UNDESIRED PARKING SPACES AND CONTRACTIBLE PIECES OF THE NONCROSSING PARTITION LINK

MICHAEL DOUGHERTY AND JON MCCAMMOND

Abstract. There are two natural simplicial complexes associated to the noncrossing partition lattice: the order complex of the full lattice and the order complex of the lattice with its bounding elements removed. The latter is a complex that we call the *noncrossing partition link* because it is the link of an edge in the former. The first author and his coauthors conjectured that various collections of simplices of the noncrossing partition link (determined by the undesired parking spaces in the corresponding parking functions) form contractible subcomplexes. In this article we prove their conjecture by combining the fact that the star of a simplex in a flag complex is contractible with the second author's theory of noncrossing hypertrees.

The *noncrossing partition lattice* NC_n is a fundamental object in modern combinatorics and from it one can construct two natural simplicial complexes: the order complex of the full lattice and the order complex of the lattice with its bounding elements removed. The former has a natural piecewise euclidean metric that has been the subject of some study, particularly for its connections to curvature of the n-strand braid group [BM10, HKS16, DMW]. The latter has a natural piecewise spherical metric and we call it the *noncrossing partition link* $\textsc{Link}(\mathrm{NC}_n)$ because it is the link of an edge in the former [BM10]. Since the top-dimensional simplices in the noncrossing partition link are in natural bijection with the maximal chains in NC_n, the factorizations of the n-cycle $(1, 2, \ldots, n)$ into $n-1$ transpositions, the parking functions of length $n-1$ and the properly ordered noncrossing trees on n labeled vertices, any of these sets can be used as labels on these simplices [Sta97, McC]. The first result we prove is the following.

Theorem A (Unused boundary edge)**.** *The simplices of the noncrossing partition link labeled by properly ordered noncrossing trees that omit a fixed boundary edge e form a contractible subcomplex.*

Our proof combines an easy result about flag complexes with the second author's theory of noncrossing hypertrees [McC]. Theorem A can also be stated in terms of parking functions.

Date: July 20, 2017.

2010 *Mathematics Subject Classification.* 05A18, 05E45, 20F36.

Key words and phrases. parking functions, noncrossing partitions, noncrossing hypertrees, flag complexes, undesired parking spaces.

Theorem B (Undesired last parking space)**.** *The simplices of the noncrossing partition link labeled by parking functions where no car wants to park in the last parking space form a contractible subcomplex.*

In this form, Theorem B is one part of the conjecture that the first author and his coauthors made in [BDH$^+$16]. The more general conjecture involves filtering the collection of parking functions by the last parking space that is undesired. We also prove the general version.

Theorem C (Undesired parking space)**.** *The simplices of the noncrossing partition link labeled by the collection of parking functions for which k is the number of the largest undesired parking space form a contractible subcomplex.*

Theorem C quickly follows from Theorem B and a few elementary lemmas. These results have also been proved by Henri Mühle using a completely different approach, namely, by embedding the relevant poset into a supersolvable lattice. See [Müh] for details.

The structure of the article is as follows. After proving that the star of a simplex in a flag complex is contractible, we review the theory of noncrossing hypertrees and prove Theorem A. When re-interpreted in terms of parking functions Theorem A becomes Theorem B and then Theorem C is derived as an easy corollary.

1. Simplices

This section recalls an easy way to deform a simplex onto a subspace.

Definition 1.1 (Simplices)**.** A *simplex* σ is the convex hull of a finite set S of points in general position in some euclidean space and the convex hull of any proper subset of S is a *proper subsimplex of* σ. For any simplex σ and proper subsimplex τ let $\sigma \setminus \tau$ denote the subspace of σ that is the union of all the subsimplices of σ that do not contain all of the vertices of τ. If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are the vertices of σ, then each point $\mathbf{p}$ in σ has a unique description $\mathbf{p} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n$ with each $a_i \geq 0$ and $\sum_i a_i = 1$. These unique scalars a_i are the *barycentric coordinates* of $\mathbf{p}$. For each point $\mathbf{p}$, the vertices $\mathbf{v}_i$ corresponding to the non-zero a_i coordinates determine the unique smallest subsimplex of σ that contains $\mathbf{p}$. The points in $\sigma \setminus \tau$ are precisely those where at least one of the vertices in τ has 0 as its barycentric coordinate. The *barycenter* of a subsimplex τ is the average of its vertices. Concretely, if τ has vertices $\mathbf{v}_1, \ldots, \mathbf{v}_k$, then $\mathbf{p} = \frac{1}{k}(\mathbf{v}_1 + \cdots + \mathbf{v}_k)$ is its barycenter.

Proposition 1.2 (Removing a proper face)**.** *Let σ be a simplex. For each proper subsimplex $\tau \subset \sigma$ there is a deformation retraction from σ to $\sigma \setminus \tau$.*

Proof. Since τ is a proper subsimplex, there are vertices of σ not in τ and the collection of all such vertices spans a disjoint subsimplex that we call τ'. Let $\mathbf{v}_{\tau,\tau'}$ be the vector from the barycenter of τ to the barycenter of τ'. For each point $\mathbf{p}$ in σ the line $\mathbf{p} + t \cdot \mathbf{v}_{\tau,\tau'}$ through $\mathbf{p}$ in the $\mathbf{v}_{\tau,\tau'}$ direction intersects the subcomplex $\sigma \setminus \tau$ in a single point. The function $f : \sigma \to \sigma$ that sends each point in σ to this well-defined point in $\sigma \setminus \tau$ is a continuous retraction and, since σ is convex, there is a straight-line homotopy between the identity map and f. This is a deformation retraction from σ to $\sigma \setminus \tau$ which leaves the points in the subcomplex fixed throughout. □

Remark 1.3 (Removing a proper face). The retraction f is easy to describe in barycentric coordinates. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be the vertices of τ, let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_\ell$ be the vertices of τ' and let

$$\mathbf{p} = a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k + b_1\mathbf{u}_1 + \cdots + b_\ell\mathbf{u}_\ell$$

(with all $a_i, b_j \geq 0$ and $\sum_i a_i + \sum_j b_j = 1$) be an arbitrary point in σ. Then $\mathbf{v}_{\tau,\tau'} = -\frac{1}{k}(\mathbf{v}_1 + \mathbf{v}_2 + \ldots + \mathbf{v}_k) + \frac{1}{\ell}(\mathbf{u}_1 + \mathbf{u}_2 + \ldots + \mathbf{u}_\ell)$ and the unique point on the line $\mathbf{p} + t \cdot \mathbf{v}_{\tau,\tau'}$ that lies in $\sigma \setminus \tau$ occurs when $t = m \cdot k$, where m is the minimum value of the set $\{a_1, a_2, \ldots, a_k\}$ and k is the number of vertices in τ. For smaller values of t, all of the a_i coordinates remain positive and the point is not yet in $\sigma \setminus \tau$. For larger values of t, at least one of the a_i coordinates is negative and the point lies outside the simplex σ.

Example 1.4 (Removing a proper face)**.** Two of these deformations are shown in Figure 1. In the top row τ is the edge ab, τ' is the edge cd and the direction of deformation is the vector from the midpoint of τ to the midpoint of τ'. In the bottom row, τ is the triangle abc, τ' is the point d and the direction of deformation is the vector from the center of τ to the point d.

2. Flag Complexes

In this section we prove that the star of a simplex in a flag complex is contractible (Proposition 2.6). This is surely a well-known fact but since we have been unable to locate a reference in the literature, we provide an easy proof using the deformation retractions described in Section 1.

Definition 2.1 (Stars and links)**.** Let ρ be a simplex in a simplicial complex X. The *star of* ρ is the smallest subcomplex $\text{Star}(\rho)$ in X which contains every simplex that has a nontrivial intersection with ρ. We distinguish three types of simplices in $\text{Star}(\rho)$ depending on whether it is contained in ρ, disjoint from ρ, or neither. A simplex τ disjoint from ρ is a *link simplex* and the set of all link simplices form a simplicial complex $\text{Link}(\rho)$ called the *link of* ρ. A simplex σ that is neither in the link nor contained in ρ is called a *connecting simplex*. Note that by the minimality condition in the definition of $\text{Star}(\rho)$, every link simplex τ is a proper subsimplex of some connecting simplex σ.

Recall that a simplical complex X is a *flag complex* when each complete graph in the 1-skeleton of X is the 1-skeleton of a simplex in X.

Lemma 2.2 (Maximal connecting simplices)**.** *Let ρ be a simplex in a flag complex X. For every simplex $\tau \in \text{Link}(\rho)$ there is a unique maximal*

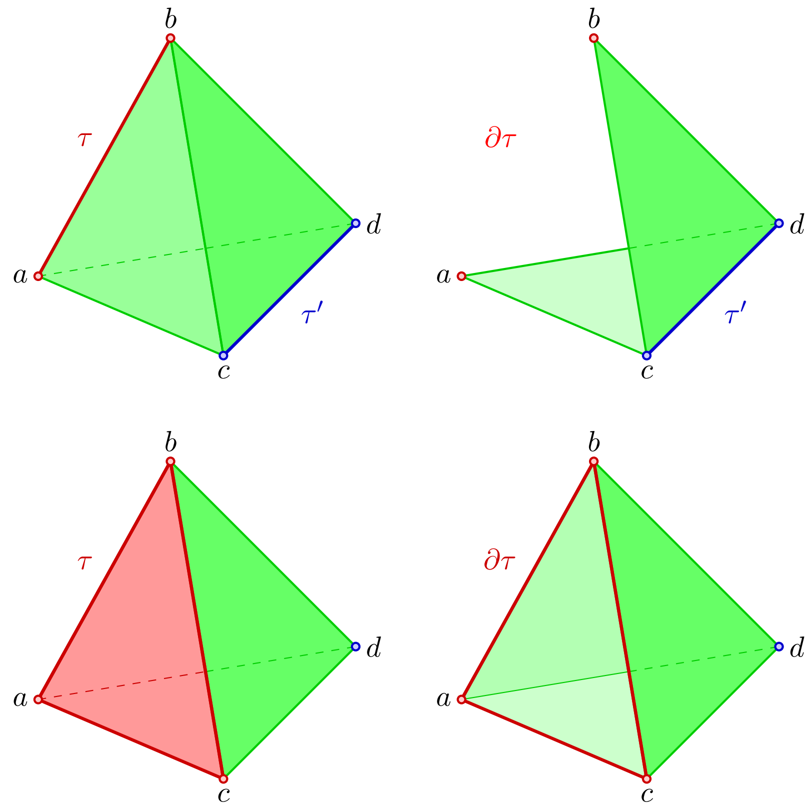

FIGURE 1. Let σ be the tetrahedron $abcd$. The top row shows the deformation retraction from σ to $\sigma \backslash \tau$ when τ is the edge ab. The bottom row shows the deformation retraction when τ is the triangle abc.

simplex $\sigma \in \textsc{Star}(\rho)$ *such that* $\sigma \cap \textsc{Link}(\rho) = \tau$. *In particular,* $\sigma' \in \textsc{Star}(\rho)$ *and* $\sigma' \cap \textsc{Link}(\rho) = \tau$ *implies* $\sigma' \subset \sigma$ *for this maximal* σ. *Moreover,* σ *is a connecting simplex that properly contains* τ.

Proof. Let V be the set of vertices in τ together with any vertex of ρ that is connected to every vertex of τ by an edge. Note that the restriction of the 1-skeleton of X to the vertex set V is a complete graph by the definition of V and by the fact that both τ and ρ are simplices. Because X is a flag complex, there is a simplex σ with V as its vertex set and its maximality properties are immediate. Finally, by the remark in Definition 2.1, there is some connecting simplex σ' that contains τ and by replacing σ' by a subsimplex if necessary we may also assume that the vertices of σ' are contained in the set V. Thus at least one vertex in ρ is connected to every vertex of τ, which means that σ is a connecting simplex properly containing τ. □

Definition 2.3 (Filtrations)**.** Let ρ be a simplex in a simplicial complex X, let $L = \textsc{Link}(\rho)$ and let $S = \textsc{Star}(\rho)$. For each integer k let $L^{(k)}$ be

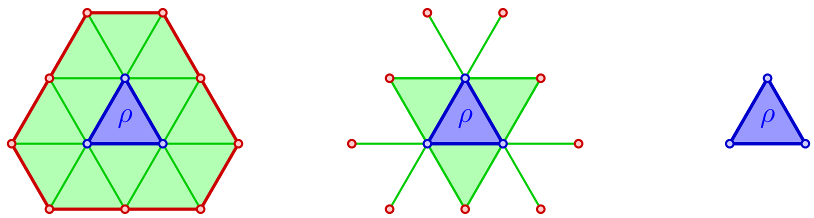

FIGURE 2. The natural filtration of the star of a triangle ρ in the standard triangular tiling of the plane.

the k-skeleton of L and let $S^{(k)}$ be the largest subcomplex of S such that $S^{(k)} \cap L = L^{(k)}$. Note that when $k = -1$, $L^{(-1)} = \emptyset$ and $S^{(-1)} = \rho$. These subcomplexes allow us to write L and S as nested unions $L = \cup_k L^{(k)}$ and $S = \cup_k S^{(k)}$ which we call the *natural filtrations* of L and S. We should note that when $1 \leq k < \dim(L)$, neither $L^{(k)}$ nor $S^{(k)}$ is a flag complex.

Example 2.4 (Filtrations)**.** Let ρ be a triangle in the standard triangular tiling of the plane. The natural filtration of the star of ρ is shown in Figure 2. The subcomplex $S^{(1)} = \text{STAR}(\rho)$ is shown on the left, the subcomplex $S^{(0)}$ is shown in the middle and the subcomplex $S^{(-1)} = \rho$ is shown on the right.

Lemma 2.5 (Local deformations)**.** *Let ρ be a simplex in a flag complex X and let $S = \cup_k S^{(k)}$ be the natural filtration of $S = \text{STAR}(\rho)$. For every $k \geq 0$ there is a deformation retraction from $S^{(k)}$ to $S^{(k-1)}$.*

Proof. Let $L^{(k)}$ be the k-skeleton of $\text{LINK}(\rho)$. For each k-simplex τ in $L^{(k)}$ with corresponding maximum connecting simplex σ as described by Lemma 2.2, we associate a deformation retraction from σ onto the subcomplex $\sigma \setminus \tau$ as in Proposition 1.2. In fact, we claim that the deformation retractions associated to each k-simplex in $L^{(k)}$ are compatible and can be performed simultaneously. To see this, note that the only points that move under the deformation from σ to $\sigma \setminus \tau$ are those contained in simplices that contain all of the vertices of τ and because τ is a maximal simplex in $L^{(k)}$, the only simplices in $S^{(k)}$ that contain τ are contained in σ by Lemma 2.2. Hence, the points moved by different deformations are pairwise disjoint. □

In Figure 2 it is easy to visualize the deformation retractions between one step in the natural filtration of the star of ρ and the next.

Proposition 2.6 (Stars contract)**.** *If ρ is a simplex in a flag complex X, then $\text{STAR}(\rho)$ is a contractible subcomplex.*

Proof. Let $\text{STAR}(\rho) = S = \cup_k S^{(k)}$ be the natural filtration of the star of ρ. By iteratively applying Lemma 2.5, we see that each subcomplex $S^{(k)}$ deformation retracts to the simplex $S^{(-1)} = \rho$ which is itself contractible. Since each $S^{(k)}$ is contractible, the star of ρ is a nested union of contractible simplicial complexes and thus contractible. □

Easy counterexamples show that the flag complex requirement is crucial.

Example 2.7 (Noncontracting stars)**.** If X is the boundary of a simplex with at least 3 vertices and ρ is any proper subsimplex containing at least 2 vertices, then the star of ρ is all of X, which is not contractible since it is homeomorphic to a sphere.

3. Noncrossing Hypertrees

In this section we apply Proposition 2.6 to the complex of noncrossing hypertrees. We follow [McC] and call two edges, or more generally two polygons, *weakly noncrossing* when they are disjoint or only intersect at a single common vertex. We begin with the notion of a noncrossing tree.

Definition 3.1 (Noncrossing trees)**.** A *tree* is a connected simplicial graph T with no nontrivial cycles. A *noncrossing tree* is a tree T whose vertices are the vertices of a convex polygon P, each edge is the convex hull of its endpoints and distinct edges of T are weakly noncrossing.

Noncrossing trees are examples of noncrossing hypertrees.

Definition 3.2 (Noncrossing hypertrees)**.** A *hypergraph* consists of a set V of vertices and a collection of subsets of V of size at least 2 whose elements are called *hyperedges*. A *hypertree*, roughly speaking, is a connected hypergraph with no hypercycles. See [McC] for a precise definition. A *noncrossing hypertree* is a hypertree whose vertices are those of a convex polygon P, each hyperedge is drawn as the convex hull of the vertices it contains, and distinct hyperedges are weakly noncrossing. The set of all noncrossing hypertrees on P is a poset under refinement. In particular, one hypertree is below another if every hyperedge of the first is a subset of some hyperedge of the second. In this ordering the noncrossing trees are its minimal elements and the noncrossing hypertree with only one hyperedge containing all of the vertices is its unique maximum element.

The lefthand side of Figure 3 is a noncrossing hypertree. Noncrossing hypertrees are in bijection with certain types of polygon dissections.

Definition 3.3 (Polygon dissections)**.** Let P be a convex polygon in the plane. A *diagonal of* P is the convex hull of two vertices of P that are not adjacent in the boundary cycle of the polygon. Two diagonals are *noncrossing* when they are weakly noncrossing as edges and a collection of pairwise noncrossing diagonals is called a *dissecton of* P. A dissection of an even-sided polygon P is itself *even-sided* if the diagonals partition P into smaller even-sided polygons. An example is shown on the righthand side of Figure 3. The collection of all even-sided dissections of an even-sided polygon P can be turned into a poset by declaring that one dissection is below another if the set of diagonals defining the first is a subset of the set of diagonals defining the second. Since ideals in this poset are boolean lattices, it corresponds to a simplicial complex. See [McC] for further details.

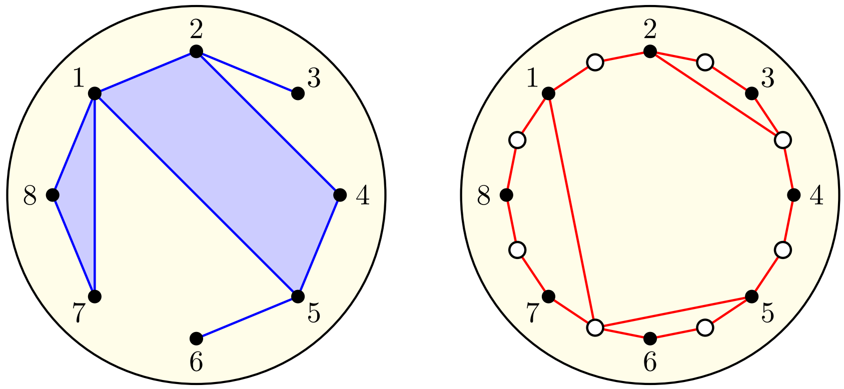

FIGURE 3. A noncrossing hypertree with 8 vertices and the corresponding even-sided dissection of a 16-gon.

There is a natural bijection between noncrossing hypertrees in an n-gon and the even-sided dissections of a $2n$-gon.

Theorem 3.4 (Noncrossing hypertrees and polygon dissections)**.** *There is a natural order-reversing bijection between the poset of noncrossing hypertrees on a fixed number of vertices and the poset of dissections of an even-sided polygon with twice as many vertices into even-sided subpolygons through the addition of pairwise noncrossing diagonals.*

Proof. [McC, Theorem 3.4]. □

Example 3.5 (Hypertrees and dissections)**.** The rough idea behind Theorem 3.4 is easy to describe. Given an even-sided subdivision of a $2n$-gon such as the one shown on the righthand side of Figure 3, we alternately color the vertices black and white and then create a hypertree using the convex hulls of the black vertices in each even-sided subpolygon. In the other direction we add a white dot in between every pair of vertices of the n-gon and then includes a diagonal corresponding to each pair of hyperedges that share a vertex and are adjacent in the linear ordering of hyperedges that share that vertex. The other end of the diagonal is the unique white dot to which it can be connected without crossing a hyperedge.

Definition 3.6 (Noncrossing hypertree complex)**.** Let P be a convex polygon. The identification given in Theorem 3.4 can be used to define a simplicial complex X called the *noncrossing hypertree complex of* P. Its simplices are the noncrossing hypertrees on P with more than one hyperedge. The vertices of X are labeled by the noncrossing hypertrees with exactly two hyperedges and these correspond to polygonal dissections with only one diagonal. At the other extreme, the maximal simplices of X are labeled by noncrossing trees.

From the polygon dissection viewpoint, the following result is immediate.

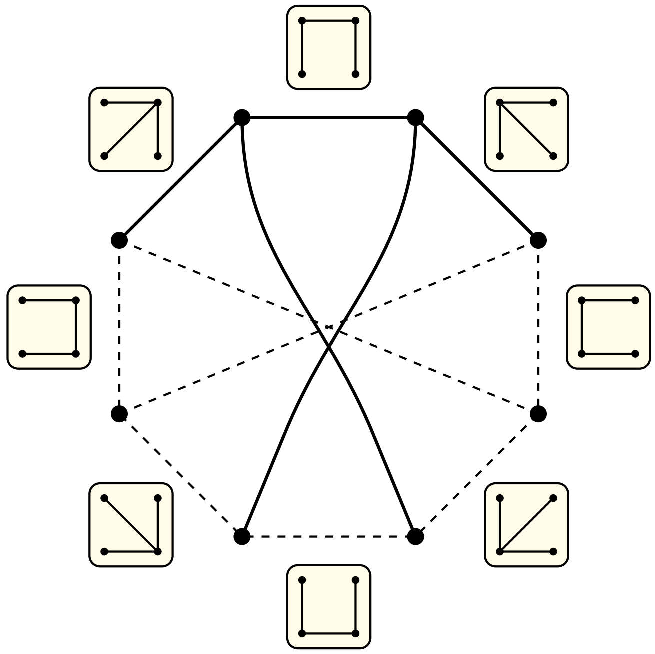

FIGURE 4. The noncrossing hypertree complex X of a square P. The edges of X are dashed when the tree T labeling the edge contains the boundary edge e along the bottom of the square P and solid when T does not contain e. The top edge of the figure is the edge labeled by the tree T_e.

Lemma 3.7 (Flag)**.** *The noncrossing hypertree complex is a flag complex.*

Proof. The vertices correspond to subdivisions containing a single diagonal and two vertices are connected by an edge iff the corresponding diagonals are noncrossing. In particular, any complete graph in the 1-skeleton has vertices labeled by dissections with diagonals that can be overlaid to obtain a new polygon dissection which contains all of them and this dissection corresponds to the noncrossing hypertree that labels the simplex with this complete graph as its 1-skeleton. □

Let P be a convex polygon. We call an edge e connecting two vertices of P a *boundary edge* if it is contained in the boundary of P.

Example 3.8 (Unused boundary edge)**.** Let P be a square with horizontal and vertical sides, let e be the bottom edge of P and let X be the noncrossing hypertree complex of P. In this case the complex X is the nonplanar graph with 8 vertices and 12 edges shown in Figure 4. The edges of X are labeled by noncrossing trees and the labels for 8 of these edges are shown. The remaining 4 edges in X are labeled by noncrossing trees that look like an N or a Z up to rotation and/or reflection. An edge has been drawn as a dashed line when its label contains the boundary edge e and it has been drawn as a solid line when its label does not contain e. Let T_e be the tree formed by the 3 sides of P other than e; T_e is the label for the edge along the top of the graph X, and the star of this edge is depicted as the subgraph formed by the solid edges.

The noncrossing trees that do not contain a particular boundary edge e always form a subcomplex that is the star of a simplex.

Lemma 3.9 (Unused boundary edge)**.** *Let P be a convex polygon and let X be the noncrossing hypertree complex of P. If e is a boundary edge of P and T_e is the noncrossing tree consisting of all boundary edges other than e, then the star of the simplex labeled by T_e in X contains the simplex labeled by a noncrossing tree T if and only if T does not contains the edge e.*

Proof. Let T is a noncrossing tree in P. First note that T does not contain the boundary edge e if and only if the corresponding polygon dissection contains a diagonal with one endpoint at the white dot between the endpoints of e. Moreover, the set of all of the diagonals ending at this white dot are pairwise noncrossing and form the polygon dissection that corresponds to the tree T_e. And since single diagonals label the vertices of the noncrossing hypertree complex, this means that T does not contain the boundary edge e if and only if the simplex labeled T contains a vertex of the simplex labeled T_e, which is equivalent to the simplex labeled T being in the star of the simplex labeled T_e. □

We are now ready to apply Proposition 2.6.

Theorem 3.10 (Unused boundary edge)**.** *The simplices of the noncrossing hypertree complex labeled by noncrossing trees that omit a fixed boundary edge e form a contractible subcomplex.*

Proof. By Lemma 3.7 the noncrossing hypertree complex is a flag complex and by Lemma 3.9 the subcomplex under discussion is the star of a simplex. Proposition 2.6 completes the proof. □

4. Noncrossing Partitions

In this section we review the connection between noncrossing partitions to noncrossing hypertrees and prove Theorem A.

Definition 4.1 (Noncrossing partitions)**.** Let P be a convex n-gon. A *noncrossing partition of P* is a partition of its vertex set so that the convex hulls of the blocks of the partition are pairwise disjoint. The noncrossing partitions of P are ordered by refinement so that one partition is below another if every block of the first is a subset of some block of the second. The result is a bounded graded lattice called the *noncrossing partition lattice.* Since the combinatorics only depend on the integer n and not on the choice of polygon P, the noncrossing partition lattice of P is denoted NC_n. The *noncrossing partition link* is the order complex of the noncrossing partition lattice with its bounding elements removed. This means that the maximal simplices in the noncrossing partition link are labeled by the maximal chains in the noncrossing partition lattice.

Definition 4.2 (Noncrossing permutations)**.** Noncrossing partitions in a convex polygon P can be reinterpreted as permutations of the vertices of P. To do this for a particular partition π simply send each vertex v of P to the next vertex in the clockwise order of the vertices in the boundary cycle of the convex hull of the block of π containing v. The result is called a *noncrossing permutation*. Next we assign a (noncrossing) permutation to each pair of comparable elements in the noncrossing partition lattice by multiplying the permutation associated to the bigger element by the inverse of the permutation associated to the smaller element. Under this procedure the permutations assigned to the covering relations are transpositions. And the collection of the transposition labels on the adjacent elements in a maximal chain correspond to the edges of a noncrossing hypertree in P [McC].

Definition 4.3 (Properly ordered noncrossing trees)**.** Let T be a noncrossing tree in a polygon P. When T comes equipped with a linear ordering of its edges, we call T an *ordered noncrossing tree*. The ordering is a *proper ordering* when it extends the local linear orderings of the edges that share any particular vertex [McC]. This is equivalent to saying that the product of the transpositions corresponding to the edges multiplied in this order produces the single cycle that corresponds to the boundary cycle of P.

Lemma 4.4 (Factorizations and trees)**.** *There are natural bijections between the maximal chains in the noncrossing partition lattice* NC_{n+1}*, factorizations of the* $(n+1)$*-cycle* $(1, 2, \ldots, n+1)$ *into* n *transpositions, and properly ordered noncrossing trees with* $n+1$ *vertices and* n *edges.*

Proof. [McC, Theorem 7.8, Corollary 7.9]. □

Since the maximal simplices of the noncrossing hypertree complex are labeled by noncrossing trees and the maximal simplices of the noncrossing partition link are labeled by *properly ordered* noncrossing trees, it is not too surprising that there is a close relationship between the two simplicial complexes. In fact, they are homeomorphic to each other, and this is one of the main results proved in [McC].

Theorem 4.5 (Hypertrees and partitions)**.** *For any convex polygon* P *there is a natural homeomorphism between the noncrossing partition link on* P *and the noncrossing hypertree complex on* P*. Moreover, the maximal simplices of the former labeled by the various proper orderings of a fixed noncrossing tree form a subcomplex that is identified with a single maximal simplex of the latter labeled by the common underlying noncrossing hypertree.*

Proof. See [McC, Section 8]. □

Theorem 4.5 allows us to identify subcomplexes of the noncrossing hypertree complex with their images in the noncrossing partition link. In

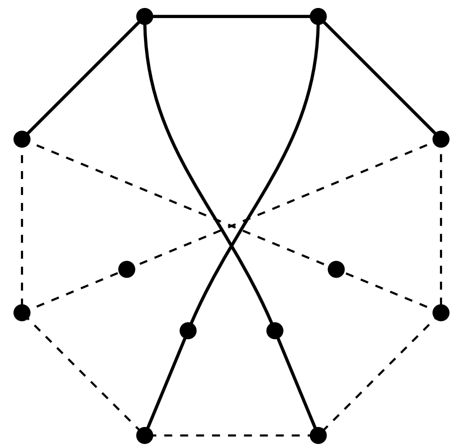

FIGURE 5. The noncrossing partition link of a square P. The edges of the link are dashed when the properly ordered noncrossing tree T labeling the edge contains the boundary edge e along the bottom of the square P and solid when T does not contain e.

particular, Theorem A is an immediate consequence of Theorem 3.10 and Theorem 4.5. We conclude this section with a concrete example to show what Theorem 4.5 looks like when the polygon P is a square.

Example 4.6 (Noncrossing partition link of a square)**.** Let P be a square with horizontal and vertical sides and let e be the bottom edge of P. The noncrossing partition link of P is a nonplanar graph with 12 vertices and 16 edges. See Figure 5. The edges of the complex are labeled by properly ordered noncrossing trees. Most noncrossing trees on 4 vertices have a unique proper ordering so the outside portion of the figure looks the same as in Figure 4. The noncrossing trees that look like an N or a Z up to rotation and/or reflection have two proper orderings. As a consequence the single edges crossing through the middle of the image in Figure 4 become two edges in Figure 5. An edge has been drawn as a dashed line when its label contains the boundary edge e and it has been drawn as a solid line when its label does not contain e. Let T_e be the tree formed by the 3 sides of P other than e. It has only one proper ordering and is the label for the edge along the top of the graph X, but note that the subgraph formed by the solid edges is no longer the star of the edge labeled T_e. It is, however, homeomorphic to the star of the edge labeled T_e in the noncrossing hypertree complex.

5. Parking Functions

In this section we recall Stanley's elegant bijection between parking functions and maximal chains in the noncrossing partition lattice and we use this to prove explicit versions of Theorems B and C.

Definition 5.1 (Parking functions and undesired spaces)**.** Recall that an n-tuple $(a_1, \ldots, a_n)$ of positive integers is called a *parking function* if, once its entries have been sorted into weakly increasing order, the i-th entry is at most i. The set of all parking functions of length n is denoted PF_n. In [BDH+16] the first author and his coauthors introduced a filtration of parking functions by undesired spaces. An integer k is called an *undesired parking space* for a specific parking function $(a_1, \ldots, a_n) \in \mathrm{PF}_n$ if k does not appear in this n-tuple. The language refers to the model where the i-th car wants to park in the a_i-th parking space on a one-way street. Let $\mathrm{PF}_{n,k}$ be the set of parking functions of length n with the property that k is the largest undesired parking space. Notice that every parking function is either a permutation of the numbers 1 through n or it belongs to $\mathrm{PF}_{n,k}$ for some unique positive integer k.

Definition 5.2 (Stanley's bijection)**.** Let P be a convex $(n+1)$-gon with vertex labels $1, \ldots, n+1$ in clockwise-increasing order and note that the noncrossing permutation associated to trivial partition with only one block is the $(n+1)$-cycle $(1, 2, \ldots, n+1)$. For each factorization of c into n transpositions, we can read off the smallest number in each transposition to produce an n-tuple with entries in the set $\{1, 2, \ldots, n\}$. Although it is not immediately obvious, Richard Stanley proved that this procedure establishes a natural bijection between the maximal chains in NC_{n+1} thought of as factorizations of c into n transpositions, and the set PF_n of length n parking functions [Sta97]. By Lemma 4.4 the set PF_n is also in bijection with the set of properly ordered noncrossing trees on $n+1$ vertices.

The parking functions where the last space is undesired can be characterized as those factorizations that avoid a particular boundary transposition.

Lemma 5.3 (Trees and parking functions)**.** *Let P be a convex $(n+1)$-gon with vertex labels $1, \ldots, n+1$ in clockwise-increasing order and let e be the boundary edge connecting the vertices labeled n and $n+1$. The properly ordered noncrossing trees that omit e correspond to parking functions in $\mathrm{PF}_{n,n}$, i.e. the parking functions where no car wants to park in the last parking space.*

Proof. Under Stanley's bijection (Definition 5.2), it is clear that n occurs in the parking function if and only if the transposition $(n, n+1)$ occurs in the factorization of the $(n+1)$-cycle $c = (1, 2, \ldots, n+1)$ into n transpositions, and this is true if and only if the corresponding properly ordered noncrossing tree contains the boundary edge e (Lemma 4.4). $\square$

If we consider the Hasse diagram of a finite bounded poset as a directed graph whose vertices are its elements and with a directed edge for each covering relation, then the maximal chains in the poset correspond to directed paths from its unique minimum element to its unique maximum element.

Definition 5.4 (Parking functions, posets and subcomplexes)**.** For each subset $A \subset \mathrm{PF}_n$ we define a poset $\textsc{Poset}(A)$ as the poset whose Hasse diagram is the union of the directed paths that correspond to the parking functions in A under Stanley's bijection. For any bounded poset P we write $\textsc{Link}(P)$ for the order complex of P with its bounding elements removed. With this definition, the complex $\textsc{Link}(\textsc{Poset}(A))$ is the subcomplex of the noncrossing partition link $\textsc{Link}(\mathrm{NC}_{n+1})$ formed by the simplices labeled by parking functions in A.

The following result is an explicit version of Theorem B.

Theorem 5.5 (Undesired last parking space)**.** *The simplices labeled by the parking functions in* $\mathrm{PF}_{n,n}$ *form a subcomplex* $\textsc{Link}(\textsc{Poset}(\mathrm{PF}_{n,n}))$ *in the noncrossing partition link* $\textsc{Link}(\mathrm{NC}_{n+1})$ *that is contractible.*

Proof. This is an immediate consequence of Theorem A and Lemma 5.3. □

In order to extend this result to the other collections $\mathrm{PF}_{n,k}$ we need to quote a decomposition result from [BDH+16].

Lemma 5.6 (Decomposition)**.** *For all positive integers* $k + \ell = n$ *with* $k > 1$*, the poset* $\textsc{Poset}(\mathrm{PF}_{n,k})$ *is isomorphic to the product* $\textsc{Poset}(\mathrm{PF}_{k,k}) \times \textsc{Bool}_\ell$*, where* $\textsc{Bool}_\ell$ *is the Boolean lattice of rank* ℓ*.*

Proof. [BDH+16, Theorem 3.5]. □

When a poset splits as a direct product of two smaller posets, the link of the whole can be constructed from the link of each factor.

Lemma 5.7 (Links and products)**.** *If* $P = P_1 \times P_2$ *is a bounded graded poset that splits as a product of smaller bounded graded posets, then* $\textsc{Link}(P)$ *is homeomoprhic to* $\textsc{Link}(P_1) * \textsc{Link}(P_2)$*, the spherical join of the smaller links. As a consequence, when* $\textsc{Link}(P_1)$ *is contractible, so is* $\textsc{Link}(P)$*.*

Proof. The first assertion is an easy exercise. For the second assertion let $L_i = \textsc{Link}(P_i)$. Since L_1 is contractible, it is homotopy equivalent to a point. Thus $L_1 * L_2$ is homotopy equivalent to the spherical join of a point and L_2, which is homeomorphic to the cone on L_2, which is contractible. □

Using Lemma 5.6 and Lemma 5.7 we prove the following explicit version of Theorem C.

Theorem 5.8 (Undesired parking space)**.** *For all integers* $1 < k \leq n$*, the simplices labeled by the parking functions in* $\mathrm{PF}_{n,k}$ *form a contractible subcomplex of the noncrossing partition link* $\textsc{Link}(\mathrm{NC}_{n+1})$*.*

Proof. For $k = n$, this follows from Theorem 5.5. For $1 < k < n$, let $\ell = n - k > 0$. By Lemma 5.6 the poset $\textsc{Poset}(\mathrm{PF}_{n,k})$ splits as a product of $\textsc{Poset}(\mathrm{PF}_{k,k})$ and the Boolean lattice $\textsc{Bool}_\ell$ and since $\textsc{Link}(\textsc{Poset}(\mathrm{PF}_{k,k}))$ is contractible by Theorem 5.5, $\textsc{Link}(\textsc{Poset}(\mathrm{PF}_{n,k}))$ is contractible by Lemma 5.7. □

References

[BDH+16] Melody Bruce, Michael Dougherty, Max Hlavacek, Ryo Kudo, and Ian Nicolas. A decomposition of parking functions by undesired spaces. *Electron. J. Combin.*, 23(3):Paper 3.32, 22, 2016.

[BM10] Tom Brady and Jon McCammond. Braids, posets and orthoschemes. *Algebr. Geom. Topol.*, 10(4):2277–2314, 2010.

[DMW] Michael Dougherty, Jon McCammond, and Stefan Witzel. Braid groups are nonpositively curved. In preparation.

[HKS16] Thomas Haettel, Dawid Kielak, and Petra Schwer. The 6-strand braid group is CAT(0). *Geom. Dedicata*, 182:263–286, 2016.

[McC] Jon McCammond. Noncrossing hypertrees. Preprint.

[Müh] Henri Mühle. Two posets of noncrossing partitions coming from undesired parking spaces. Preprint `arXiv:1701.02109 [math.CO]`.

[Sta97] Richard P. Stanley. Parking functions and noncrossing partitions. *Electron. J. Combin.*, 4(2):Research Paper 20, approx. 14, 1997. The Wilf Festschrift (Philadelphia, PA, 1996).

E-mail address: `dougherty@math.ucsb.edu`

Department of Mathematics, UC Santa Barbara, Santa Barbara, CA 93106

E-mail address: `jon.mccammond@math.ucsb.edu`

Department of Mathematics, UC Santa Barbara, Santa Barbara, CA 93106