

FACTORING EUCLIDEAN ISOMETRIES

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ABSTRACT. Every isometry of a finite dimensional euclidean space is a product of reflections and the minimum length of a reflection factorization defines a metric on its full isometry group. In this article we identify the structure of intervals in this metric space by constructing, for each isometry, an explicit combinatorial model encoding all of its minimal length reflection factorizations. The model is largely independent of the isometry chosen in that it only depends on whether or not some point is fixed and the dimension of the space of directions that points are moved.

Every good geometry book proves that each isometry of euclidean n -space is a product of at most $n+1$ reflections and several more-advanced sources include Scherk's theorem which identifies the minimal length of such a reflection factorization from the basic geometric attributes of the isometry under consideration [Sch50, Die71, ST89, Tay92]. The structure of the full set of minimal length reflection factorizations, on the other hand, does not appear to have been given an elementary treatment in the literature even though the proof only requires basic geometric tools.¹ In this article we construct, for each isometry, an explicit combinatorial model encoding all of its minimal length reflection factorizations. The model is largely independent of the isometry chosen in that it only depends on whether or not some point is fixed and the dimension of the space of directions that points are moved.

Analogous results for spherical isometries already exist and are easy to state: when w is an orthogonal linear transformation of \mathbb{R}^n only fixing the origin, for example, there is a natural bijection between minimal length factorizations of w into reflections fixing the origin and complete flags of linear subspaces in \mathbb{R}^n [BW02]. In other words, the structure of all such factorizations is encoded in the lattice of linear subspaces of \mathbb{R}^n with one factorization for each maximal chain. We construct a

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¹The only discussions of this issue that we have found in the literature use Wall's parameterization of the orthogonal group and the main results are stated in terms of inclusions of nondegenerate subspaces under an asymmetric bilinear form derived from the original isometry [Tay92, Wal63].

similar poset for euclidean isometries but its structure is more complicated. The motivation for constructing these combinatorial models is to use them to analyze the structure of euclidean Artin groups. See [McC] and [MS] for details. It is with this application in mind that we include an examination of the extent to which these posets are lattices.

The article is structured as follows. The first four sections establish basic definitions, the middle sections define the combinatorial models and establish our main result, and the final sections explore the extent to which the constructed posets fail to be complete lattices.

1. EUCLIDEAN GEOMETRY

In this section we review some elementary euclidean geometry with a special emphasis on notation. The main thing to note is that we sharply distinguish between points and vectors as in [ST89] since this greatly clarifies the arguments used latter in the article.

Definition 1.1 (Points and vectors). Throughout the article, V denotes an n -dimensional real vector space with a positive definite inner product and E denotes its affine analog where the location of the origin has been forgotten. The elements of V are *vectors* and the elements of E are *points*. We use greek letters for vectors and roman letters for points. There is a uniquely transitive action of V on E . Thus, given a point x and a vector λ there is a unique point y with $x + \lambda = y$ and given two points x and y there is a unique vector λ with $x + \lambda = y$. We say that λ is the vector from x to y . For any $\lambda \in V$, the map $x \mapsto x + \lambda$ is a *translation* isometry t_λ of E and since $t_\mu t_\nu = t_{\mu+\nu} = t_\nu t_\mu$, the set $T = \{t_\lambda \mid \lambda \in V\}$ is an abelian group. For any point $x \in E$, the map $\lambda \mapsto x + \lambda$ is a bijection that identifies V and E but the isomorphism depends on this initial choice of a *basepoint* x in E . Lengths of vectors and angles between vectors are calculated using the usual formulas and distances and angles in E are defined by converting to vector-based calculations.

Definition 1.2 (Linear subspaces of V). A *linear subspace* of V is a subset closed under linear combination and every subset of V is contained in a unique minimal linear subspace called its *span*. Each linear subspace U has an *orthogonal complement* U^\perp consisting of those vectors in V orthogonal to all the vectors in U and there is a corresponding orthogonal decomposition $V = U \oplus U^\perp$. The *codimension* of U is the dimension of U^\perp .

Definition 1.3 (Affine subspaces of E). An *affine subspace* of E is any subset B that contains every line determined by distinct points in B

and every subset of E is contained in a unique minimal affine subspace called its *affine hull*. Associated with any affine subspace B is its (linear) *space of directions* $\text{DIR}(B) \subset V$ consisting of the collection of vectors connecting points in B . The *dimension* and *codimension* of B is that of its space of directions and a set of $k + 1$ points in E is in *general position* when its affine hull has dimension k .

Definition 1.4 (Barycentric coordinates). Let x_0, x_1, \dots, x_k be $k + 1$ points in general position in E , and let B be its k -dimensional affine hull. If we identify E and V by picking a basepoint then each point in B can be expressed as a linear combination of the vectors from this origin to x_i where the coefficients sum to 1. Since these coefficients turn out to be independent of our choice of basepoint, we can unambiguously write $\sum c_i x_i = c_0 x_0 + c_1 x_1 + \dots + c_k x_k$ (with $c_i \in \mathbb{R}$ and $\sum c_i = 1$) to represent each point in B without making such an identification. These are called *barycentric coordinates* on B .

The intrinsic way barycentric coordinates are defined means when $\{x_i\}$ and $\{y_i\}$ are points in general position with affine hulls B and C and an isometry of E sends each x_i to y_i , then it also sends the point $\sum c_i x_i \in B$ to the point $\sum c_i y_i \in C$ with the same coefficients.

Definition 1.5 (Affine subspaces of V). An *affine subspace* of V is a translation of one of its linear subspaces. In particular, every affine subspace M can be written in the form $t_\mu(U) = U + \mu = \{\lambda + \mu \mid \lambda \in U\}$ where U is a linear subspace of V . This representation is not unique, (since $U + \mu = U$ for all $\mu \in U$), but it can be made unique if we insist that μ is of minimal length or, equivalently, that μ be a vector in U^\perp , in which case we say $U + \mu$ is the *standard form* of M .

Under any identification of E and V , the affine subspaces of E are identified with those of V . More precisely, each affine subspace B in E corresponds to some $M = U + \mu$. The linear subspace $U = \text{DIR}(B)$ is canonical but the vector $\mu \in U^\perp$ depends on the choice of basepoint.

2. POSETS AND LATTICES

For posets and lattices we generally follow [Sta97] and [DP02].

Definition 2.1 (Posets). Let P be a partially ordered set. When a minimum or a maximum element exists in P , it is denoted $\mathbf{0}$ and $\mathbf{1}$, respectively, and posets containing both are *bounded*. The *dual* P^* of a poset P has the same underlying set but with the order reversed, and a poset is *self-dual* when it and its dual are isomorphic. For each $Q \subset P$ there is an induced *subposet* structure on Q which is simply

the restriction of the order on P . A subposet C in which any two elements are comparable is called a *chain* and its length is $|C| - 1$. Every finite chain is bounded and its maximum and minimum elements are its *endpoints*. If a finite chain C is not a subposet of a strictly larger finite chain with the same endpoints, then C is *saturated*. Saturated chains of length 1 are called *covering relations*. If every saturated chain in P between the same pair of endpoints has the same finite length, then P is *graded*. The *rank* of an element p is the length of the longest chain with p as its upper endpoint and its *corank* is the length of the longest chain with p as its lower endpoint, assuming such chains exists.

Definition 2.2 (Lattices). Let Q be a subset of a poset P . A lower bound for Q is any $p \in P$ with $p \leq q$ for all $q \in Q$. When the set of lower bounds for Q has a unique maximum element, this element is the *greatest lower bound* or *meet* of Q . Upper bounds and the *least upper bound* or *join* of Q are defined analogously. The meet and join of Q are denoted $\bigwedge Q$ and $\bigvee Q$ in general and $u \wedge v$ and $u \vee v$ if u and v are the only elements in Q . When every pair of elements has a meet and a join, P is a *lattice* and when every subset has a meet and a join, it is a *complete lattice*. An easy argument shows that every bounded graded lattice is complete.

The main posets we need are the linear and affine subspace posets.

Definition 2.3 (Linear subspaces). The linear subspaces of V partially ordered by inclusion form a poset we call $\text{LIN}(V)$. It is graded, bounded, self-dual and a complete lattice. The bounding elements are clear, the grading is by dimension (in that a k -dimensional subspace has rank k and corank $n - k$), the meet of a collection of subspaces is their intersection and their join is the span of their union. And finally, the map sending a linear subspace to its orthogonal complement is a bijection that establishes self-duality.

Definition 2.4 (Affine subspaces). The *affine subspaces* of E partially ordered by inclusion form a poset we call $\text{AFF}(E)$. For each n , it is a graded poset that is bounded above but not below since distinct points are distinct minimal elements. It is neither self-dual nor a lattice. There is, however, a well-defined rank-preserving poset map $\text{AFF}(E) \rightarrow \text{LIN}(V)$ sending each affine subspace B to its space of directions $\text{DIR}(B)$. Chains in $\text{LIN}(V)$ and $\text{AFF}(E)$ are traditionally called *flags* and a maximal chain, starting at a minimal element and ending at the whole space, is a *complete flag*.

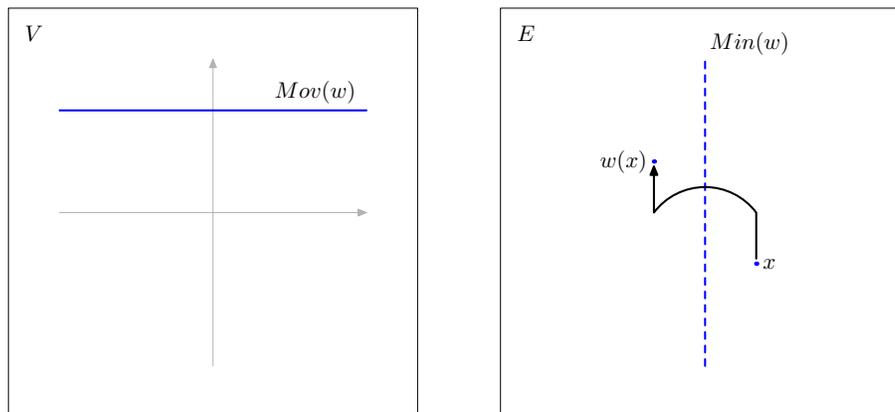


FIGURE 1. The basic invariants of a glide reflection.

3. EUCLIDEAN ISOMETRIES

The symmetries of E that preserve distances and angles are isometries and they form a group that we call $W = \text{ISOM}(E)$. The choice of the letter W reflects the fact that we treat $\text{ISOM}(E)$ like a continuous version of an affine Weyl group. We begin by highlighting two sets associated with any isometry.

Definition 3.1 (Invariants). Let $w \in W$ be an isometry of E . If λ is the vector from x to $w(x)$ then we say x is *moved by λ under w* . The collection $\text{MOV}(w) = \{\lambda \mid x + \lambda = w(x), x \in E\} \subset V$ of all such vectors is the *move-set* of w . As we show below, a move-set is an affine subspace and thus has standard form $U + \mu$ where U is a linear subspace and μ is a vector in U^\perp . The points in E that are moved by μ under w are those that are moved the shortest distance. The collection $\text{MIN}(w)$ of all such points is called the *min-set* of w . The sets $\text{MOV}(w) \subset V$ and $\text{MIN}(w) \subset E$ are the *basic invariants* of w . The basic invariants of a glide reflection of the plane are illustrated in Figure 1.

Proposition 3.2 (Invariants are affine). *For each isometry $w \in W$, its move-set $\text{MOV}(w)$ is an affine subspace of V and for each affine subspace $M \subset \text{MOV}(w)$, the points moved by some $\lambda \in M$ under w form an affine subspace of E . In particular, the min-set $\text{MIN}(w)$ is an affine subspace of E .*

Proof. For each $\mu, \nu \in \text{MOV}(w)$ choose x and y so that $w(x) = x + \mu$ and $w(y) = y + \nu$. Because w is an isometry, it sends the line through x and y to the line through $w(x)$ and $w(y)$. In barycentric coordinates, for each $c, d \in \mathbb{R}$ with $c + d = 1$, the point $cx + dy$ is sent to $c(x + \mu) +$

$d(y + \nu) = (cx + dy) + (c\mu + d\nu)$. In particular, every point on the affine line through μ and ν in V is the motion of some point in E showing that $\text{MOV}(w)$ is affine. Similarly, if x and y are both moved by λ and μ in an affine subspace M under w , then for each $c, d \in \mathbb{R}$ with $c + d = 1$, $cx + dy$ is sent to $c(x + \lambda) + d(y + \mu) = (cx + dy) + (c\lambda + d\mu)$. Thus every point on the line through x and y is moved by some vector in M under w and the set of all such points is an affine subspace. \square

Definition 3.3 (Elliptic and hyperbolic). Let w be an isometry of E and let $U + \mu$ be the standard form of its move-set $\text{MOV}(w)$. There are points fixed by w iff μ is trivial iff $\text{MOV}(w)$ is a linear subspace. Under these conditions w is *elliptic* and the min-set $\text{MIN}(w)$ is just the *fix-set* $\text{FIX}(w)$ of points fixed by w . Similarly, w has no fixed points iff μ is nontrivial iff $\text{MOV}(w)$ a nonlinear affine subspace of V . Under these conditions w is *hyperbolic*. The names come from a tripartite classification of isometries [BH99]. The third type, parabolic, does not occur in this context.

Translations and reflections are the simplest examples of hyperbolic and elliptic isometries, respectively. The translations t_λ , defined in Definition 1.1, can alternatively be characterized as the only isometries whose move-set is a single point or whose min-set is all of E . They are the essential difference between elliptic and hyperbolic isometries. There is a special factorization of a hyperbolic isometry w that we call its *standard splitting*.

Proposition 3.4 (Standard splittings). *If w is a hyperbolic isometry whose move-set has standard form $\text{MOV}(w) = U + \mu$, then the unique isometry u that satisfies the equation $w = t_\mu u$, is an elliptic isometry with $\text{MOV}(u) = \text{DIR}(\text{MOV}(w))$ and $\text{FIX}(u) = \text{MIN}(w)$.*

Proof. First note that $\text{DIR}(\text{MOV}(w)) = U$ and translations translate move-sets. Next, a point x is fixed by u iff x is moved by μ under w and such points do exist. Thus u is elliptic and $\text{FIX}(u) = \text{MIN}(w)$. \square

The standard splitting can be used to show that min-sets and move-sets have complementary dimensions.

Lemma 3.5 (Complementary invariants). *For each isometry $w \in W$ the dimensions of $\text{MOV}(w)$ and $\text{MIN}(w)$ add up to the dimension of E .*

Proof. Suppose w is elliptic and $U = \text{MOV}(w)$. If we choosing a base-point in $\text{FIX}(w)$ to identify E with V , then w corresponds to a linear transformation of V and the map $x \mapsto w(x) - x$ is linear as well. Its image is $\text{MOV}(w)$, its kernel is $\text{FIX}(w)$ and the desired equation is just

the rank-nullity theorem. When w is hyperbolic, we use Proposition 3.4 to find an elliptic isometry u with $\text{MOV}(u) = \text{DIR}(\text{MOV}(w))$ and $\text{FIX}(u) = \text{MIN}(w)$. The result holds for u and thus for w . \square

There is also a stronger version of Lemma 3.5.

Lemma 3.6 (Orthogonal invariants). *For each isometry $w \in W$, there is an orthogonal decomposition $V = \text{DIR}(\text{MOV}(w)) \oplus \text{DIR}(\text{MIN}(w))$.*

Proof. By Lemma 3.5 the dimensions of these subspaces add up to n so it is sufficient to prove that $\text{DIR}(\text{MOV}(w))$ is subset of $\text{DIR}(\text{MIN}(w))^\perp$. When w is elliptic, $\text{DIR}(\text{MOV}(w)) = \text{MOV}(w)$ and $\text{DIR}(\text{MIN}(w)) = \text{DIR}(\text{FIX}(w))$ are linear subspaces of V . To see that they are orthogonal, let $B = \text{FIX}(w)$ and let $U = \text{DIR}(B)^\perp$. For each $x \in E$, there is a unique $x_0 \in B$ closest to x and the vector μ from x to x_0 is orthogonal to $\text{DIR}(B)$, i.e. μ is in U . Also, since w preserves distances and fixes x_0 , the point in B closest to $w(x)$ is again x_0 . Thus the vector from x to $w(x)$ is a combination of two vectors in U and $\text{MOV}(w) \subset U$. By the dimension count $\text{MOV}(w) = U$. When w is hyperbolic with $\text{MOV}(w) = U + \mu$, there is an elliptic isometry u with $\text{MOV}(u) = U = \text{DIR}(\text{MOV}(w))$ and $\text{MIN}(u) = \text{MIN}(w)$ (Proposition 3.4). The decomposition derived from u completes the proof. \square

The standard splitting also can be used to prove a handy characterization of the min-set of an isometry.

Proposition 3.7 (Identifying min-sets). *If w is a hyperbolic isometry whose move-set has standard form $\text{MOV}(w) = U + \mu$, then the min-set of w is the unique affine subspace B of E stabilized by w whose dimension is the codimension of U and where all points in B experience the same motion under w .*

Proof. The min-set of w has these properties since its points under the same motion by definition and the other aspects follow from Lemma 3.6. Thus we only need to show that $\text{MIN}(w)$ is the only such subspace. Let B be an affine subspace satisfying these conditions and choose an affine subspace C in E with $\text{DIR}(C) = U$. The points in C parameterize the affine subspaces B' with $\text{DIR}(B') = U^\perp$ in the following sense: every such B' intersects C in a single point x and each point $x \in C$ determines a unique such subspace B' . Call these the U^\perp subspaces of E . Next, let $w = t_\mu u$ be the standard splitting of w and note that because $\text{MOV}(u) = U$, u stabilizes C . Moreover, because isometries send rectangles to rectangles, points in the same U^\perp subspace undergo the same motion under the action of u and thus the same motion under w . Thus C contains a representative of every possible motion under

u and because the dimension of U and C agree, it contains a unique representative of each motion. In particular, C contains a unique point that is fixed under u . This also means that the converse also holds: points undergoing the same motion under u (or equivalently the same motion under w) belong to the same U^\perp subspace. As a consequence B is a subspace of a U^\perp subspace and because their dimensions agree, B is a U^\perp subspace. Finally, $\mu \in U^\perp$ means that t_μ stabilizes each U^\perp subspace, and thus w stabilizes such a subspace iff u stabilizes it. But the only U^\perp subspace stabilized by u is the one corresponding to the unique point it fixes in C . These are exactly the points moved by μ under w proving that B is $\text{MIN}(w)$. \square

Definition 3.8 (Reflections in V). A *hyperplane* H in V is a linear subspace of codimension 1 and for each hyperplane there is a unique nontrivial isometry r that fixes H pointwise called a *reflection*. Let L be the 1-dimensional orthogonal complement of H and note that it contains exactly two unit vectors $\pm\alpha$ called the *roots* of r . Since L , H , and r can be recovered from α , we write $L = L_\alpha$, $H = H_\alpha$ and $r = r_\alpha$.

Definition 3.9 (Reflections in E). A *hyperplane* H in E is an affine subspace of codimension 1 and, as above, for each hyperplane there is a unique nontrivial isometry r that fixes H pointwise called a *reflection*. The set R of all reflections generates $W = \text{ISOM}(E)$. The space of directions is a hyperplane $\text{DIR}(H) = H_\alpha$ in V and $\pm\alpha$ are called the *roots* of r . Note that $\text{MOV}(r)$ is the line $L_\alpha \subset V$ spanned by α and $\text{MIN}(r) = \text{FIX}(r) = H \subset E$.

4. INTERVALS IN MARKED GROUPS

A *marked group* is a group G with a fixed generating set S which, for convenience, we assume is symmetric and injects into G . Thus, $s \in S$ iff $s^{-1} \in S$ and we can view S as a subset of G . Fixing a generating set defines a natural metric on the group.

Definition 4.1 (Metrics on groups). Let G be a group generated by a set S . The (right) *Cayley graph of G with respect to S* is a labeled directed graph denoted $\text{CAY}(G, S)$ with vertices indexed by G and edges indexed by $G \times S$. The edge $e_{(g,s)}$ has *label* s , it starts at v_g and ends at $v_{g'}$ where $g' = g \cdot s$. There is a natural faithful, vertex-transitive, label and orientation preserving left action of G on its Cayley graph. Moreover these are the only label and orientation preserving graph automorphisms of $\text{CAY}(G, S)$, making the identity automorphism the unique automorphism of this type that fixes a vertex. The *distance* $d(g, h)$ is the combinatorial length of the shortest path in the Cayley

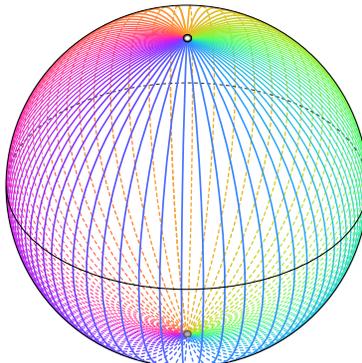


FIGURE 2. The interval between two antipodal points of \mathbf{S}^2 .

graph from v_g to v_h . Note that the symmetry assumption allows us to restrict attention to directed paths. This defines a metric on G and distance from the identity defines a *length function* $\ell_S: G \rightarrow \mathbb{N}$. The value $\ell_S(g) = d(1, g)$ is called the *S-length of g* and it is also the length of the shortest factorization of g in terms of elements of S . Because Cayley graphs are homogeneous, metric properties of the distance function translate into properties of ℓ_S . Symmetry and the triangle inequality, for example, imply that $\ell_S(g) = \ell_S(g^{-1})$, and $\ell_S(gh) \leq \ell_S(g) + \ell_S(h)$.

Next recall the notion of an interval in a metric space.

Definition 4.2 (Intervals in metric spaces). Let x, y and z be points in a metric space (X, d) . Borrowing from euclidean plane geometry we say that z is *between* x and y whenever the triangle inequality degenerates into an equality. Concretely z is between x and y when $d(x, z) + d(z, y) = d(x, y)$. The *interval* $[x, y]$ is the collection of points between x and y and this includes both x and y . Intervals can also be endowed with a partial ordering by declaring that $u \leq v$ whenever $d(x, u) + d(u, v) + d(v, y) = d(x, y)$.

As an illustration, consider points x and y on the 2-sphere with its usual metric. If they are not antipodal, then the only points between them are those on the unique shortest geodesic connecting x to y with the usual ordering along paths. But if they are antipodal, say x is the south pole and y is the north pole, then the interval $[x, y]$ is all of \mathbf{S}^2 and $u < v$ iff u and v lie on a common line of longitude connecting x to y with the latitude of u below the latitude of v as shown in Figure 2. Because marked groups are metric spaces, they have intervals.

Definition 4.3 (Intervals in groups). Let g and h be distinct elements in a marked group G . The *interval* $[g, h]$ is the poset of group elements

between g and h with $g' \in G$ is in $[g, h]$ when $d(g, g') + d(g', h) = d(g, h)$ and $g' \leq g''$ when $d(g, g') + d(g', g'') + d(g'', h) = d(g, h)$. In the Cayley graph $g' \in [g, h]$ means that $v_{g'}$ lies on some minimal length path from v_g to v_h and $g' < g''$ means that $v_{g'}$ and $v_{g''}$ both occur on a common minimal length path v_g to v_h with $v_{g'}$ occurring before $v_{g''}$.

Proposition 4.4 (Posets in Cayley graphs). *If g and h are distinct elements in a group G generated by a set S then the interval $[g, h]$ is a bounded graded poset whose Hasse diagram is embedded as a subgraph of the Cayley graph $\text{CAY}(G, S)$.*

Proof. The interval $[g, h]$ is bounded below by g , bounded above by h and graded by the distance from g . To see the Hasse diagram of $[g, h]$ inside the Cayley graph of G note that its vertices correspond to the elements between g and h and its coverings relations correspond to those directed edges in the Cayley graph that occur in some shortest directed path from v_g to v_h . \square

Since the structure of a graded poset can be recovered from its Hasse diagram, we let $[g, h]$ denote the edge-labeled directed graph that is visible as a subgraph of the Cayley graph $\text{CAY}(G, S)$.

Remark 4.5 (Isomorphic intervals). The left action of a group on its right Cayley graph preserves labels and distances. Thus the interval $[g, h]$ is isomorphic (as a edge-labeled directed graph) to the interval $[1, g^{-1}h]$. In other words, every interval in the Cayley graph of G is isomorphic to one that starts at the identity.

We call $g^{-1}h$ the *type* of the interval $[g, h]$ and note that intervals are isomorphic iff they have the same type. The distance ordering on G creates a single poset that contains every type of interval.

Definition 4.6 (Distance ordering). The *distance ordering* on a marked group G is defined by setting $g' \leq g$ iff $g' \in [1, g]$. By Remark 4.5, this gives G a poset structure that contains an interval of every type that occurs in the metric space on G .

5. REFLECTION LENGTH

In the language of the previous section, our goal is to establish the poset structure of intervals in the group $W = \text{ISOM}(E)$ generated by the set R of reflections. The key to analyzing these intervals is to have a good understanding of the length function. In this section we recall how the reflection length of an isometry is determined from its basic invariants (Theorem 5.7), a result known as Scherk's theorem [ST89]. The lower bounds are straight-forward.

Proposition 5.1 (Lower bounds). *If $w = r_1 r_2 \cdots r_k$ is a product of reflections, then the dimension of $\text{MOV}(w)$ is at most k and it is equal to k iff the roots of these reflections are linearly independent. As a consequence $\ell_R(w) \geq \dim(\text{MOV}(w))$. In addition, because linear independence of the roots implies w is elliptic, the stronger lower bound $\ell_R(w) > \dim(\text{MOV}(w))$ holds when the isometry w is hyperbolic.*

Proof. Let $\pm\alpha_i$ be the roots of the reflection r_i . Because r_i only moves points in the α_i direction, the cumulative motion of any point x under w is a linear combination of the α_i 's. Thus $\text{MOV}(w)$ is contained in their span, proving the first assertion and its consequence. The final part follows from the fact that hyperplanes with linearly independent normal vectors have a common point of intersection. Such a point is not moved by any of the r_i and thus is fixed by w . \square

The easy way to establish an upper bound on reflection length is to construct a factorization. For this we need a few lemmas.

Lemma 5.2 (Fix-sets and reflections). *If w is an elliptic isometry and r is a reflection whose hyperplane intersects $\text{FIX}(w)$, then $w' = rw$ is elliptic and the dimensions of $\text{FIX}(w)$ and $\text{FIX}(w')$ are at most 1 apart.*

Proof. Let H be the hyperplane of r and note that the hypothesized point in $H \cap \text{FIX}(w)$ shows that $w' = rw$ is elliptic and that $\text{FIX}(w)$ and $\text{FIX}(w')$ have points in common. Moreover, $\text{FIX}(w')$ certainly contains $\text{FIX}(r) \cap \text{FIX}(w) = H \cap \text{FIX}(w)$, a space that is either $\text{FIX}(w)$ or a codimension 1 subspace of $\text{FIX}(w)$. Thus the dimension of the fix-set decreases by at most 1. Finally, since reflecting twice is trivial, $rw' = w$ and w' and r satisfy the same hypotheses as w and r . Reversing the roles of w and w' shows the dimension increases by at most 1. \square

Lemma 5.3 (Fixing points). *Let w be a nontrivial elliptic isometry of E whose fix-set is a k -dimensional affine subspace B . For each $(k+1)$ -dimensional affine subspace C containing B , there is a unique reflection r such that $w' = rw$ and $\text{FIX}(w') = C$.*

Proof. Any point x in $C \setminus B$ is not fixed by w and the set of points equidistant from x and $w(x)$ is a hyperplane H . The reflection r that fixes H is the unique reflection sending $w(x)$ to x and thus the unique reflection where $w' = rw$ fixes $x \in C$. In other words this is the only reflection for which the assertion might be true. Next, since w is an isometry $d(x, y) = d(w(x), w(y)) = d(w(x), y)$ for all $y \in \text{FIX}(w)$. Thus $B \subset H$ and all of B is fixed by $w' = rw$. In particular, $\text{FIX}(w')$ contains x and B and by Proposition 3.2 it contains their affine hull which is C . By Lemma 5.2 $\text{FIX}(w')$ cannot be an affine subspace properly containing C . \square

These lemmas make it easy to construct reflection factorizations.

Proposition 5.4 (Elliptic upper bound). *Every elliptic isometry w with a k -dimensional move-set has a length k reflection factorization.*

Proof. By Lemma 3.5 the affine subspace $B = \text{FIX}(w)$ has codimension k . Next, select a chain of affine subspaces $B = B_k \subset B_{k-1} \subset \cdots \subset B_1 \subset B_0 = E$ where the subscript indicates its codimension. By Lemma 5.3 there is a reflection r_k such that $w_{k-1} = r_k w$ is an elliptic with $\text{FIX}(w_{k-1}) = B_{k-1}$. Iteratively applying Lemma 5.3 we can find reflections r_{k-1}, \dots, r_2, r_1 and elements w_{k-2}, \dots, w_1, w_0 where $w_{i-1} = r_i w_i$ is an elliptic with $\text{FIX}(w_{i-1}) = B_{i-1}$ for $i = k, \dots, 2, 1$. In the end, $w_0 = r_1 w_1 = \cdots = r_1 r_2 \cdots r_k w$ but w_0 is the identity since it fixes all of E . Rearranging shows $w = r_k \cdots r_2 r_1$. \square

Proposition 5.5 (Hyperbolic upper bound). *Every hyperbolic isometry with a k -dimensional move-set has a length $k + 2$ reflection factorization.*

Proof. Let w be a hyperbolic isometry whose move-set is k -dimensional, let $\text{MOV}(w) = U + \mu$ be its standard form, and let $w = t_\mu u$ be the standard splitting of w where u is an elliptic with $\text{MOV}(u) = U$ (Proposition 3.4). By Proposition 5.4 u has a length k reflection factorization and the translation t_μ is a product of two parallel reflections. Thus w has a length $k + 2$ reflection factorization. \square

To complete the proof of Theorem 5.7, we need one more observation.

Lemma 5.6 (Parity). *The lengths of all reflection factorizations of a given element have the same parity and the lengths of two elements differing by a reflection have opposite parity. More specifically, if w and w' are isometries and r is a reflection such that $w' = rw$ then $\ell_R(w') = \ell_R(w) - 1$ or $\ell_R(w') = \ell_R(w) + 1$.*

Proof. Partitioning isometries based on whether or not they preserve orientation shows that the Cayley graph of W respect to R is a bipartite graph and this has the first assertion as a consequence. For the second assertion, note that $\ell_R(w')$ and $\ell_R(w)$ differ by at most 1 by the way reflection length is defined and parity rules out equality. \square

Theorem 5.7 (Reflection length). *The reflection length of an isometry is determined by its basic invariants. More specifically, let w be an isometry whose move-set is k -dimensional. When w is elliptic, $\ell_R(w) = k$ and when w is hyperbolic, $\ell_R(w) = k + 2$.*

Proof. For elliptic isometries, Propositions 5.1 and 5.4 complete the proof. For hyperbolic isometries, Propositions 5.1 and 5.5 show that

$\ell_R(w)$ is $k + 1$ or $k + 2$. The former is ruled out because w has a length $k + 2$ factorization and by Lemma 5.6 $\ell_R(w)$ has the same parity. \square

One corollary of Theorem 5.7 is that the factorizations produced by Propositions 5.4 and 5.5 are now known to be minimal length. We conclude this section by characterizing some of the reflections that occur in minimal length factorizations of a fixed isometry. For this we need an elementary observation.

Lemma 5.8 (Rewriting factorizations). *Let $w = r_1 r_2 \cdots r_k$ be a reflection factorization. For any selection $1 \leq i_1 < i_2 < \cdots < i_j \leq k$ of positions there is a length k reflection factorization of w whose first j reflections are $r_{i_1} r_{i_2} \cdots r_{i_j}$ and another length k reflection factorization of w where these are the last j reflections in the factorization.*

Proof. Because reflections are closed under conjugation, for any reflections r and r' there exist reflections r'' and r''' such that $rr' = r''r$ and $r'r = rr'''$. Iterating these rewriting operations allows us to move the selected reflections into the desired positions without altering the length of the factorization. \square

Definition 5.9 (Reflections below w). Let r be a reflection. By Lemma 5.8, the following conditions are equivalent: (1) $\ell_R(rw) < \ell_R(w)$ (2) r is the leftmost reflection in some minimal length factorization of w (3) r is a reflection in some minimal length factorization of w (4) r is the rightmost reflection in some minimal length factorization of w and (5) $\ell_R(wr) < \ell_R(w)$. When these hold, we say that r is a *reflection below* w .

Proposition 5.10 (Motions and reflections). *If w is an isometry and x is not fixed by w then the unique reflection r that sends x to $w(x)$ is a reflection below w .*

Proof. That r occurs in some minimal length factorization of w is immediate from the flexibility of the constructions used to prove Proposition 5.4 and 5.5. \square

6. REFLECTIONS AND INVARIANTS

In this section we characterize when a reflection r is below an isometry w in terms of their basic invariants. We begin with a corollary of Theorem 5.7.

Lemma 6.1 (Move-sets and parity). *For each reflection r and isometry w , the dimensions of $\text{MOV}(w)$ and $\text{MOV}(rw)$ have opposite parity.*

Proof. By Lemma 5.6 $\ell_R(w)$ and $\ell_R(rw)$ have opposite parity and by Theorem 5.7 the same holds for the dimensions of their move-sets. \square

Proposition 6.2 (Move-sets and reflections). *Let r be a reflection with roots $\pm\alpha$, let w be an isometry with $\text{MOV}(w) = U + \mu$, and let $U_\alpha = \text{SPAN}(U \cup \{\alpha\})$. If $\alpha \in U$ then $\text{MOV}(rw)$ is a codimension 1 subspace of $\text{MOV}(w)$. If $\alpha \notin U$ then $\text{MOV}(rw) = U_\alpha + \mu$ and it contains $\text{MOV}(w)$ as a codimension 1 subspace.*

Proof. Let L_α be the line spanned by α and let k be the dimension of U . Because r only moves points in the α direction, the move-set of rw is contained in $U_\alpha + \mu$ and it must contain at least one point from each of its L_α cosets. For $\alpha \notin U$, this implies that the dimension of $\text{MOV}(rw)$ is either k or $k + 1$. By Lemma 6.1 its dimension is $k + 1$ and we have $\text{MOV}(rw) = U_\alpha + \mu$. On the other hand, for $\alpha \in U$, we have $U_\alpha = U$ and the dimension of $\text{MOV}(rw)$ is either k or $k - 1$. By Lemma 6.1 its dimension is $k - 1$ and $\text{MOV}(rw)$ is a codimension 1 subspace of $\text{MOV}(w)$. \square

Proposition 6.2 makes it possible to determine how type and reflection length change when multiplying by a reflection.

Proposition 6.3 (Hyperbolic isometries and reflections). *Let r with reflection with hyperplane H and roots $\pm\alpha$, let w be a hyperbolic isometry with $\ell_R(w) = k$ and $\text{MOV}(w) = U + \mu$ in standard form, and let $U_\alpha = \text{SPAN}(U \cup \{\alpha\})$.*

- *If $\alpha \in U$ then rw is hyperbolic and $\ell_R(rw) = k - 1$.*
- *If $\alpha \notin U$ and $\mu \in U_\alpha$ then rw is elliptic and $\ell_R(rw) = k - 1$.*
- *If $\alpha \notin U$ and $\mu \notin U_\alpha$ then rw is hyperbolic and $\ell_R(rw) = k + 1$.*

Proof. When α is in U , by Proposition 6.2 $\text{MOV}(rw)$ is a subspace of $\text{MOV}(w)$ and since $\text{MOV}(w)$ does not contain the origin, neither does $\text{MOV}(rw)$. Thus rw is hyperbolic. Similarly, when α is in U the new move-set is $U_\alpha + \mu$ which contains the origin iff $\mu \in U_\alpha$ and this determines whether rw is elliptic or hyperbolic. In all three cases $\ell_R(rw)$ is determined by Theorem 5.7. \square

The elliptic analog of Proposition 6.3 requires more preparation.

Lemma 6.4 (Minimal elliptic factorizations). *Let $w = r_1 r_2 \cdots r_k$ be a product of reflections where r_i has hyperplane H_i . If w is elliptic and $\ell_R(w) = k$ then the roots of these reflections are linearly independent. Conversely, if the roots of these reflections are linearly independent then w is elliptic, $\ell_R(w) = k$, $\text{FIX}(w) = H_1 \cap \cdots \cap H_k$ and this is one of its minimum length reflection factorizations.*

Proof. The factorization shows $\ell_R(w) \leq k$. If w is elliptic and $\ell_R(w) = k$ then by Theorem 5.7 its move-set is k -dimensional and by Proposition 5.1 the roots of the reflections are linearly independent. Conversely, if the roots are linearly independent, then their hyperplanes intersect in a codimension k subspace $B = H_1 \cap \cdots \cap H_k$ that is fixed by w . Thus w is elliptic. Moreover, if we start at the identity and multiply the reflections one at a time in order, then the linear independence of the roots and Proposition 6.2 implies that the move-set of these partial products steadily increase. Thus $\text{MOV}(w)$ has dimension equal to k and by Lemma 3.5 $\text{FIX}(w)$ has codimension k . Since we have already found a subspace fixed by w of this dimension, $\text{FIX}(w) = B$, $\ell_R(w) = k$ by Theorem 5.7, and this factorization has minimal length. \square

Lemma 6.5 (Fix-sets and reflections). *If w and w' are elliptic isometries and r is a reflection such that $w' = rw$ then the hyperplane of r intersects both $\text{FIX}(w)$ and $\text{FIX}(w')$.*

Proof. By Lemma 5.6 $\ell_R(w') = \ell_R(w) \pm 1$ and by relabeling if necessary we can assume that $\ell_R(w') = \ell_R(w) + 1$. In particular, if we set $r_1 = r$ and $w = r_2 \cdots r_k$ is a minimal length reflection factorization of w then $w' = rw = r_1 r_2 \cdots r_k$ is a minimal length reflection factorization of w' . By Lemma 6.4 $\text{FIX}(w') = H_1 \cap \cdots \cap H_k$ and $\text{FIX}(w) = H_2 \cap \cdots \cap H_k$ where H_i is the hyperplane of r_i . This means that $H = H_1$ contains $\text{FIX}(w')$ and, since $\text{FIX}(w')$ is nonempty, it intersects $\text{FIX}(w)$. \square

Proposition 6.6 (Elliptic isometries and reflections). *Let r be a reflection with hyperplane H and roots $\pm\alpha$ and let w be an elliptic isometry with $\ell_R(w) = k$, $\text{MOV}(w) = U$ and $\text{FIX}(w) = B$.*

- If $\alpha \notin U$ then rw is elliptic and $\ell_R(rw) = k + 1$.
- If $\alpha \in U$ and $B \subset H$ then rw is elliptic and $\ell_R(rw) = k - 1$.
- If $\alpha \in U$ with $B \not\subset H$, then B and H are disjoint, rw is hyperbolic and $\ell_R(rw) = k + 1$.

Proof. For $\alpha \notin U$, rw is elliptic because $\text{MOV}(rw) = \text{SPAN}(U \cup \{\alpha\})$ by Proposition 6.2 and this subspace contains the origin. When $\alpha \in U$ and $B \subset H$, rw is elliptic because B is fixed by rw . Finally, suppose $\alpha \in U$ and $B \not\subset H$. By Lemma 3.6, $\text{DIR}(B) \subset \text{DIR}(H)$ so the only way B is not a subset of H is if it is completely disjoint from H . The isometry rw is hyperbolic since by Lemma 6.5 it is not elliptic. In all three cases $\ell_R(rw)$ is determined by Theorem 5.7. \square

The only situations where the length goes down are the following.

Proposition 6.7 (Hyperbolic descents). *Let w be a hyperbolic isometry and let r be a reflection below w . If rw is hyperbolic then $\text{MOV}(rw)$*

is a codimension 1 subspace of $\text{MOV}(w)$, and if rw is elliptic, then $\text{DIR}(\text{FIX}(rw))^\perp = \text{SPAN}(\text{MOV}(w))$.

Proof. When rw is hyperbolic this follows from Propositions 6.2 and 6.3. When rw is elliptic, these same propositions imply that $\text{MOV}(rw) = \text{SPAN}(\text{MOV}(w))$ and by Lemma 3.6 $\text{DIR}(\text{FIX}(rw))$ is its orthogonal complement. \square

Proposition 6.8 (Elliptic descents). *If w is a elliptic isometry and r is a reflection below w , then rw is elliptic and $\text{FIX}(rw)$ contains $\text{FIX}(w)$ as a codimension 1 subspace.*

Proof. This follows from Propositions 6.2 and 6.6. \square

7. COMBINATORIAL MODELS

In this section we construct an abstract poset P whose elements are indexed by affine subspaces of V and E . Its subposets are used to establish Theorem 8.7, our main result.

Definition 7.1 (Global poset). We construct a *global poset* P from two types of elements. For each nonlinear affine subspace M in V , P contains a *hyperbolic* element h^M and for each affine subspace B in E , P contains an *elliptic* element e^B . We also define an *invariant map* $\text{inv}: W \rightarrow P$ that sends w to $h^{\text{MOV}(w)}$ when w is hyperbolic and to $e^{\text{FIX}(w)}$ when w is elliptic. This explains the names and the notation. The elements of P are ordered as follows. First, hyperbolic elements are ordered by inclusion and elliptic elements by reverse inclusion: $h^M \leq h^{M'}$ iff $M \subset M'$ and $e^B \leq e^{B'}$ iff $B \supset B'$. Next, no elliptic element is ever above a hyperbolic element. And finally, $e^B < h^M$ iff $M^\perp \subset \text{DIR}(B)$. The reader should be careful to note that because M is by definition a nonlinear subspace of V , the vectors orthogonal to all of M are also orthogonal to its span, a linear subspace whose dimension is $\dim(M) + 1$. Transitivity is an easy exercise.

When W is viewed as a poset under the distance ordering, the invariant map is an order-preserving map between posets.

Proposition 7.2 (Order-preserving). *If w is an isometry and r is a reflection with $\ell_R(rw) < \ell_R(w)$ then $\text{inv}(rw) < \text{inv}(w)$ in P . As a consequence, when W is viewed as a poset using the distance ordering, the map $\text{inv}: W \rightarrow P$ is a rank-preserving homomorphism between posets.*

Proof. This is an immediate consequence of Propositions 6.7 and 6.8 and the observation that the image of a covering relation in W is a covering relation in P . \square

Definition 7.3 (Model posets). Let $w \in W$ be an isometry. By Proposition 7.2, the invariant map is order-preserving and thus it sends isometries in interval $[1, w]$ to elements less than or equal to $\text{inv}(w)$. Let $P(w)$ denote the subposet of P induced by restricting to those elements less than or equal to $\text{inv}(w)$. We call $P(w)$ the *model poset for w* and it is a *hyperbolic poset* or an *elliptic poset* depending on the type of w . Since it is also useful to have a notation for these subposets in the absence of an isometry, let P^M denote the subposet induced by restricting to those elements less than or equal to h^M for a nonlinear affine subspace $M \subset V$ and let P^B denote the subposet induced by restricting to those elements less than or equal to e^B for an affine subspace $B \subset E$. (This notation is unambiguous because M and B are affine subspaces of V and E , respectively.) As should be clear, when w is hyperbolic $P(w) = P^{\text{Mov}(w)}$ and when w is elliptic $P(w) = P^{\text{Fix}(w)}$.

The structure of an elliptic poset is straightforward.

Proposition 7.4 (Elliptic posets). *For each affine subspace B in E , the elliptic poset P^B is isomorphic to $\text{LIN}(U)$ where $U = \text{DIR}(B)^\perp$.*

Proof. The poset P^B is essentially the poset of affine subspaces of E that contain B under reverse inclusion which is isomorphic to linear subspaces of V that contain $\text{DIR}(B)$ under reverse inclusion and thus isomorphic to linear subspaces of $\text{DIR}(B)^\perp = U$ under inclusion. \square

The structure of a hyperbolic poset is only slightly more complicated.

Remark 7.5 (Hyperbolic posets). Every hyperbolic poset can be decomposed into two subposets whose structure is easy to describe. Let M be a nonlinear affine subspace of V and for the moment assume that M has codimension 1. From the definition it is clear that the hyperbolic elements in the hyperbolic poset P^M form an induced subposet isomorphic to $\text{AFF}(M)$ and the elliptic elements in P^M form an induced subposet isomorphic to $\text{AFF}(E)^*$ where the asterisk indicates that this is the dual of $\text{AFF}(E)$ with reverse inclusion instead of inclusion. This is because $\text{SPAN}(M)$ is all of V and $M^\perp = \text{SPAN}(M)^\perp$ is the trivial subspace. Thus every affine subspace B in E satisfies $M^\perp \subset \text{DIR}(B)$. A similar structure is present even when M is not codimension 1 since the elliptics e^B below h^M are uniquely determined by the intersection of B with any fixed affine subspace C in E with $\text{DIR}(C) = M^\perp$. Thus, in this case, the hyperbolic and elliptic elements of P^M induce subposets that look like $\text{AFF}(M)$ and $\text{AFF}(C)^*$. From these two basic pieces the whole poset is described by declaring that some elements in $\text{AFF}(M)$ are above specific elements in $\text{AFF}(C)^*$.

8. MODELS FOR INTERVALS

In this section we prove that the map $inv: [1, w] \rightarrow P(w)$ is an isomorphism of posets. This should be slightly surprising since the invariant map is far from injective in general. There are many distinct rotations, for example, that rotate around the same codimension 2 subspace. We begin with three lemmas about local situations.

Lemma 8.1 (From elliptic to elliptic). *If w is an elliptic isometry with $inv(w) = e^B$ and $C \subset E$ is an affine subspace containing B as a codimension 1 subspace, then there is a reflection r such that $u = rw$ and $inv(u) = e^C$.*

Proof. This is a restatement of Lemma 5.3 in the new terminology. \square

Lemma 8.2 (From hyperbolic to elliptic). *If w is a hyperbolic isometry with $inv(w) = h^M$ and B is an affine subspace of E with $M^\perp = \text{SPAN}(M)^\perp = \text{DIR}(B)$, then there is a reflection r such that $u = rw$ and $inv(u) = e^B$.*

Proof. If x is a point in B then, since w does not fix x , the set of points equidistant from x and $w(x)$ form a hyperplane H . Let r be the reflection that fixes H , let $\pm\alpha$ be its roots and let $u = rw$. Since some scalar multiple of α lies in M and M is nonlinear, α is not in $U = \text{DIR}(M)$. By Proposition 6.2, $\text{MOV}(u) = \text{SPAN}(M)$ and by Lemma 3.6, $\text{DIR}(\text{FIX}(u)) = M^\perp = \text{SPAN}(M)^\perp$. Since the affine subspace B is determined by one of its points and its space of directions, $\text{FIX}(u) = B$ and $inv(u) = e^B$ as required. \square

Lemma 8.3 (From hyperbolic to hyperbolic). *If w is a hyperbolic isometry with $inv(w) = h^M$ and M' is a codimension 1 affine subspace of $M = \text{MOV}(w)$, then there is a reflection r such that $u = rw$ and $inv(u) = h^{M'}$.*

Proof. Let B be the set of all points in E moved by some $\lambda \in M'$. By Proposition 3.2 B is an affine subspace and because M' is a proper subspace of M , B is a proper subspace of E . In fact, because M' has codimension 1 in M , B has codimension 1 in E , i.e. a hyperplane. Let r be the corresponding reflection and note that its roots $\pm\alpha$ are the unit vectors orthogonal to $\text{DIR}(M')$ inside $\text{DIR}(M)$. By Proposition 6.2 $\text{MOV}(rw)$ is a codimension 1 subspace of M but since r does not move B , $\text{MOV}(rw) \supset M'$ and thus $\text{MOV}(rw) = M'$. \square

Proposition 8.4 (Surjective). *Let $w \in W$ be an isometry. For each maximal chain in $P(w)$ from e^E to $inv(w)$ of length k , there is a factorization of w as a product of k reflections, $w = r_1 r_2 \cdots r_k$, so*

that the suffixes of this factorization are sent under the invariant map to the elements in this maximal chain. As a consequence, the map $inv: [1, w] \rightarrow P(w)$ is surjective.

Proof. In what follows we describe the elements in the given maximal chain in descending order so that $inv(w)$ is its first element and e^E is its last. By Lemma 8.1, Lemma 8.2 or Lemma 8.3 depending on the types of the first and second elements in the chain, there is a reflection r_1 where $inv(r_1w)$ is the second element in the chain. Applying the same lemmas to r_1w means there is a reflection r_2 such that $inv(r_2r_1w)$ is the third element in the chain, and so on. After k repetitions, we have found $\{r_i\}$ so that $inv(r_k \cdots r_2r_1w) = e^E$ but this means that $r_k \cdots r_2r_1w$ fixes all of E , it is the identity and rewriting yields $w = r_1r_2 \cdots r_k$. The intermediate stages are $r_i \cdots r_2r_1w = r_{i+1} \cdots r_k$ and these are sent by the invariant map to the correct elements in the maximal chain by construction. \square

Proposition 8.5 (Injective). *For each isometry $w \in W$ the map $inv: [1, w] \rightarrow P(w)$ is injective.*

Proof. Let w be an isometry with $\ell_R(w) = k$ and let $u, u' \in [1, w]$ be isometries with $inv(u) = inv(u')$. By definition this means we can write $w = uv = u'v'$ with $\ell_R(w) = \ell_R(u) + \ell_R(v) = \ell_R(u') + \ell_R(v')$. Because the invariant map is rank-preserving (Proposition 7.2), $\ell_R(u) = \ell_R(u')$ and the proof is by induction on $j = \ell_R(w) - \ell_R(u) = \ell_R(w) - \ell_R(u') = \ell_R(v) = \ell_R(v')$. The base step is trivial since $w = u = u'$ when $j = 0$. The inductive step splits into two cases. Case 1: suppose that u and u' are elliptic with $inv(u) = inv(u') = e^B$. There must be a point x in B not fixed by w because $j > 0$ implies w either has a smaller fix-set or it is hyperbolic and fixes no points at all. Since both u and u' fix x , both v and v' send x to $w(x)$. As a consequence, both v and v' have minimal length factorizations that include the unique reflection r sending x to $w(x)$ (Proposition 5.10) and by Lemma 5.8 we can write $v = v_0r$ and $v' = v'_0r$. Thus, u and u' are both below $wr = uv_0 = u'v'_0$ of length $\ell_R(wr) = \ell_R(w) - 1$ and by induction $u = u'$. Case 2: suppose that u and u' are hyperbolic with $inv(u) = inv(u') = h^M$ and let $M = U + \mu$ be its standard form. By Proposition 3.4 we can write $w = t_\mu w_0$, $u = t_\mu u_0$ and $u' = t_\mu u'_0$ where w_0, u_0 and u'_0 are elliptics. Since u_0 and u'_0 are below w_0 , both $\text{FIX}(u_0)$ and $\text{FIX}(u'_0)$ contain $\text{FIX}(w_0)$ and have directions $\text{DIR}(u_0) = \text{DIR}(u'_0) = \text{DIR}(U)^\perp$. Since they have points in common and the same set of directions, $\text{FIX}(u_0) = \text{FIX}(u'_0)$. The previous case, applied to w_0, u_0 and u'_0 shows that $u_0 = u'_0$ and thus $u = u'$. \square

Proposition 8.6 (Inverse). *If $w \in W$ is an isometry and $u, u' \in [1, w]$ are isometries with $\text{inv}(u) < \text{inv}(u')$ in $P(w)$, then $u < u'$ in $[1, w]$.*

Proof. Extend the chain $e^E \leq \text{inv}(u) < \text{inv}(u') \leq \text{inv}(w)$ to a maximal chain in $P(w)$ and apply Proposition 8.4. The result is a factorization $w = r_1 r_2 \cdots r_k$ where there is some suffix $w_i = r_{i+1} \cdots r_k$ with $\text{inv}(w_i) = \text{inv}(u')$ and a shorter suffix $w_j = r_{j+1} \cdots r_k$ sent to $\text{inv}(w_j) = \text{inv}(u)$. By Proposition 8.5, $w_i = u'$ and $w_j = u$. This means that $u' = r_{i+1} \cdots r_j u$ and $u < u'$ in $[1, w]$. \square

Theorem 8.7 (Model posets). *For each isometry w , the poset structure of the interval $[1, w]$ is isomorphic to the model poset $P(w)$. As a consequence, the minimum length reflection factorizations of w are in bijection with the maximal chains in $P(w)$.*

Proof. By Propositions 7.2, 8.4 and 8.5 the invariant map is an order-preserving bijection from $[1, w]$ to $P(w)$ and by Proposition 8.6 its inverse is also order-preserving. \square

9. INTERVALS AND LATTICES

As mentioned in the introduction, a discussion of whether or not the intervals in W are lattices is included here because these results are needed elsewhere. By Theorem 8.7 this reduces to the question of which model posets are lattices. The elliptic case is straightforward.

Theorem 9.1 (Elliptic posets are lattices). *For each affine subspace B in E , the elliptic poset P^B is a complete lattice.*

Proof. This is an immediate consequence of Proposition 7.4. \square

The hyperbolic posets are usually not lattices. To prove this, we begin by quoting a definition and a proposition from [BM10].

Definition 9.2 (Bowtie). We say that a poset P contains a *bowtie* if there exists a 4-tuple $(a, b : c, d)$ of distinct elements such that a and b are minimal upper bounds for c and d and c and d are maximal lower bounds for a and b . The name reflects the fact that when edges are drawn to show that a and b are above c and d , the configuration looks like a bowtie. See Figure 3.

Proposition 9.3 (Lattice or bowtie). *A bounded graded poset P is a lattice iff P contains no bowties.*

Thus one only needs to determine whether P^M contains any bowties. Note that maximal lower bounds are easy to calculate.

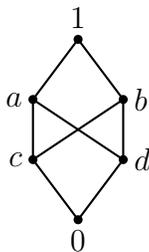


FIGURE 3. A bounded graded poset that is not a lattice.

Remark 9.4 (Maximal lower bounds). Let M be a nonlinear affine subspace of V . Most pairs of elements in P^M have a unique maximal lower bound. For example, $e^{B_1} \wedge e^{B_2} = e^C$ where C is the affine hull of $B_1 \cup B_2 \subset E$ and $h^{M_1} \wedge e^{B_1} = e^C$ where C is the unique affine subspace containing B_1 with $\text{DIR}(C)$ equal to the smallest linear subspace of V containing $\text{DIR}(B_1)$ and M_1^\perp . In other words, $\text{DIR}(C) = \text{SPAN}(\text{DIR}(B_1) \cup M_1^\perp)$. Also, so long as $M_3 = M_1 \cap M_2$ is nonempty, $h^{M_1} \wedge h^{M_2} = h^{M_3}$.

The only case not mentioned in Remark 9.4 is the following one.

Proposition 9.5 (Distinct maximal lower bounds). *If M is a nonlinear affine subspace of V and two elements in P^M do not have a unique maximal lower bound, then they are hyperbolic elements h^{M_1} and h^{M_2} with M_1 and M_2 disjoint and their maximal lower bounds are elliptic elements of the form e^B where $\text{DIR}(B)^\perp = \text{DIR}(M_1) \cap \text{DIR}(M_2)$.*

Proof. The cases other than two hyperbolics h^{M_1} and h^{M_2} with M_1 and M_2 are disjoint are ruled out by Remark 9.4. The common lower bounds for these two are elliptic elements of the form e^B where $\text{DIR}(B)^\perp$ is contained in $\text{SPAN}(M_1) \cap \text{SPAN}(M_2)$ and e^B is maximal when $\text{DIR}(B)^\perp = \text{SPAN}(M_1) \cap \text{SPAN}(M_2)$, but this expression simplifies. Because M_1 and M_2 are disjoint subsets of M , $\text{SPAN}(M_1)$ and $\text{SPAN}(M_2)$ only intersect inside the linear subspace $\text{DIR}(M)$. Also $\text{SPAN}(M_i) \cap \text{DIR}(M) = \text{DIR}(M_i)$, so we have $\text{SPAN}(M_1) \cap \text{SPAN}(M_2) = \text{DIR}(M_1) \cap \text{DIR}(M_2)$. \square

Theorem 9.6 (Hyperbolic posets are not lattices). *Let M be a nonlinear affine subspace of V . The poset P^M contains a bowtie and is not a lattice iff $\text{DIR}(M)$ contains a proper non-trivial linear subspace U , which is true iff the dimension of M is at least 2. More precisely, for every such subspace and for every choice of distinct elements h^{M_1} and h^{M_2} with $\text{DIR}(M_1) = \text{DIR}(M_2) = U$ and distinct elements e^{B_1} and e^{B_2}*

with $\text{DIR}(B_1) = \text{DIR}(B_2) = U^\perp$, these four elements form a bowtie. Conversely, all bowties in P^M are of this form.

Proof. Since it is easy to check that the elements listed form a bowtie, we focus on establishing the converse. If a and c are elements in P^M with distinct maximal lower bounds then by Proposition 9.5 they are both hyperbolic with disjoint move-sets, say $a = h^{M_1}$ and $b = h^{M_2}$ with M_1 and M_2 disjoint. Moreover, their distinct maximal lower bounds are elliptic elements of the form e^B where $\text{DIR}(B)^\perp = \text{DIR}(M_1) \cap \text{DIR}(M_2) = U$. Let $c = e^{B_1}$ and $d = e^{B_2}$ be two such elements with B_1 and B_2 distinct and note that because $\text{DIR}(B_1) = \text{DIR}(B_2)$, distinct implies disjoint. If we further assume that a and b are minimal upper bounds for c and d then we can conclude that $\text{DIR}(M_1) = \text{DIR}(M_2) = U$. The distinctness of a and b implies U is a proper subspace of $\text{DIR}(M)$ and the distinctness of c and d implies B_i is a proper subspace of E , $\text{DIR}(B_i) = U^\perp$ is a proper subspace of V , and U is nontrivial. \square

10. LATTICE COMPLETIONS

It is well-known that Dedekind used a method of cuts to complete the rationals to the reals. Less well-known is that H. M. MacNeille was able to generalize this technique of ‘‘Dedekind cuts’’ to show that every partially ordered set embeds in a complete lattice in an essentially unique and minimal way. The resulting complete lattice is called its Dedekind-MacNeille completion. We begin by reviewing its construction as described in [DP02] and then apply these results to the hyperbolic posets that fail to be lattices.

Definition 10.1 (Dedekind-MacNeille completion). Let P be a poset and for any subset Q in P , let Q^u and Q^ℓ denote the set of upper bounds for Q and lower bounds for Q , respectively. The *Dedekind-MacNeille completion* of P is the collection of subsets Q of P satisfying $Q = (Q^u)^\ell$ ordered by set inclusion.

Theorem 10.2 (Properties of $\text{DM}(P)$). *For any poset P , its Dedekind-MacNeille completion $\text{DM}(P)$ is a complete lattice. Moreover, there is an order-preserving embedding $\varphi: P \rightarrow \text{DM}(P)$ of P into its Dedekind-MacNeille completion given by sending each element p of P to $\{p\}^\ell$.*

The Dedekind-MacNeille completion of a poset P can be difficult to construct from the given definition, particularly when P is infinite, but there is a characterization theorem which enables one to recognize $\text{DM}(P)$ once it has been constructed by other means.

Definition 10.3 (Join-dense and Meet-dense). Let Q be a subset of a poset P . We say that Q is *join-dense* in P if every element of P is

the join of some subset of Q . Similarly, Q is *meet-dense* in P if every element of P is the meet of some subset of Q .

The following is a restatement of Theorem 2.36 in [DP02].

Theorem 10.4 (Characterizing $\text{DM}(P)$). *Let P be an ordered set and let $\varphi: P \rightarrow \text{DM}(P)$ be the order-embedding of P into its Dedekind-MacNeille completion defined above. The image of P under φ is join-dense and meet-dense in the complete lattice $\text{DM}(P)$. Conversely, if L is a complete lattice and P is a subset of L which is both join-dense and meet-dense in L , then L is order-isomorphic to $\text{DM}(P)$ via an order-isomorphism which agrees with φ on P .*

Using this result we are ready to construct the Dedekind-MacNeille completion of P^M where M is a nonlinear affine subspace of V of dimension at least 2. Given Proposition 9.3 and Theorem 9.6, it should not be too surprising that the additional elements are closely related to the locations of the bowties in P^M .

Definition 10.5 (Augmenting P^M). Let P^M be a hyperbolic poset with M a nonlinear affine subspace of V . We define a poset \tilde{P}^M that contains P^M as an induced subposet. The additional elements are of the form n^U where U is a proper nontrivial linear subspace of $\text{DIR}(M)$ and they are ordered by inclusion, i.e. $n^U < n^{U'}$ iff $U \subset U'$. We also set $n^U < h^{M'}$ iff $U \subset \text{DIR}(M')$ and $e^B < n^U$ iff $\text{DIR}(B)^\perp \subset U$.

Proposition 10.6 (Complete lattice). *Every augmented hyperbolic poset is a complete lattice.*

Proof. This is straightforward. As an illustration we sketch the proof that arbitrary meets exist and leave the existence of arbitrary joins as an easy exercise. Let $Q \subset \tilde{P}^M$ be a collection of hyperbolic elements h^{M_i} , new elements n^{U_j} and elliptic elements e^{B_k} . If there is at least one elliptic in Q then $\bigwedge Q$ is the elliptic e^C where C is the smallest affine subspace containing each B_k and where $\text{DIR}(C)$ must contain certain directions determined by the M_i and U_j . If there are only hyperbolics in Q and the M_i 's have a point in common then $\bigwedge Q = h^{M'}$ where $M' = \bigcap M_i$. Finally, if the M_i 's do not have a common point or if there is at least one new element in Q , then either $\bigwedge Q = n^U$ when $U = (\bigcap U_j) \cap (\bigcap \text{DIR}(M_i))$ is nontrivial or $\bigwedge Q = e^E$ when the subspace U defined in this way is trivial. \square

Theorem 10.7 (Intervals and lattice completions). *For each affine subspace M in E , the augmented poset \tilde{P}^M is a complete lattice containing P as a meet-dense and join-dense subset. As a consequence, \tilde{P}^M is the Dedekind-MacNeille completion of P^M .*

Proof. By Proposition 10.6 and Theorem 10.4 we only need to show that the new elements are meets and joins of elements in P^M , but this is trivially true by Theorem 9.6 since each new element n^U is associated with a bowtie in P^M . In the terminology used there, n^U is the meet of h^{M_1} and h^{M_2} and it is the join of e^{B_1} and e^{B_2} . \square

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