties and Algorithmic Properties

Remark (Closure Properties)

The class of CAT(0) groups is closed under direct products, free products with amalgamation, HNN extensions along finite subgroups, and free products with amalgamation along virtually cyclic subgroups.

Remark (Algorthmic Properties)

Groups that are CAT(0) have

1. a solvable word problem,
2. a quadratic Dehn function,
3. a solvable conjugacy problem,
4. an algorithm to test if an element has finite order,
5. an algorithm to test if torsion elements are conjugate.

Semi-simple actions

Definition (Semi-simple isometries)

An elliptic isometry fixes some point, a hyperbolic isometry moves every point at least $\epsilon > 0$ and both types are semi-simple.

Remark (Semi-simple isometries)

If H is a fin. gen. group that acts properly (but not necessarily cocompactly) by semi-simple isometries on X, then:

1. Every polycyclic subgroup of H is virtually abelian.
2. Every fin. gen. abelian subgroup of H is quasi-isometrically embedded (with respect to any choice of word metrics).
3. H does not contain $BS(p, q)$ subgroups with $|p| \neq |q|$.
4. If $A \cong \mathbb{Z}^n$ is central in H then there exists a subgroup of finite index in H that contains A as a direct factor.

Roman Wall in Regensburg

Old Foundations

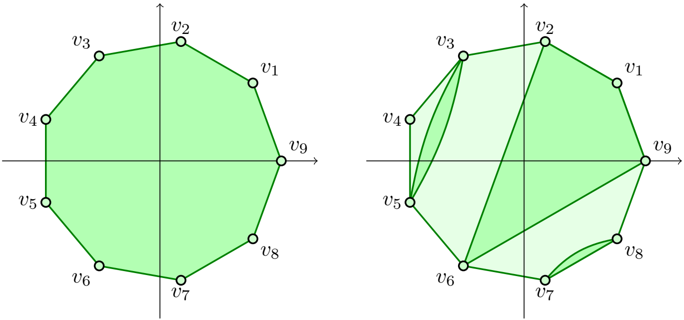
Remark (Main Goal)

The main goal of today's talk is to outline the proof that the braid groups are CAT(0), but like the building on the previous slide, the foundations for the proof are very old. It is only the final insight that is new.

Remark (Notation)

Recall that for each $A \subset [n]$ of size $k > 1$ with $B = [n] - A$ we have defined a subset of vertices V_A, a subdisk P_A, a rotation δ_A and a subgroup $\textsc{Braid}_A = \textsc{Fix}(B)$ isomorphic to $\textsc{Braid}_k$. And for every noncrossing partition Π we have defined a dual braid δ_Π that is a product of commuting rotations.

Subsets, Subdisks, Rotations and Dual Braids



If $\Pi = \{\{\{1,2,6,9\},\{3,5\},\{7,8\}\}$, then $\delta_\Pi = \delta_{\{1,2,6,9\}}\delta_{\{3,5\}}\delta_{\{7,8\}}$

Dual braid complex with the orthoscheme metric

Definition (Dual braid complex with the orthoscheme metric)

The dual braid complex for $\textsc{Braid}_n$ with the orthoscheme metric starts with the Cayley graph of $\textsc{Braid}_n$ with respect to the dual braid generating set. The length of each edge labeled by δ_Π is $\sqrt{k}$ where $k = \textsc{rank}(\Pi)$ in the noncrossing partition lattice. An orthoscheme with the correct edge lengths is added to each complete subgraph. The resulting PE simplicial complex is denoted $\textsc{Cplx}(\textsc{Braid}_n)$.

The proof scheme is by induction and the base cases are true.

Theorem (Previous results)

The k-strand dual braid complex with the orthoscheme metric is $\mathrm{CAT}(0)$ *when* $k \leq 6$.

Dual Parabolic Subcomplexes

So suppose that $\textsc{Cplx}(\textsc{Braid}_k)$ is known to be $\mathrm{CAT}(0)$ for all $k < n$ and consider the space $X = \textsc{Cplx}(\textsc{Braid}_n)$. We already know several subcomplexes that are $\mathrm{CAT}(0)$.

Definition (Subcomplexes and Subsets)

For each $S \subset \textsc{Braid}_k$ let $\textsc{Cplx}(S)$ be the full subcomplex of $\textsc{Cplx}(\textsc{Braid}_k)$ whose vertex set is indexed by S.

Proposition (Dual parabolic subcomplexes)

For every $A \subset [n]$ of size k, the isomorphism $\textsc{Braid}_k \to \textsc{Braid}_A$ extends to an isometric embedding of $\textsc{Cplx}(\textsc{Braid}_k)$ into $X = \textsc{Cplx}(\textsc{Braid}_n)$ with image $\textsc{Cplx}(\textsc{Braid}_A)$. In particular, $\textsc{Cplx}(\textsc{Braid}_A)$ is a $\mathrm{CAT}(0)$ subspace of X by induction.

Local criterion

Recall the local criterion from Talk 1.

Theorem (Local criterion)

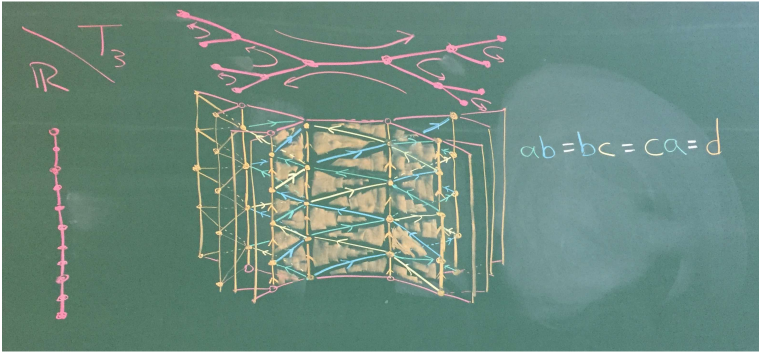
Let G be a group acting vertex-transitively by isometries on a connected and simply-connected euclidean cell complex X with finitely many shapes. If X contains a $\mathrm{CAT}(0)$ subcomplex that contains a neighborhood of a vertex, then X is a $\mathrm{CAT}(0)$ space.

If we can create such a subspace Y and vertex v, then the induction will be complete.

Remark (Parabolics are too small)

The dual parabolic subspaces are insufficient because they have strictly smaller dimensions than X. We need a subcomplex Y with $\dim(Y) = \dim(X)$ to use the local criterion.

The 3-strand dual braid complex



The copies of $\mathrm{CPLX}(\mathrm{BRAID}_2)$ inside $\mathrm{CPLX}(\mathrm{BRAID}_3)$ are lines.

Action of δ on X: Left vs. Right

Recall that $\delta = \delta_{[n]}$ is the rotation of all the vertices of P.

Remark (Left action of δ)

For $n = 3$, the left action of δ on $\textsc{Cplx}(\textsc{Braid}_3)$ translates the vertical line through the identity vertex but rotates the 3-valent tree through a $2\pi/3$ rotation. For all n it is a loxodromic isometry of $\textsc{Cplx}(\textsc{Braid}_n)$ with a single line as its min-set.

Remark (Right action of δ)

For $n = 3$ the right action of δ on the vertices of $\textsc{Cplx}(\textsc{Braid}_n)$ (surprisingly!) extends to a cell map on all of $\textsc{Cplx}(\textsc{Braid}_n)$ that merely moves every vertex $\sqrt{2}$ in the vertical direction. But note that this map does not preserve edge labels. For all n, right multiplication by δ extends to a cellular isometry.

Full-dimensional Subcomplexes

Definition (Maximal Dual Parabolic Subgroups)

For each $i \in [n]$ let $F_i = \textsc{Fix}(\{i\})$ which is equal to $\textsc{Braid}_A$ with $A = [n] - \{i\}$. These are simply the maximal dual parabolic subgroups of $\textsc{Braid}_n$.

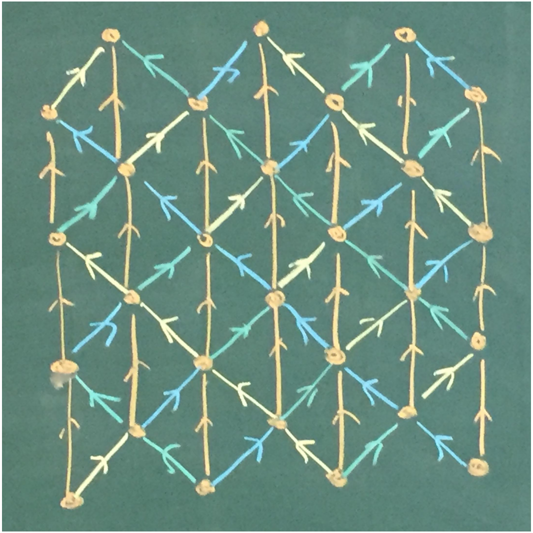
Definition (Vertical Shifts)

Let $\Delta = \langle \delta \rangle = \{\delta^\ell \mid \ell \in \mathbb{Z}\}$ be the copy of $\mathbb{Z}$ generated by $\delta = \delta_{[n]}$. Right multiplying by elements in Δ is an isometry of $X = \textsc{Cplx}(\textsc{Braid}_n)$ that we describe as a vertical shift.

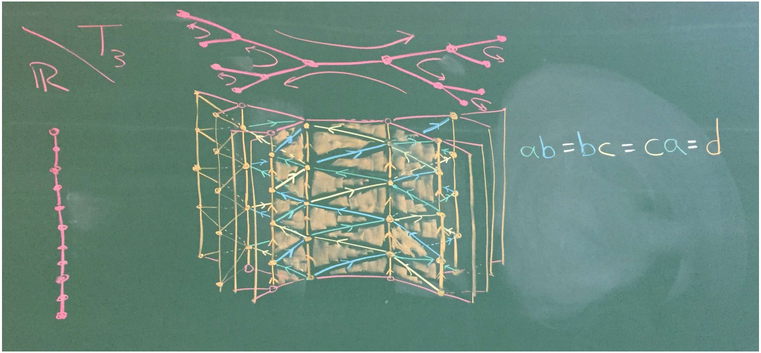
Definition (Full-dimensional Subcomplexes)

For each $i \in [n]$ consider the set $F_i \cdot \Delta = \{\alpha \cdot \delta^\ell \mid \alpha \in F_i, \ell \in \mathbb{Z}\}$. This is a subset but not a subgroup of $\textsc{Braid}_n$. We define $Y_i = \textsc{Cplx}(F_i \cdot \Delta) \subset X$ and note that $\dim(Y_i) = \dim(X)$.

Full-dimensional subcomplex of CPLX(BRAID$_3$))



The 3-strand dual braid complex



$Y = Y_1 \cup Y_2 \cup Y_3$ contain a neighborhood of the identity vertex.

Neighborhood of the Identity Vertex

Proposition (Neighborhood of a Vertex)

The subcomplex $Y = Y_1 \cup \cdots \cup Y_n$ *contains a neighborhood of the identity vertex in* $X = \textsc{Cplx}(\textsc{Braid}_n)$.

Proof sketch.

Let v be the vertex indexed by the identity of $\textsc{Braid}_n$. Every simplex containing v is contained in a maximal simplex containing v, which is part of a column, which is determined by a maximal simplex σ starting at v. This reduction uses the fact that X and all of the Y_i are invariant under the right action by δ and that they are unions of columns of orthoschemes.

The vertices of σ can be described by a non-crossing tree, every non-crossing tree has a boundary edge, and from this we show that every vertex of σ is contained in some Y_i. □

The Hard Part

Recall the gluing lemma from the first talk.

Lemma (Gluing n subspaces)

Let $Y = Y_1 \cup \cdots \cup Y_n$ be a metric space. If for each $\varnothing \neq B \subset [n]$, the corresponding intersection $Y_B = \cap_{i \in B} Y_i$ is a non-empty complete $\mathrm{CAT}(0)$ space, then Y is a complete $\mathrm{CAT}(0)$ space.

The material up to this point is pretty well-known.

Remark (The Hard Part)

By induction there are subcomplexes Y_i with $i \in [n]$ that should be $\mathrm{CAT}(0)$ (and they are) and their union Y contains a neighborhood of a vertex in $X = \mathrm{CPLX}(\mathrm{BRAID}_n)$. By the gluing lemma and the local criterion, the induction will be complete IF we can understand the intersections of the subcomplexes Y_i and show that they are $\mathrm{CAT}(0)$. This is the hard part.

Lowering the Dimension

The standard approach to this problem has been to lower the dimension of the space one needs to analyze.

Definition (Cross-section Complex)

Since every top-dimensional orthoscheme of X belongs to a unique column of orthoschemes, the complex X splits as metric direct product $X = C \times \mathbb{R}$ where C is a complex built out of $\widetilde{A}$ Coxeter shapes. We call C the cross-section complex.

Definition (Vertex Links)

All of the vertex links in C are isometric so it is sufficient to show that just one of them is $\mathrm{CAT}(1)$. Let L be the link of a vertex in C. If L is $\mathrm{CAT}(1)$ then C is $\mathrm{CAT}(0)$, X is $\mathrm{CAT}(0)$ and $\textsc{Braid}_n$ is a $\mathrm{CAT}(0)$ group. Note that $\dim(L) = \dim(X) - 2$.

Dimension Lowering Distorts Metrics

Remark (Distortions)

For small values of n, L can be visualized and this is an effective method for proving results. For general values of n, passing from X to C to L distorts the geometry of the subspaces Y_i - for example, squares become rhombi - and this makes the exact relationship between this subspaces hard to analyse and understand. The intersections in the cross-section C or in the link L look like they are $\text{CAT}(0)$ or $\text{CAT}(1)$ but proving this in this context was hard.

Remark (The Structure of the Link)

The link L is covered by subcomplexes that are metric unit spheres, it is a subcomplex of a spherical building and it can be described as a union of apartments in this spherical building.

Noncrossing hypertrees

Remark (The First Breakthrough)

In the summer of 2015, I found a way to simplify the cell structure of the vertex link L using combinatorial structures I call noncrossing hypertrees. There is a preprint on the arxiv that discusses this simplification and its uses.

Remark (From L to C to X)

Using the noncrossing hypertree simplification, we were able to start to understand large portions of the cross-section complex C. Working in C instead of L has some advantages and the arguments simplified. Eventually, we started working directly in the orthoscheme complex X instead of the cross-section complex C and the arguments simplified even more.

A Strand in the Boundary

Remark (The Second Breakthrough)

The second breakthrough came this summer when Michael computed an explicit intersection of two Y_i's inside $\textsc{Braid}_5$. As we tried to make sense of this concrete example, the true nature of the set $F_i \cdot \Delta$ emerged. After the second breakthrough, noncrossing hypertrees were no longer needed.

Recall that $F_i = \textsc{Fix}(\{i\}) = \textsc{Braid}_A$ where $A = [n] - \{i\}$, $\Delta = \langle \delta \rangle$ and $Y_i = \textsc{Cplx}(F_i \cdot \Delta)$.

Lemma (A Strand in the Boundary)

For each $\alpha \in F_i \cdot \Delta$, there is a representative f of α such that the $(i, \cdot)$-strand of f is contained in ∂P, the boundary of the polygon P. In fact, this condition characterizes the braids in set $F_i \cdot \Delta$.

Boundary Braids

Definition (Boundary Braids)

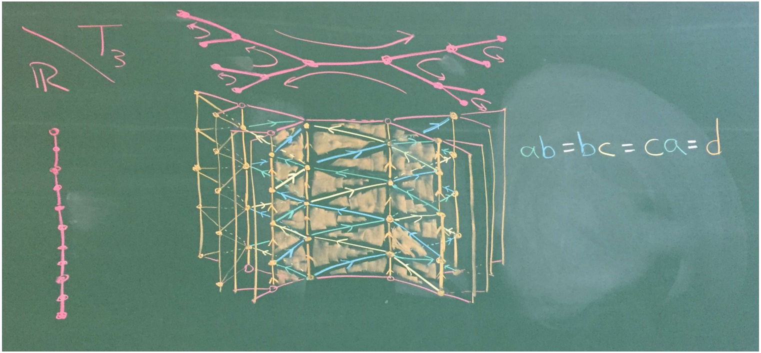
For each $B \subset [n]$ let $\textsc{Bndry}(B)$ be the set of braids with a representative f where for all $i \in B$, the $(i, \cdot)$-strand of f is contained in ∂P. In other words, the strands that start in positions V_B stay in ∂P. We call $\textsc{Bndry}(B)$ the set of B-boundary braids.

Remark ($\{i\}$-Boundary Braids)

By the lemma $F_i \cdot \Delta$ is the set of $\{i\}$-boundary braids and Y_i is the complex of the $\{i\}$-boundary braids.

One might guess that $\textsc{Bndry}(C) \cap \textsc{Bndry}(D) = \textsc{Bndry}(C \cup D)$ and this is indeed the case. From this we also prove that $Y_B = \cap_{i \in B} Y_i$ is $\textsc{Cplx}(\textsc{Bndry}(B))$ and that $Y_C \cap Y_D = Y_{C \cup D}$.

The 3-strand dual braid complex



When $n = 3$, Y_B is the same as $\text{CPLX}(\text{BNDRY}(B))$, and their intersections behave as expected.

Locally Marked Strands

The final step is to show that the intersection $Y_B = \cap_{i \in B} Y_i$ is CAT(0) for each $B \subset [n]$.

Definition (Locally Marked Strands)

Fix $B \subset [n]$ and let $\alpha = [f]$ be a braid. The strands of f that start at V_B end at a set $V_{B'}$ with $|B| = |B'|$. Let v_α be the vertex of the Cayley graph indexed by α. We assign the set B' to this vertex and say that $V_{B'}$ is the set of locally marked strands at v_α.

As we analyzed the structure of $\textsc{Bndry}(B)$, it became clear that there were two types of basic changes: those that move the locally marked strands and those that do not. And this realization leads to an unexpected product structure.

Basic Moves

Definition (Basic Moves)

Let v_α be a vertex in the Cayley graph with marked strands B'. A basic move at v_α is a dual braid δ_Π with three properties: (1) it must move at least one marked strand in B', (2) all of the marked strands that move under δ_Π must stay in ∂P, and (3) it cannot be factored into dual braids so that one of the factors fails to move a marked strand.

Definition (Move graph)

Fix a set $B \subset [n]$ of marked strands at the identity vertex and then locally mark strands at every vertex of X. The move graph with respect to B is a subgraph of the Cayley graph with the same vertex set and it contains an edge e if and only if the label of e is a basic move at the start vertex of e.

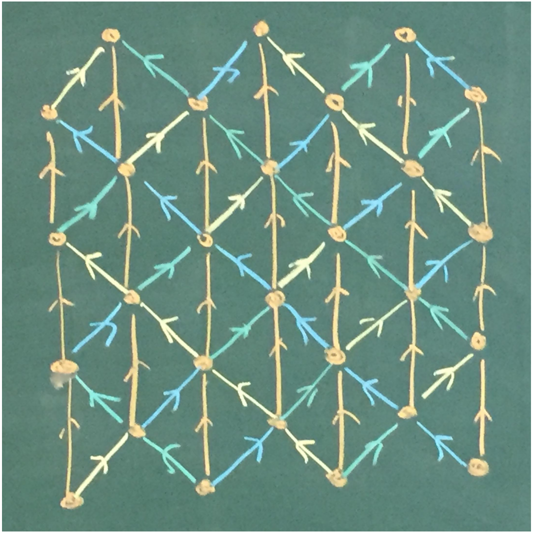
Move sets

Definition (Move sets)

The set $\textsc{Move}(B) \subset \textsc{Braid}_n$ is defined to be the set of braids α indexing vertices in the connected component of the identity vertex in the move graph with respect to B. In other words, α is in $\textsc{Move}(B)$ if and only if it can be represented as an undirected path in the Cayley graph with respect to the dual braid generators so that every directed edge of the path is a basic move at its start vertex.

The braids in $\textsc{Move}(B)$ are in some precise sense orthogonal to those in $\textsc{Fix}(B)$. This is easy to see in the full-dimensional subcomplexes of $\textsc{Cplx}(\textsc{Braid}_3)$.

Full-dimensional subcomplex of CPLX(BRAID$_3$))



Product Structure

Proposition (Product Structure)

The set $\textsc{Bndry}(B)$ *is equal to the set* $\textsc{Fix}(B) \cdot \textsc{Move}(B)$. *This means that every braid* γ *in* $\textsc{Bndry}(B)$ *can be written as product* $\gamma = \alpha \cdot \beta$ *with* $\alpha \in \textsc{Fix}(B)$ *and* $\beta \in \textsc{Move}(B)$, *and that every braid* $\gamma = \alpha \cdot \beta \in \textsc{Fix}(B) \cdot \textsc{Move}(B)$ *is in* $\textsc{Bndry}(B)$.

On the level of subcomplexes the structure is even nicer.

Proposition (Metric Products)

The natural map from (an orthoscheme subdivision of) $\textsc{Cplx}(\textsc{Fix}(B)) \times \textsc{Cplx}(\textsc{Move}(B))$ *to* $\textsc{Cplx}(\textsc{Bndry}(B)) = Y_B$ *is an isometry of metric simplicial complexes.*

As a consequence $\textsc{Cplx}(\textsc{Bndry}(B))$ is $\mathrm{CAT}(0)$ if and only if both factor complexes are $\mathrm{CAT}(0)$.

Move sets and Robots

The complex $\textsc{Cplx}(\textsc{Fix}(B))$ is the complex of a dual parabolic subgroup and it is $\mathrm{CAT}(0)$ by our induction hypothesis. And the complex $\textsc{Cplx}(\textsc{Move}(B))$ is one that we saw in Talk 2.

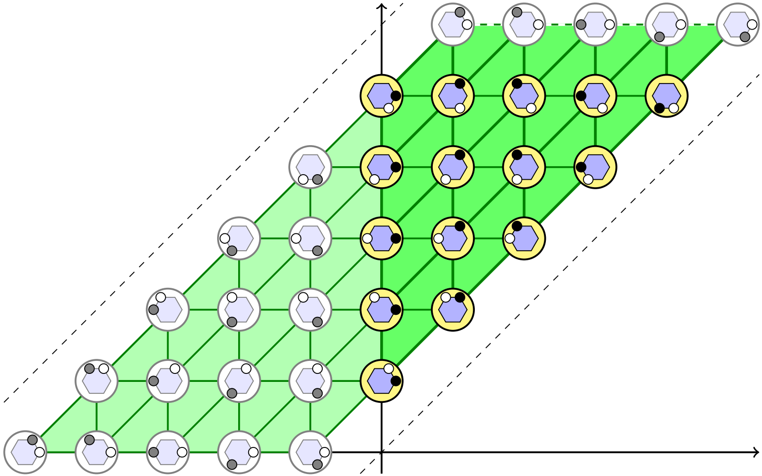
Proposition (Move sets and Robots)

The complex $\textsc{Cplx}(\textsc{Move}(B))$ is isometric to the universal cover of the configuration space of $\ell = |B|$ robots on an n-cycle. In particular, it can be identified with a dliated column in $\mathbb{R}^\ell$ and it is a $\mathrm{CAT}(0)$ space.

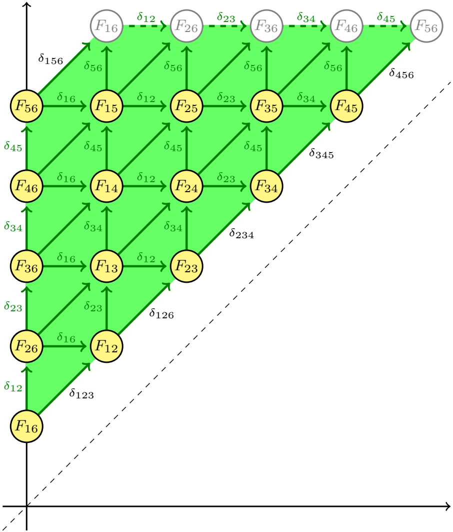
Remark (Universal Cover)

It is the universal cover rather than the configuration space because moving all of the marked robots m times around ∂P using basic moves is a non-trivial braid unless $m = 0$.

A Dilated Column



A Labeled Projection Map



Assembling the Pieces

Remark (Projection Map)

There is even a projection map from $\text{CPLX}(\text{BNDRY}(B))$ to $\text{CPLX}(\text{MOVE}(B))$ where the preimage of each vertex is the complex of some dual parabolic subgroups.

And we can now assemble the pieces.

Proof (Assembling the Pieces).

For each $B \subset [n]$, $\text{CPLX}(\text{MOVE}(B))$ is $\text{CAT}(0)$ by the proposition and $\text{CPLX}(\text{FIX}(B))$ is $\text{CAT}(0)$ by our induction hypothesis. Thus their metric product $\text{CPLX}(\text{BNDRY}(B)) = Y_B$ is $\text{CAT}(0)$. Since every intersection $Y_B = \cap_{i \in B} Y_i$ is nonempty complete $\text{CAT}(0)$, the union $Y = Y_1 \cup \cdots \cup Y_n$ is $\text{CAT}(0)$ (Gluing Lemma). And since Y contains a neighborhood of the identity vertex, X is $\text{CAT}(0)$ (Local Criterion). □

Next Steps and Future Directions

The very next step is to finish writing the article! All of the proofs have been written down. We are currently working to streamline the notation and improve the exposition. There should be an article ready to post sometime this fall.

Remark (Future directions)

There are several obvious avenues to pursue.

- other Artin groups
- other Garside groups
- other surface braid groups with boundary
- other robot configuration spaces

And I can say more about each of these if there is still time.

Thank You for your Attention!