THE INFINITE CYCLOHEDRON AND ITS AUTOMORPHISM GROUP

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Abstract. Cyclohedra are a well-known infinite familiy of finite-dimensional polytopes that can be constructed from centrally symmetric triangulations of even-sided polygons. In this article we introduce an infinite-dimensional analogue and prove that the group of symmetries of our construction is a semidirect product of a degree 2 central extension of Thompson's infinite finitely presented simple group T with the cyclic group of order 2. These results are inspired by a similar recent analysis by the first author of the automorphism group of an infinite-dimensional associahedron.

Associahedra and cyclohedra are two well-known families of finite-dimensional polytopes with one polytope of each type in each dimension. In [Fos] the first author constructed an infinite-dimensional analogue As^∞ of the associahedra As^n and proved that its automorphism group is a semidirect product of Richard Thompson's infinite finitely presented simple group T with the cyclic group of order 2. In this article we extend this analysis to an infinite-dimensional analogue Cy^∞ of the cyclohedra Cy^n. Its symmetry group is a semidirect product of a degree 2 central extension of Thompson's group T with the cyclic group of order 2. In atlas notation (which we recall in Section 5), the symmetry group of the infinite associahedron has shape $T.2$ and the symmetry group of the infinite cyclohedron has shape $2.T.2$.

The article is structured as follows. The first two sections recall basic properties of associahedra and cyclohedra, especially the way their faces correspond to partial triangulations of convex polygons. Section 3 reviews the Farey tessellation of the hyperbolic plane as a convenient tool for organizing our constructions. The infinite-dimensional polytopes As^∞ and Cy^∞ are constructed in Section 4, and the final sections use the close connection between Thompson's group T and the Farey tessellation to analyze the symmetry group of Cy^∞.

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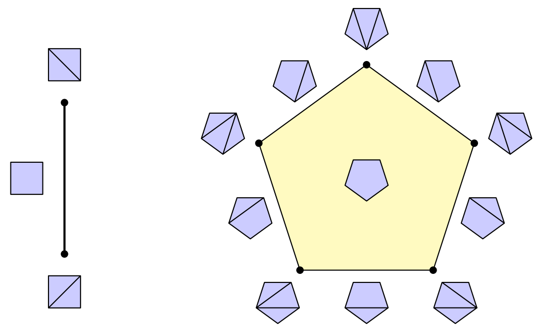

FIGURE 1. The low dimensional associahedra As^1 and As^2.

1. Associahedra

In this section we recall the basic properties of associahedra. We begin with the construction of their face lattice.

Definition 1.1 (Diagonals)**.** Let P be a convex polygon with n vertices. Of the $\binom{n}{2}$ edges connecting distinct vertices of P, n of them are *boundary edges* connecting vertices that occur consecutively in the boundary of P. The other edges are called *diagonals*. Two diagonals are said to *cross* if they intersect in the interior of P and they are *noncrossing* otherwise, even if they have an endpoint in common.

Definition 1.2 (Triangulations)**.** A *partial triangulation of* P is given by its boundary edges plus a possibly empty collection of pairwise noncrossing diagonals. A *triangulation of* P is a maximal partial triangulation, a condition equivalent to the requirement that every complementary region is a triangle. The number of triangulations is given by the Catalan numbers. We order the partial triangulations of P by reverse inclusion so that the triangulations are minimal elements and the partial triangulation with no diagonals is the unique maximal element.

Although it is not immediately obvious, it turns out that the poset of partial triangulations of a convex n-sided polygon is the face lattice of a simple $(n-3)$-dimensional polytope called an *associahedron*. In low-dimensions we see that a square has two triangulations, a pentagon has five and a hexagon has fourteen. The corresponding 1-, 2-, and 3-dimensional associahedra with 2, 5 and 14 vertices, respectively, are shown in Figures 1 and 2. Although there have been many distinct polytopal realizations of associahedra over the years, we confine ourselves to a brief description of one particular construction: the associahedron as the secondary polytope of a regular convex polygon, a very general construction developed by Gelfand, Kapranov and Zelevinsky [GZK89, GZK90, GKZ94].

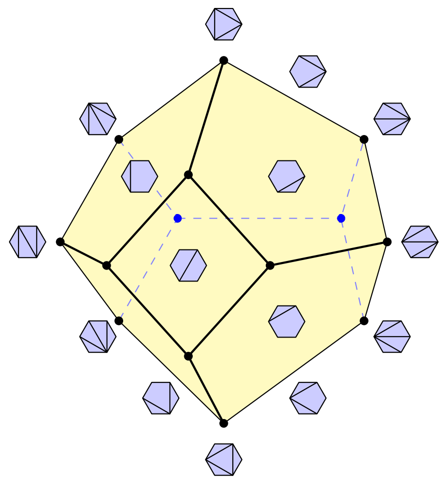

FIGURE 2. The 3-dimensional associahedron As^3 with a some of its face labels.

Definition 1.3 (Secondary polytopes)**.** The general construction of a secondary polytope goes as follows. Let P be a d-dimensional convex polytope with $n + d + 1$ vertices. For each triangulation t of P, one defines a *GKZ vector* $v(t) \in \mathbb{R}^{n+d+1}$ as the vector whose i-th coordinate is the sum of the volumes of the top-dimensional simplices in the triangulation t that contain the i-th vertex of P. The convex hull of the GKZ vectors for all possibly triangulations of P produces an n-dimensional polytope called its *secondary polytope*.

In their articles, Gelfand, Kapranov and Zelevinsky prove that the secondary polytope of a convex polygon is an associahedron. The reader should note that the indices shift during the construction: the regular polygon with n vertices produces the associahedron of dimension $n - 3$. Secondary polytopes were later generalized by Billera and Sturmfels to produce fiber polytopes [BS92] and both of these constructions are described in Ziegler's book on polytopes [Zie95]. For a survey of other polytopal realizations of associahedra and for further references to the literature, see the excellent recent article by Ceballos, Santos and Ziegler [CSZ]. In this article, we shall insist that our associahedra are constructed as metric secondary polytopes from regular polygons of a specific size.

Definition 1.4 (Associahedra)**.** A *standard regular polygon* is one obtained as the convex hull of equally spaced points on the unit circle (i.e. a regular polygon circumscribed by the unit circle) and a *standard associahedron* is the metric polytopal realization of the associahedron obtained by applying the secondary polytope construction to a standard regular polygon. We write As^n to denote the standard associahedron of dimension n, following the convention that uses superscripts to indicate the dimension of a polytope.

By standardizing metrics in this way, the standard associahedra are symmetric polytopes and there exist some isometric embeddings between them.

Proposition 1.5 (Isometries and embeddings)**.** *Every isometry of the standard regular polygon P_n extends to an isometry of the standard associahedron As^{n-3}, which means that the dihedral group of size $2n$ acts isometrically on As^{n-3}. In addition, the natural inclusion map $P_n \hookrightarrow P_{2n}$ which sends the vertices of P_n to every other vertex of P_{2n} induces an isometric inclusion of As^{n-3} into As^{2n-3} as one of its faces.*

Proof. The first assertion is immediate from the definition of the construction. For the second assertion we identify the (partial) triangulations of P_n with (partial) triangulations of P_{2n} that include the diagonals that are images of the boundary edges of P_n. In the coordinates of the construction, this involves including $\mathbb{R}^n$ into $\mathbb{R}^{2n}$ and translating by a vector which records the contributions of the n triangles that are outside the image of P_n. Since the same n exterior triangles always occur, this is a pure translation and the embedding is an isometry. □

Remark 1.6 (Other isometric embeddings)**.** There exists an isometric embedding of As^{n-3} into As^{m-3} whenever n divides m, but when $\frac{m}{n}$ is greater than 2, one needs to first choose a consistent triangulation of the portion of P_m exterior to the image of P_n in order to define the map. In other words, there are multiple isometric embeddings of As^{n-3} into As^{m-3} in this case as opposed to the canonical embedding described in the proposition.

2. Cyclohedra

In this section we shift our attention from associahedra to cyclohedra. The cyclohedra were initially investigated by Raul Bott and Clifford Taubes in [BT94] in a slightly different guise before being rediscovered by Rodica Simion in [Sim03]. In the same way that the faces of an associahedron correspond to (partial) triangulations of a polygon, the faces of a cyclohedron correspond to centrally symmetric (partial) triangulations of an even-sided polygon.

Definition 2.1 (Centrally symmetric triangulations)**.** Let $P = P_{2n}$ be a standard $2n$-sided polygon and consider the π-rotation of P that

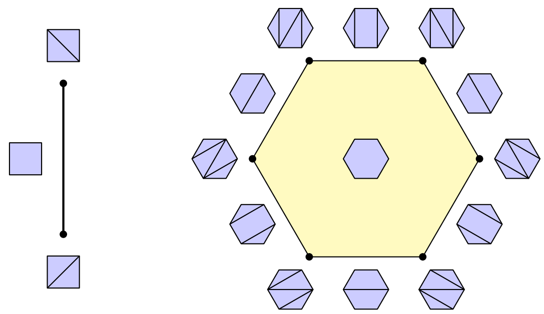

FIGURE 3. The low-dimensional cyclohedra Cy^1 and Cy^2.

sends the vertex v_i to the vertex v_{i+n} (where vertices are numbered modulo $2n$ in the order they occur in the boundary of P). As an isometry it sends every vector to its negative, i.e. it sends $x + iy$ to $-x - iy$. A partial triangulation of P is said to be *centrally symmetric* when it is invariant under this π-rotation. The centrally symmetric partial triangulations form a poset as before under reverse inclusion on the sets of diagonals used to define them. The minimal elements of this poset are the *centrally symmetric triangulations*.

The poset of centrally symmetric partial triangulations of P_{2n} is the face lattice of a simple $(n-1)$-dimensional polytope called a *cyclohedron*.

Definition 2.2 (Cyclohedra)**.** As should be clear from the definitions, the vertices of the cyclohedron can be identified with a subset of the vertices of the associahedron built from the same regular polygon. More precisely, the triangulations of a standard polygon with $2n$ sides index the vertices of the associahedron As^{2n-3} and the subset of centrally symmetric ones index the vertices of cyclohedron whose dimension is merely $n-1$. An additional reason for using the secondary polytope construction to build the associahedron As^{2n-3} is that the convex hull of its vertices with centrally symmetric labels is a polytopal realization of the $(n-1)$-dimensional cyclohedron. We call this the *standard cyclohedron* in dimension $n-1$ and we write Cy^{n-1} for this polytope.

We should note that for many of the polytopal realizations of the associahedron, the convex hull of the vertices with centrally symmetric labels is not a cyclohedron. The convex hull has a different combinatorial type and often a different and higher dimension [HL07]. Using the secondary polytope metric, the cyclohedron Cy^{n-1} inside the associahedron As^{2n-3} can also be described in terms of its isometries.

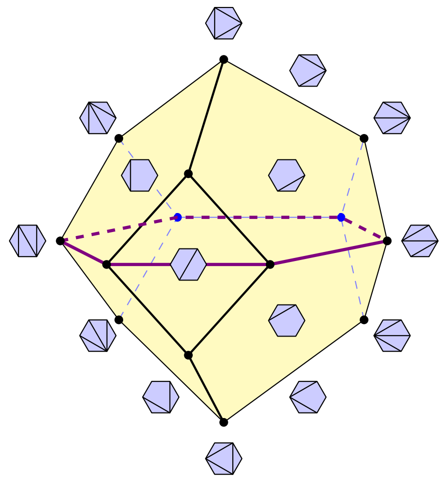

FIGURE 4. The involution τ acts on the 3-dimensional associahedron As^3 as a vertical reflection through the equator. The 2-dimensional hexagonal cyclohedron Cy^2 is visible as the intersection of As^3 with the fixed set of τ.

Proposition 2.3 (Fixed subpolytope)**.** *Let P be a standard $2n$-sided polygon and let As^{2n-3} be the standard associahedron constructed from the triangulations of P. The π-rotation of P induces an isometric involution τ that acts on $\mathbb{R}^{2n}$ by permuting coordinates, it stablizes $\mathsf{As}^{2n-3} \subset \mathbb{R}^{2n}$ and the intersection of the fixed set of τ with As^{2n-3} is the cyclohedron Cy^{n-1}.*

The sharp drop in dimension from the associahedron to the cyclohedron is caused by the geometry of the involution τ.

Remark 2.4 (Dimension drop)**.** In the secondary polytope coordinate system, the involution τ systematically switches coordinates x_i and x_{i+n}. Thus, geometrically, it fixes an n-dimensional subspace of $\mathbb{R}^{2n}$ and it acts as the antipodal map on its n-dimensional orthogonal complement. The associahedron As^{2n-3} lives in an affine subspace of $\mathbb{R}^{2n}$ of dimension $2n-3$ and τ fixes its center. If we translate the coordinate system so that the center of As^{2n-3} is the new origin and restrict τ to the subspace containing As^{2n-3}, then τ fixes a subspace of dimension $n-1$ and acts as the antipodal map on its orthogonal complement of dimension $n-2$. Hence the dimension of the corresponding cyclohedron.

In low-dimensions we see that both triangulations of a square are centrally symmetric so that Cy^1 and As^1 are identical. A hexagon has exactly six centrally symmetric triangulations plus six centrally symmetric partial triangulations, so Cy^2 is a hexagon. See Figure 3. We can also see Cy^2 as a polytope contained in the associahedron As^3. See Figure 4. The top and bottom vertices of As^3 are swapped by τ and the ones fixed by τ are the six vertices that appear to be on the equator relative to the north/south pole through the top and bottom vertices. The 3-dimensional cyclohedron Cy^3 is embedded in the 5-dimensional associahedron As^5. Its vertices are labelled by the twenty centrally symmetric triangulations of a regular octogon, it has thirty edges and twelve 2-dimensional faces of which four are squares, four are pentagons and four are hexagons.

As an immediate consequence of Proposition 2.3 we have the following, whose proof is identical to that of Proposition 1.5.

Proposition 2.5 (Isometries and embeddings)**.** *Every isometry of the standard regular polygon P_{2n} extends to an isometry of the standard cyclohedron Cy^{n-1}, which means that the dihedral group of size $4n$ acts isometrically on Cy^{n-1}. In addition, the natural inclusion map $P_{2n} \hookrightarrow P_{4n}$ which sends the vertices of P_{2n} to every other vertex of P_{4n} induces an isometric inclusion of Cy^{n-1} into Cy^{2n-1} as one of its faces.*

3. Farey tessellations

In this section we describe the well-known Farey tessellation of the hyperplane plane and some slight variations. We begin with the traditional version.

Definition 3.1 (Rational Farey tessellation)**.** As is well-known, the group $PSL_2(\mathbb{Z})$ naturally acts on circle $\mathbb{R}\mathrm{P}^1$ sitting inside the 2-sphere $\mathbb{C}\mathrm{P}^1$ and it acts transitively on points in $\mathbb{Q}\mathrm{P}^1$. In addition, the action preserves the orientation of the circle $\mathbb{R}\mathrm{P}^1$ and thus stabilizes each hemisphere of $\mathbb{C}\mathrm{P}^1$. In other words, $PSL_2(\mathbb{Z})$ acts on the upper-half plane model of the hyperbolic plane. The orbit of the bi-infinite geodesic connecting the points $0 = \frac{0}{1}$ and $\infty = \frac{1}{0}$ in the boundary under this action divides the plane into geodesics and ideal triangles. This is what we call the *rational Farey tessellation of the upper-half plane* $\mathbb{U}$ and a portion of it is shown in Figure 5.

The dyadic Farey tessellation of the upper-half plane $\mathbb{U}$ is combinatorially equivalent to the rational Farey tessellation, but metrically distinct.

Definition 3.2 (Dyadic Farey tessellation)**.** The *dyadic Farey tessellation of the upper-half plane* $\mathbb{U}$ is formed by first connecting each integer n in the boundary to $\infty = \frac{1}{0}$ and to the integer $n+1$. Afterwards, intervals in the boundary are iteratively evenly subdivided and additional

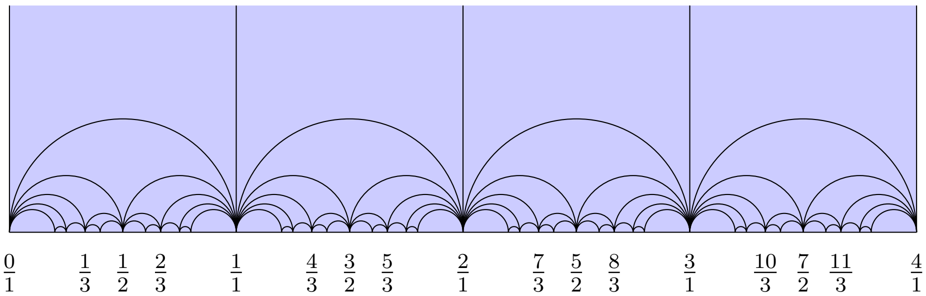

Figure 5. A portion of the rational Farey tessellation of the upper-half plane $\mathbb{U}$. In the interval between 0 and 4, the arcs where both endpoints have denominator at most 5 are drawn and the points with denominator at most 3 are labeled.

ideal triangles are drawn. See Figure 6. Instead of having all rational numbers as endpoints, the modified tessellation has endpoints at the dyadic rationals $\mathbb{Z}[\frac{1}{2}]$ plus ∞.

Although this version looks more symmetric than the rational Farey tessllation, it is the rational version which is invariant under a vertex transitive group of hyperbolic isometries.

Remark 3.3 (Literature)**.** Both versions have been used to study topics related to the focus of this article. The dyadic version of the Farey tessellation is closely related to the Belk and Brown forest diagrams for elements of Thompson's group F [BB05] and the rational version is at the heart of the cluster algebra structure on the hyperbolic plane that Sergey Fomin, Michael Shapiro, and Dylan Thurston produced by a series of edge-flips [FST08]. An edge-flip is a modification of a triangulation which removes a edge and then adds in the other diagonal of the resulting quadrilateral. Finite sequences of edge-flips lead to what one might call a *finite retriangulation*. It is also worth noting along these lines that the theory of cluster algebras of finite type is a major source of polytopal realizations of both associahedra and cyclohedra, although they are distinct from the realizations that we are using.

Of more immediate interest are the analogous tessellations of the disc model.

Definition 3.4 (Tessellations of the disc)**.** The *rational Farey tessellation of the unit disc* $\mathbb{D}$ is the image of the rational Farey tessellation of the upper-half plane model when translated to the Poincaré disc model by the linear fractional transformation that sends 0, 1 and ∞

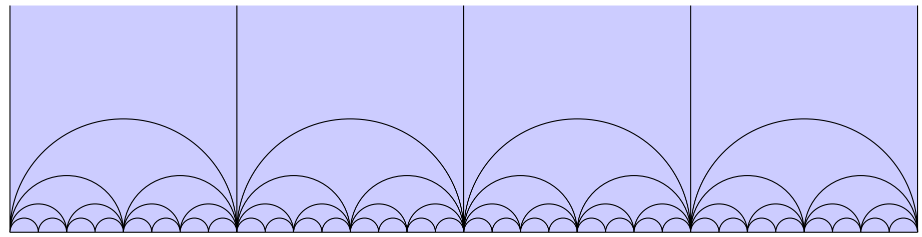

FIGURE 6. A portion of the dyadic Farey tessellation of the upper-half plane $\mathbb{U}$. In the interval between 0 and 4, the arcs where both endpoints have denominator at most 8 are drawn and the points with denominator at most 4 are labeled.

to 1, i and -1, respectively. Under this transformation, the point $\frac{1}{3}$ is sent to the point $\frac{3}{5} + \frac{4}{5}i$, which lies on the unit circle precisely because $3^2 + 4^2 = 5^2$, and more generally this transformation establishes a bijection between the points in $\mathbb{Q}\mathrm{P}^1$ and the rational points on the unit circle viewed as rescaled Pythagorean triples. The *dyadic Farey tessellation of the unit disc* $\mathbb{D}$ is a combinatorially equivalent but metrically distinct tessellation where the modifications are similar to the ones described above. After placing the inital geodesic connecting 1 and -1, additional boundary points are added by evenly dividing boundary arcs and adding in a new ideal triangle. We call the arcs that occur in the dyadic Farey tessellation, *dyadic Farey arcs*. Both the rational and the dyadic Farey tessellations of the unit disc $\mathbb{D}$ are shown in Figure 7. As with the upper-half plane tessellations, the dyadic version appears more symmetric, but it is the rational version which is invariant under a vertex transitive group of hyperbolic isometries.

Remark 3.5 (Tessellations and triangulations)**.** The Farey tessellation, in any of its various forms, can also be viewed as a 2-dimensional simplicial complex. One adds a vertex for each endpoint of a bi-infinite geodesic (indexed by $\mathbb{Q}P^1 = \mathbb{Q} \cup \{\infty\}$ in the rational case and by $\mathbb{Z}[\frac{1}{2}] \cup \{\infty\}$ in the dyadic case), an edge for every bi-infinite geodesic and a triangle for each complementary region. Note that each vertex link in this 2-complex looks like the standard simplicial structure on $\mathbb{R}$ with one vertex for each integer and edges connecting adjacent integers. The group of combinatorial automorphisms of the reals with this cell structure is the infinite dihedral group and the combinatorial automorphisms of the full 2-complex associated to the Farey tessellation

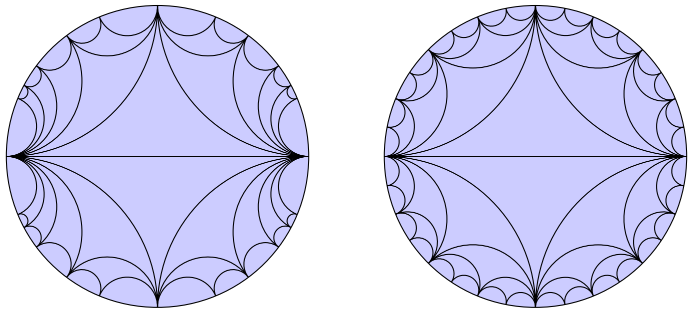

FIGURE 7. A portion of the rational Farey tessellation of the unit disc $\mathbb{D}$ is shown on the left and a portion of the dyadic Farey tessellation of the unit disc $\mathbb{D}$ is shown on the right.

is the non-oriented version of $PSL_2(\mathbb{Z}) \rtimes \mathbb{Z}_2$ which can be identified with $PGL_2(\mathbb{Z})$. One way to see this identification is as follows. The natural action of $PGL_2(\mathbb{Z})$ on $\mathbb{C}P^1$ is orientation-preserving. The images of the positive imaginary axis under this action form two copies of the Farey tessellation, one in the upper-half plane and another in the lower-half plane. If we identify numbers with their complex conjugates, then $PGL_2(\mathbb{Z})$ acts on the upper-half plane and those elements represented by matrices with a negative determinant become the orientation-reversing isometries of the Farey tessellation.

And finally, notice that when the dyadic Farey tessellation is depicted in the Klein disc model of the hyperbolic plane, it looks like a nested union of standard n-sided regular polygons where $n = 2^k$ for all $k > 1$. It is this close connection between standard regular polygons and the dyadic Farey tessellation of the unit disc which enables us to construct an infinite-dimensional version of a cyclohedron and to compute the structure of its automorphism group.

4. Infinite-dimensional polytopes

In this section we define what we mean by an infinite-dimensional polytope and illustrate the definition by constructing the polytopes that we call As^∞ and Cy^∞.

Definition 4.1 (Infinite-dimensional polytopes)**.** Let $n_1 < n_2 < \cdots$ be an increasing list of positive integers, let P^{n_i} be an n_i-dimensional metric polytope and for each i select a specific inclusion map $P^{n_i} \hookrightarrow P^{n_{i+1}}$ which isometrically embeds P^{n_i} as a face of $P^{n_{i+1}}$. Finally, let P^∞ denote the union of such a nested sequence. We call P^∞ an *infinite-dimensional polytope.*

$$\begin{tikzcd}
\mathsf{As}^1 \arrow[r, hook] \arrow[d, equal] & \mathsf{As}^5 \arrow[r, hook] & \mathsf{As}^{13} \arrow[r, hook] & \cdots \arrow[r, hook] & \mathsf{As}^{2n-3} \arrow[r, hook] & \cdots \arrow[r, phantom, "="] & \mathsf{As}^\infty \\
\mathsf{Cy}^1 \arrow[r, hook] & \mathsf{Cy}^3 \arrow[r, hook] \arrow[u, hook] & \mathsf{Cy}^7 \arrow[r, hook] \arrow[u, hook] & \cdots \arrow[r, hook] & \mathsf{Cy}^{n-1} \arrow[r, hook] \arrow[u, hook] & \cdots \arrow[r, phantom, "="] & \mathsf{Cy}^\infty \arrow[u, hook]
\end{tikzcd}$$

FIGURE 8. The nested embeddings of finite-dimensional polytopes used to construct As^∞ and Cy^∞. The integer n in the diagram ranges over the proper powers of 2.

We believe the name is reasonable because these infinite-dimensional polytopes share many properties with finite-dimensional polytopes. For example, infinite-dimensional polytopes as defined here are the convex hull of a countable set of points in a countable direct sum of copies of $\mathbb{R}$ with only finitely many points in any finite-dimensional affine subspace. Also, the union P^∞ has a well-defined set of faces and its face lattice is the union of the nested face lattices for the P^{n_i}. Moreover, in the same way that every finite polytope can be viewed as a contractible regular cell complex with one cell for each element of the face lattice, P^∞ is a contractible regular cell complex with one cell for each element in its face lattice. Contractibility of the union is just a special case of the standard fact that a cell complex constructed as a nested union of contractible cell complexes is itself contractible. (Recall the standard proof: any map of a sphere into the union has a compact image which is contained in a finite subcomplex which is contained in one of the finite contractible stages in the union, and thus this map is homotopic to a constant map. This means that all of the homotopy groups of the union are trivial and by Whitehead's theorem the union is contractible.)

Definition 4.2 (As^∞ and Cy^∞)**.** For each integer $n = 2^k$ with $k > 1$ we consider the standard regular n-sided polygon and the corresponding associahedra As^{2n-3} and cyclohedra Cy^{n-1}. As a consequence of Propositions 1.5, 2.3, and 2.5 we have the isometric embeddings shown in Figure 8 and the unions of these nested embeddings define infinite dimensional polytopes that we call the *infinite associahedron* As^∞ and the *infinite cyclohedron* Cy^∞.

From this construction as a union of polytopes, it should be clear that the vertices of As^∞ are indexed by finite retriangulations of the unit disc (in the sense described in Remark 3.3) where all but finitely many of the arcs of the triangulation are dyadic Farey arcs, and the vertices of Cy^∞ are those labeled by such finite retriangulations that are invariant under the π-rotation of the disc. In fact, as in the finite-dimensional case, the π-rotation of the disc induces an involution of As^∞ and the portion of As^∞ fixed by this involution is Cy^∞.

5. Thompson's group T

In this section we recall the definition of Richard Thompson's infinite finitely presented simple group T and its close connection with the Farey tessellation of the hyperbolic plane. In 1965 Richard Thompson defined three groups conventionally denoted F, T and V. The group F is the easiest to define and T and V were the first known example of infinite finitely-presented simple groups [CFP96]. Thompson's group F is the group of piecewise-linear homeomorphisms of the unit interval where every slope is a power of 2 and all of the break points occur at dyadic rationals. This group is also often studied in terms of pairs of rooted finite planar binary trees. Thompson's group T is an extension of F where the endpoints of the interval are identified and certain rotations are allowed. Since we do not need a detailed description for our main result, we shall merely sketch the key ideas.

Definition 5.1 (Thompson's group T)**.** For our purposes, the main result we need about Thompson's group T is that its elements can be identified with pairs of ideal Farey m-gons in the dyadic Farey tessellation of the unit disc with an indication of how the corners of the one are sent to the corners of the other. This description induces piecewise-linear maps on the boundary circle (parametrized by arc length and rescaled by dividing by 2π) [FKS12, Fos12]. Alternatively, its elements can be identified with the group of finite retriangulations and renormalizations of the Farey tessellation of the disc. See Sonja Mitchell Gallagher's dissertation for precise definitions and further details of this description [Gal13].

There are several relevant groups that are closely related to T and for these it is convenient to introduce the following notation.

Definition 5.2 (Atlas notation)**.** The atlas of finite groups popularized many simple notational conventions for describing groups that are closely related to simple groups [CCN+85]. For example, when G is a group that has a normal subgroup isomorphic to a group A and the quotient group G/A is isomorphic to a group B, one writes $G = A.B$ and says that G has *shape* $A.B$. When iterated, the convention is to left associate. Thus $G = A.B.C$ means that $G = (A.B).C$, i.e. G has a normal subgroup of shape $A.B$ with quotient isomorphic to C. Finally, the ATLAS writes m to denote the finite cyclic group $\mathbb{Z}_m$.

One extension of T we wish to discuss is its non-oriented version.

Definition 5.3 (Non-oriented T)**.** The map that sends every complex number $x+iy$ to its complex conjugate $x-iy$ is an orientation-reversing map of order 2 that preserves the Farey tessellation of the unit disc. The *non-oriented version of Thompson's group* T is the automorphism group of T and it is generated by T acting on itself by conjugation and the automorphism of T induced by the complex conjugation map of the Farey tessellation. It is denoted T^{no} in [Fos] and its structure is isomorphic to $T \rtimes \mathbb{Z}_2$. In other words, it has shape $T.2$.

The other groups related to T that we need to discuss are the subgroups of T which centralize elements of finite order. These torsion elements of T are easy to describe and classify.

Definition 5.4 (Torsion elements)**.** Thompson's group T contains torsion elements of every order and all finite cyclic subgroups of the same order are conjugate. More explicitly, torsion elements are created by, essentially, finding an m-sided ideal polygon in the Farey tessellation whose boundary arcs are Farey arcs and then retriangulating this m-gon so that it looks like a $\frac{2\pi}{k}$ rotation of the original triangulation for some k that divides m. This is an element whose order divides k and all finite cyclic subgroups of order k are generated by a $\frac{2\pi}{k}$ rotation of some Farey m-gon in this way. This characterization can be used to prove that all such subgroups are conjugate [BCST09, Fos12].

In our classification of automorphisms of the infinite cyclohedron, we need to understand the centralizer of a particular order 2 element in T. A proof of the following result can be found in [Mat08, Chapter 7].

Proposition 5.5 (Centralizers)**.** *Let g be an element in T of order m and let G be the centralizer of g in T, i.e. the elements in T that commute with g. The cyclic group $\mathbb{Z}_m$ generated by g is a normal subgroup of G and the quotient group $G/\mathbb{Z}_m$ is isomorphic to T itself. In other words, the subgroup $G \subset T$ has shape $m.T$.*

The fact that T has proper subgroups of shape $m.T$ for every m means that it is a very self-similar group.

6. Automorphism groups

In this final section we analyze the structure of the automorphism group of the infinite polytope Cy^∞. We first define what we mean by an automorphism of an infinite polytope. For ordinary finite-dimensional convex polytopes, there are various notions of equivalence. Convex polytopes can be considered as metric objects, or as objects up to equivalence under affine transformations, or as purely combinatorial objects. When constructing the infinite cyclohedron inside the infinite associahedron we adopted the metric viewpoint, but when we consider its automorphism group we adopt a combinatorial one.

Definition 6.1 (Automorphisms)**.** By a *combinatorial automorphism* of a convex polytope P, in finite or infinite dimensions, we mean an automorphism of its associated face lattice. Thus vertices are sent to vertices, edges to edges, and so on, but there is no requirement that parallel faces be send to parallel faces as there would be if we required the existence of an underlying affine transformation of the ambient affine space. The set of all combinatorial automorphisms clearly form a group $\mathsf{Aut}(P)$, which we call its *combinatorial automorphism group.*

In this notation, the main result proved by the first author in [Fos] is the following.

Theorem 6.2 (Associahedral automorphisms)**.** *The combinatorial automorphism group of the infinite associahedron is isomorphic to* $T \rtimes \mathbb{Z}_2$*, the non-oriented version of Thompson's group* T*. In other words,* $\mathsf{Aut}(\mathsf{As}^\infty)$ *has shape* $T.2$*.*

It might seem remarkable that the union of a nested sequence of polytopes which only have dihedral symmetries would have such a large and complicated automorphism group, but the situation is analogous to the way in which the intervals $[-n, n]$ only have two symmetries each (because the origin must be fixed) while their union, the real line, has a much larger symmetry group which includes translations. The corresponding fact for Thompson's group T, proved by the first author in [Fos11], is that it contains the group $PSL_2(\mathbb{Z})$ as an undistorted subgroup. When $PSL_2(\mathbb{Z})$ is viewed as the group of orientation-preserving maps of the hyperbolic plane that preserve the rational Farey tessellation, the connection with T is clear and the undistorted natural of the inclusion map is understandable.

A key step in the proof of Theorem 6.2 is the analysis of (in our language) the structure of (the 3-skeleton of) a neighborhood of a vertex in As^∞. The first author and Maxime Nguyen show in [FN12] that if ϕ and ψ are two combinatorial automorphisms of As^∞ and there is a vertex v such that $\phi(v) = w = \psi(v)$ and the ϕ and ψ agree on the link of v, then $\phi = \psi$. In other words, combinatorial automorphisms are completely determined by their behavior in the neighborhood of a single vertex. Moreover, each feasible local isometry is realized by an element of the non-oriented version of Thompson's group T, so these are the only combinatorial automorphisms. What we need is an analogous fact.

Proposition 6.3 (Extending automorphisms)**.** *Every combinatorial automorphism of the infinite cyclohedron* Cy^∞ *comes from an automorphism of the infinite associahedron* As^∞ *that commutes with the* π*-rotation* τ*.*

Proof. The proof is similar to the one in Section 4 of [FN12] but with minor variations. The vertices of Cy^∞ can be identified with finite centrally symmetric retriangulations of the Farey tessellation. The edges of Cy^∞ correspond to a flip either of the diagonal edge (the unique arc that passes through the origin of the disc) or a pair of flips of arcs symmetric with respect to the π-rotation. Purely from the combinatorics of the flips in the neighborhood of a fixed vertex, one can recover the underlying structure of the tessellation. For example, the 2-cells that contain a fixed vertex v are either squares, pentagons or hexagons. Squares correspond to flips involving arcs that do not bound a common triangle. Pentagons correspond to flips involving two pairs of centrally symmetric arcs where one arc in the first pair bounds a common triangle with one arc in the second pair. Hexagons correspond to the diagonal flip and one of the two flips involving centrally symmetric arcs bounding a common triangle with the diagonal arc. In other words, the vertex v belongs to the boundary of exactly two hexagons and these hexagons share a unique edge, the edge corresponding to the diagonal flip.

Building from this identification of the diagonal flip, the pentagons can be used to identify arcs sharing a common triangle and looking at how they are included into the 3-cells lets one see which endpoint they share. Continuing in this way, one uses the purely combinatorial data in the low dimensional skeleton of the neighborhood of a vertex v to identify each edge leaving v with the flip of a specific arc (or centrally symmetric pair of arcs) in the Farey tesselation corresponding to v. And once one has reconstructed the tessellation, we see that combinatorial automorphisms are once again completely determined by their behavior in the neighborhood of a single vertex. Moreover, each feasible local isometry is realized by an element of the non-oriented version of Thompson's group T which commutes with the π-rotation τ, so these are the only combinatorial automorphisms. $\square$

We are now ready to prove our main result.

Theorem 6.4 (Cyclohedral automorphisms). *The combinatorial automorphism group of the infinite cyclohedron is isomorphic to the centralizer of a particular involution in the combinatorial automorphism group of the infinite associahedron* $\mathsf{Aut}(\mathsf{As}^\infty)$*. And since the centralizer of an involution in* T *is a group isomorphic to a degree* 2 *central extension of* T*, the group* $\mathsf{Aut}(\mathsf{Cy}^\infty)$ *has shape* $2.T.2$*.*

Proof. By Proposition 6.3 it suffices to find which elements in $T \rtimes \mathbb{Z}_2$ stabilize the polytope Cy^∞. In order to send centrally symmetric triangulations to centrally symmetric triangulations, the corresponding map on the boundary circle must be invariant under the π-rotation map τ. This map τ represents an element in T of order 2 and its centralizer in T, by Proposition 5.5 has shape $2.T$. This represents all of the orientation-preserving elements in $T \rtimes \mathbb{Z}_2$ that stabilize Cy^∞. Since the orientation-reversing complex conjugation map, the one which reflects across the x-axis, also commutes with the π-rotation map τ, the full subgroup of $\mathsf{Aut}(\mathsf{As}^\infty)$ that stabilizes Cy^∞ is a semidirect product of the centralizer in T and $\mathbb{Z}_2$, a group with shape $2.T.2$. □

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