

DUAL EUCLIDEAN ARTIN GROUPS AND THE FAILURE OF THE LATTICE PROPERTY

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ABSTRACT. The irreducible euclidean Coxeter groups that naturally act geometrically on euclidean space are classified by the well-known extended Dynkin diagrams and these diagrams also encode the modified presentations that define the irreducible euclidean Artin groups. These Artin groups have remained mysterious with some exceptions until very recently. Craig Squier clarified the structure of the three examples with three generators more than twenty years ago and François Digne more recently proved that two of the infinite families can be understood by constructing a dual presentation for each of these groups and showing that it forms an infinite-type Garside structure. In this article I establish that none of the remaining dual presentations for irreducible euclidean Artin groups correspond to Garside structures because their factorization posets fail to be lattices. These are the first known examples of dual Artin presentations that fail to form Garside structures. Nevertheless, the results presented here about the cause of this failure form the foundation for a subsequent article in which the structure of euclidean Artin groups is finally clarified.

There is an irreducible Artin group of euclidean type for each of the extended Dynkin diagrams. In particular, there are four infinite families, \tilde{A}_n , \tilde{B}_n , \tilde{C}_n and \tilde{D}_n , known as the *classical* types plus five remaining *exceptional* examples \tilde{E}_8 , \tilde{E}_7 , \tilde{E}_6 , \tilde{F}_4 , and \tilde{G}_2 . Most of these groups have been poorly understood until very recently. Among the few known results are a clarification of the structure of the Artin groups of types \tilde{A}_2 , \tilde{C}_2 and \tilde{G}_2 by Craig Squier in [Squ87] and two papers by François Digne [Dig06, Dig12] proving that the Artin groups of type \tilde{A}_n and \tilde{C}_n have dual presentations that are Garside structures. Our main result is that Squier's and Digne's examples are the only ones that have dual presentations that are Garside structures.

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Theorem A (Dual presentations and Garside structures). *The unique dual presentation of $\text{ART}(\tilde{X}_n)$ is a Garside structure when X is C or G and it is not a Garside structure when X is B, D, E or F . When the group has type A there are distinct dual presentations and the one investigated by Digne is the only one that is a Garside structure.*

The proof is made possible by a simple combinatorial model developed in collaboration with Noel Brady that encodes all minimal length factorizations of a euclidean isometry into reflections [BM]. Although the results here are essentially negative, they establish the foundations for positive results presented in [MS]. In this subsequent article a new class of Garside groups are constructed from crystallographic groups closely related to euclidean Coxeter groups and these new groups contain euclidean Artin groups as subgroups, thereby clarifying their algebraic structure. For a survey of all three articles see [McCb].

The article is structured as follows. The first sections give basic definitions, define dual Artin groups and dual presentations, and review the results of [BM] on factoring euclidean isometries into reflections. The middle sections apply these results to understand how Coxeter elements of irreducible euclidean Coxeter groups can be factored into reflections present in the group, thereby constructing dual presentations. The final sections record explicit results on a type-by-type basis, from which the main theorem immediately follows. As a final note, I would like to highlight the fact that the rough outline of the main theorem was established in collaboration with John Crisp several years ago while I was visiting him in Dijon. John has since left mathematics but his central role in the genesis of this work needs to be acknowledged.

1. BASIC DEFINITIONS

This short section provides some basic definitions that are included for completeness. The terminology roughly follows [DP02], [Hum90] and [Sta97].

Definition 1.1 (Coxeter groups). A *Coxeter group* is any group W that can be defined by a presentation of the following form. It has a standard finite generating set S and only two types of relations. For each $s \in S$ there is a relation $s^2 = 1$ and for each unordered pair for distinct elements $s, t \in S$ there is at most one relation of the form $(st)^m = 1$ where $m = m(s, t) > 1$ is an integer. When no relation involving s and t occurs we consider $m(s, t) = \infty$. A *reflection* in W is any conjugate of an element of S and we use R to denote the set of all reflections in W . In other words, $R = \{wsw^{-1} \mid s \in S, w \in W\}$. This

presentation is usually encoded in a labeled graph Γ called a *Coxeter diagram* with a vertex for each $s \in S$, an edge connecting s and t if $m(s, t) > 2$ and a label on this edge if $m(s, t) > 3$. The group defined by the presentation encoded in Γ is denoted $W = \text{COX}(\Gamma)$. A Coxeter group is *irreducible* when its diagram is connected.

Definition 1.2 (Artin groups). For each Coxeter diagram Γ there is an *Artin group* $\text{ART}(\Gamma)$ defined by a presentation with a relation for each two-generator relation in the standard presentation of $\text{COX}(\Gamma)$. More specifically, if $(st)^m = 1$ is a relation in $\text{COX}(\Gamma)$ then the presentation of $\text{ART}(\Gamma)$ has a relation that equates the two length m words that strictly alternate between s and t . Thus $(st)^2 = 1$ becomes $st = ts$, $(st)^3 = 1$ becomes $sts = tst$, $(st)^4 = 1$ becomes $stst = tsts$, etc. There is no relation when $m(s, t)$ is infinite.

Definition 1.3 (Posets). Let P be a partially ordered set. If P contains both a minimum element and a maximum element then it is *bounded*. For each $Q \subset P$ there is an induced *subposet* structure on Q by restricting the partial order on P . A subposet C in which any two elements are comparable is called a *chain* and its *length* is $|C| - 1$. Every finite chain is bounded and its maximum and minimum elements are its *endpoints*. If a finite chain C is not a subposet of a strictly larger finite chain with the same endpoints, then C is *saturated*. Saturated chains of length 1 are called *covering relations*. If every saturated chain in P between the same pair of endpoints has the same finite length, then P is *graded*. The *rank* of an element p is the length of the longest chain with p as its upper endpoint and its *corank* is the length of the longest chain with p as its lower endpoint, assuming such chains exist. The *dual* P^* of a poset P has the same underlying set but the order is reversed, and a poset is *self-dual* when it and its dual are isomorphic.

Definition 1.4 (Lattices). Let Q be any subset of a poset P . A lower bound for Q is any $p \in P$ with $p \leq q$ for all $q \in Q$. When the set of lower bounds for Q has a maximum element, this element is the *greatest lower bound* or *meet* of Q . Upper bounds and the *least upper bound* or *join* of Q are defined analogously. The meet and join of Q are denoted $\bigwedge Q$ and $\bigvee Q$ in general and $u \wedge v$ and $u \vee v$ if u and v are the only elements in Q . When every pair of elements has a meet and a join, P is a *lattice* and when every subset has a meet and a join, it is a *complete lattice*.

Definition 1.5 (Bowties). Let P be a poset. A *bowtie* in P is a 4-tuple of distinct elements $(a, b : c, d)$ such that a and b are minimal upper bounds for c and d and c and d are maximal lower bounds for

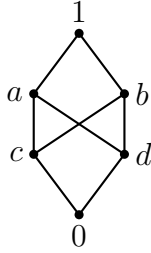


FIGURE 1. A bounded graded poset that is not a lattice.

a and b . The name reflects the fact that when edges are drawn to show that a and b are above c and d , the configuration looks like a bowtie. See Figure 1. It turns out that a bounded graded poset P is a lattice iff P contains no bowties [BM10]. This makes it easy to show that certain subposets are also not lattices. For example, if P is not a lattice because it contains a bowtie $(a, b : c, d)$ and Q is any subposet that contains all four of these elements, then Q is also not a lattice since it contains the same bowtie.

2. DUAL ARTIN GROUPS

As mentioned in the introduction, attempts to understand Artin groups of euclidean type using standard techniques have had limited success. In this article and its sequel significant progress is made through the use of dual presentations for Artin groups defined using intervals in Coxeter groups [MS].

Definition 2.1 (Intervals in metric spaces). Let x, y and z be points in a metric space (X, d) . Borrowing from euclidean plane geometry we say that z is *between* x and y whenever the triangle inequality degenerates into an equality. Concretely z is between x and y when $d(x, z) + d(z, y) = d(x, y)$. The *interval* $[x, y]$ is the collection of points between x and y and this includes both x and y . Intervals can also be endowed with a partial ordering by declaring that $u \leq v$ whenever $d(x, u) + d(u, v) + d(v, y) = d(x, y)$.

Fixing a generating set for a group defines a natural metric and this leads to the notion of an interval in a group.

Definition 2.2 (Intervals in groups). A *marked group* is a group G with a fixed generating set S which, for convenience, we assume is symmetric and injects into G . The (right) *Cayley graph of G with respect to S* is a labeled directed graph denoted $\text{CAY}(G, S)$ with vertices indexed by G and edges indexed by $G \times S$. The edge $e_{(g,s)}$ has *label* s , it starts

at v_g and ends at $v_{g'}$ where $g' = g \cdot s$. There is a natural faithful, vertex-transitive, label and orientation preserving left action of G on its Cayley graph and these are the only graph automorphisms that preserves labels and orientations. The *distance* $d(g, h)$ is the combinatorial length of the shortest path in the Cayley graph from v_g to v_h and note that the symmetry assumption allows us to restrict attention to directed paths. This defines a metric on G and from this metric we get intervals. More explicitly, for $g, h \in G$, the *interval* $[g, h]$ is the poset of group elements between g and h with $g' \in [g, h]$ when $d(g, g') + d(g', h) = d(g, h)$ and $g' \leq g''$ when $d(g, g') + d(g', g'') + d(g'', h) = d(g, h)$.

The interval $[g, h]$ is a bounded graded poset whose Hasse diagram is embedded as a subgraph of the Cayley graph $\text{CAY}(G, S)$ as the union of all minimal length directed paths from v_g to v_h . This is because $g' \in [g, h]$ means $v_{g'}$ lies on some minimal length path from v_g to v_h and $g' < g''$ means that $v_{g'}$ and $v_{g''}$ both occur on a common minimal length path from v_g to v_h with $v_{g'}$ occurring before $v_{g''}$. Because the structure of a graded poset can be recovered from its Hasse diagram, we let $[g, h]$ denote the edge-labeled directed graph that is visible inside $\text{CAY}(G, S)$. The left action of a group on its right Cayley graph preserves labels and distances. Thus the interval $[g, h]$ is isomorphic (as a labeled oriented directed graph) to the interval $[1, g^{-1}h]$. In other words, every interval in the Cayley graph of G is isomorphic to one that starts at the identity. We call $g^{-1}h$ the *type* of the interval $[g, h]$ and note that intervals are isomorphic iff they have the same type.

Definition 2.3 (Distance order). The *distance order* on a marked group G is defined by setting $g' \leq g$ iff $g' \in [1, g]$. This turns G into a poset that contains an interval of every type that occurs in the metric space on G . Next, there is a *length function* $\ell_S: G \rightarrow \mathbb{N}$ that sends each element to its distance from the identity. The value $\ell_S(g) = d(1, g)$ is called the *S-length of g* and it is also the length of the shortest factorization of g in terms of elements of S . Because Cayley graphs are homogeneous, metric properties of the distance function translate into properties of ℓ_S . Symmetry and the triangle inequality, for example, imply that $\ell_S(g) = \ell_S(g^{-1})$, and $\ell_S(gh) \leq \ell_S(g) + \ell_S(h)$.

Intervals in groups can be used to construct new groups.

Definition 2.4 (Interval groups). Let G be a group generated by a set S and let g and h be distinct elements in G . The *interval group* $G_{[g, h]}$ is defined as follows. Let S_0 be the elements of S that actually occurs as labels of edges in $[g, h]$. The group $G_{[g, h]}$ has S_0 as its generators and we impose all relations that are visible as closed loops inside the portion

of the Cayley graph of G that we call $[g, h]$. The elements in $S \setminus S_0$ are not included since they do not occur in any relation. More precisely, if they were included as generators, they would generate a free group that splits off as a free factor. Thus it is sufficient to understand the group defined above. Next note that this group structure only depends on the type of the interval so it is sufficient to consider interval groups of the form $G_{[1, g]}$. For these groups we simplify the notation to G_g and say that G_g is the interval group obtained by *pulling G apart at g* .

The interval $[1, g]$ incorporates all of the essential information about the presentation of G_g . More traditional presentations for interval groups using relations are established in [McCa] and described in the next section. Dual Artin groups are examples of interval groups.

Definition 2.5 (Dual Artin groups). Let $W = \text{COX}(\Gamma)$ be a Coxeter group with standard generating set S and reflections R . For any fixed total ordering of the elements of S , the product of these generators in this order is called a *Coxeter element* and for each Coxeter element w there is a dual Artin group defined as follows. Let $[1, w]$ be the interval in the Cayley graph of W with respect to R and let $R_0 \subset R$ be the subset of reflections that actually occur in some minimal length factorizations of w . The *dual Artin group with respect to w* is the group $W_w = \text{ART}^*(\Gamma, w)$ generated by R_0 and subject only to those relations that are visible inside the interval $[1, w]$.

Remark 2.6 (Artin groups and dual Artin groups). In general the relationship between the Artin group $\text{ART}(\Gamma)$ and the dual Artin group $\text{ART}^*(\Gamma, w)$ is not yet completely clear. It is straightforward to show using the Tits representation that the product of the elements in S that produce w is a factorization of w into reflections of minimum length which means that this factorization describes a directed path in $[1, w]$. As a consequence S is a subset of R_0 . Moreover, the standard Artin relations are consequences of relations visible in $[1, w]$ (as illustrated in [BM00]) so that the injection of S into R_0 extends to a group homomorphism from $\text{ART}(\Gamma)$ to $\text{ART}^*(\Gamma, w)$. When this homomorphism is an isomorphism, we say that the interval $[1, w]$ encodes a *dual presentation* of $\text{ART}(\Gamma)$.

Every dual Artin group that has been successfully analyzed so far is isomorphic to the corresponding Artin group and as a consequence its group structure is independent of the Coxeter element w used in its construction. It is precisely because this assertion has not been proved in full generality that dual Artin groups deserve a separate name. One reason that dual Artin groups are of interest is that they nearly satisfy

the requirements to be Garside groups. In fact, from the construction it is easy to show that interval $[1, w]$ used to define a dual Artin group has all of the properties of a Garside structure with one exception.

Proposition 2.7 (Garside structures). *Let Γ be a Coxeter diagram and let w be a Coxeter element for $W = \text{COX}(\Gamma)$. If the interval $[1, w]$ is a lattice, then the dual Artin group $\text{ART}^*(\Gamma, w)$ is a Garside group.*

The reader should note that we are using “Garside structure” and “Garside group” in the expanded sense of Digne [Dig06, Dig12] rather than the original definition that requires the generating set to be finite. In the language of the “Foundations of Garside theory” book [DDG⁺] these are “quasi-Garside” groups and structures. Since these are the only types of Garside structures considered here, the prefix “quasi” is dropped but we shall occasionally remind the reader that the interval $[1, w]$ has infinitely many elements. The grading of the interval used to define an interval group substitutes for finiteness of the generating set in forcing algorithmic processes to terminate. The standard proofs are otherwise unchanged. With the exception of the shift from finite to infinite generating sets, Proposition 2.7 was stated by David Bessis in [Bes03, Theorem 0.5.2]. For a more detailed discussion see [Bes03] and particularly the book [DDG⁺]. Interval groups appear in [DDM13] and in [DDG⁺, Chapter VI] as the “germ derived from a groupoid”. The terminology is different but the translation is straightforward.

Consequences of being a Garside group include normal forms for elements and a finite-dimensional classifying space, which imply that the group has a decidable word problem and is torsion-free [CMW04, DP99]. It was an early hope that every dual Artin group would be a Garside group, but this article provides the first explicit examples where this hope fails. Concretely, when w is Coxeter element for an irreducible euclidean Coxeter group W , we show that the interval $[1, w]$ in $\text{CAY}(W, R)$ is not a lattice except for the cases already analyzed by Squier and Digne.

3. DUAL PRESENTATIONS

The section records some known results about presentations for interval groups in general and for dual Artin groups in particular.

Definition 3.1 (Factorizations). In a group G generated by a set S , each positive word over S can be evaluated as a group element and the word *represents* the element g to which it evaluates. In the Cayley graph $\text{CAY}(G, S)$ a word represents g iff the unique directed path that starts at v_1 and corresponds to the word ends at the vertex v_g .

An element is *positive* if it is represented by some positive word. A *minimal positive factorization* of a positive element g is a word corresponding to a minimal length directed path from v_1 to v_g in the Cayley graph $\text{CAY}(G, S)$. Minimal positive factorizations are called *reduced S -decompositions* in [Bes03]. Inside an interval every directed path corresponds to a positive word whose length is equal to the distance between its endpoints. In particular, there is a bijective correspondence between directed paths in $[1, g]$ from v_1 and v_g and minimal positive factorizations of g .

Definition 3.2 (Bigons). Let g be a positive element in an S -group G and consider two directed paths in $[1, g]$ that start at the same vertex and end at the same vertex. The relation $U = V$ that equates the positive words U and V corresponding to these paths is called a *bigon relation* and it holds in G_g since UV^{-1} is visible as a closed loop inside $[1, g]$. Both positive words necessarily have the same length k which we call the *height* of the relation. A bigon relation is *big* when its height is $k = d(1, g)$, i.e. as big as possible and it is *small* when it is not a consequence of those bigon relations of strictly shorter height.

The various types of bigon relations are sufficient to define interval groups. A detailed proof of the following proposition can be found in [McCa] but we include a brief version for completeness.

Proposition 3.3 (Bigon presentations). *If g is a positive element in a group G generated by a set S , S_0 is the subset of S labeling edges in $[1, g]$, and \mathcal{R}_a , \mathcal{R}_b and \mathcal{R}_s denote the collection of all bigon, big bigon and small bigon relations, respectively, visible in the interval $[1, g]$, then $\langle S_0 \mid \mathcal{R}_a \rangle$, $\langle S_0 \mid \mathcal{R}_b \rangle$ and $\langle S_0 \mid \mathcal{R}_s \rangle$ are three presentations of G_g .*

Proof. Since \mathcal{R}_a , \mathcal{R}_b and \mathcal{R}_s are all subsets of the relations visible inside the interval $[1, g]$ it is sufficient to show that the remaining relations are consequences of these relations. First, given any closed undirected path in the portion of the Cayley graph that is $[1, g]$, one can add a path from v_1 to each vertex and show that this loop is a consequence of bigon relations. More explicitly, draw the closed loop as simple loop in the plane, place v_1 in the center and the paths to the vertices as subdivided radial line segments. Every complementary region is then a bigon relation. This shows that $\langle S_0 \mid \mathcal{R}_a \rangle$ is a presentation for G_g . Next, by extending each bigon with paths from v_1 to the start point and from the endpoint to v_g , it is clear that every bigon is a consequence via cancellation of a big bigon relation. And finally, the small bigon relations are, by definition, sufficient to establish all big bigon relations. \square

When the generating set is closed under conjugation – as is the case with the reflections inside a Coxeter group – there are many bigon relations of height 2 visible in any interval.

Definition 3.4 (Hurwitz action). If S is any subset of a group G that is closed under conjugation and S^n denotes all words of length n over S , then there is a natural action of the n -strand braid group on S^n . The standard braid generator s_i replaces the two letter subword ab in positions i and $i + 1$ with the subword ca where $c = aba^{-1} \in S$ and it leaves the letters in the other positions unchanged. It is straightforward to check that this action satisfies the relations in the standard presentation of the braid group. Of particular interest here is that every word in the same orbit under this action evaluates to the same element of G and thus there is a well-defined *Hurwitz action* of the k -strand braid group on the minimal positive factorizations of an element g where $k = d(1, g)$.

Notice that when a standard braid generator replaces ab with ca inside a minimal positive factorization of g , $ab = ca$ is a height 2 bigon relation visible in $[1, g]$. It thus makes sense to call any height 2 bigon relation of the form $ab = ca$ visible inside $[1, g]$ a *Hurwitz relation*. Relations of this form are what Bessis calls *dual braid relations* in [Bes]. When the Hurwitz action is transitive on factorizations, these relations are sufficient to define G_g .

Proposition 3.5 (Hurwitz presentations). *Let g be a positive element in a group G generated by S and let S_0 be the subset of S labeling edges in $[1, g]$. If the Hurwitz action is transitive on minimal positive factorizations of g , then $\langle S_0 \mid \mathcal{R}_h \rangle$ is a presentation of G_g where \mathcal{R}_h denotes the collection of Hurwitz relations visible in $[1, g]$.*

Proof. Let H be the group defined in the statement of the proposition. Since the Hurwitz relations are visible in $[1, g]$ they are satisfied by G_g and every relation that holds in H also holds in G_g . On the other hand, the transitivity of the action implies that every bigon relation is a consequence of Hurwitz relations and by Proposition 3.3 these are sufficient to define G_g . Thus every relation that holds in G_g holds in H and the two groups are the same. \square

Although their results are often stated in a completely different language, representation theorists have already addressed the question of whether or not the Hurwitz action is transitive on minimal length reflection factorizations of a Coxeter element in a Coxeter group in many contexts. For example, Crawley-Boevey has shown that transitivity

holds when every $m = m(s, t)$ is either 2, 3, or ∞ [CB92] and Ringel has extended this to include the crystallographic cases, i.e. where every $m = m(s, t)$ is 2, 3, 4, 6, or ∞ [Rin94]. In 2010 Igusa and Schiffler in [IS10] proved transitivity of the Hurwitz action for all Coxeter groups in complete generality and in 2014 a short proof of this general fact was posted by Baumeister, Dyer, Stump and Wegener [BDSW]. See also [IT09] and [Igu11]. As a consequence Proposition 3.5 applies to all of the dual Artin groups of euclidean type and we can use this to establish that dual euclidean Artin groups and euclidean Artin groups are isomorphic. A proof of Theorem 3.6 will be included as an appendix in the final paper in this series [MS].

Theorem 3.6 (Dual Artin groups are Artin groups). *For every choice of Coxeter element w in an irreducible euclidean Coxeter group $W = \text{COX}(\tilde{X}_n)$, the dual Artin group $\text{ART}^*(\tilde{X}_n, w)$ is naturally isomorphic to the Artin group $\text{ART}(\tilde{X}_n)$.*

We conclude this section by noting an elementary consequence of the Hurwitz action.

Lemma 3.7 (Rewriting factorizations). *Let $w = r_1 r_2 \cdots r_k$ be a reflection factorization in a Coxeter group W . For any selection $1 \leq i_1 < i_2 < \cdots < i_j \leq k$ of positions there is a length k reflection factorization of w whose first j reflections are $r_{i_1} r_{i_2} \cdots r_{i_j}$ and another length k reflection factorization of w where these are the last j reflections in the factorization.*

4. EUCLIDEAN ISOMETRIES

In order to analyze intervals in euclidean Coxeter groups, we need to establish some notations for euclidean isometries. As in [BM] and [ST89] we sharply distinguish between points and vectors.

Definition 4.1 (Points and vectors). Throughout the article, V denotes an n -dimensional real vector space with a positive definite inner product and E denotes the euclidean which is its affine analog where the vector space structure of V (in particular the location of the origin) has been forgotten. The elements of V are *vectors* and the elements of E are *points*. We use greek letters for vectors and roman letters for points. There is a uniquely transitive action of V on E . Thus, given a point x and a vector λ there is a unique point y with $x + \lambda = y$ and given two points x and y there is a unique vector λ with $x + \lambda = y$. We say that λ is the vector from x to y . For any $\lambda \in V$, the map $x \mapsto x + \lambda$ is an isometry t_λ of E that we call a *translation* and note

that $t_\mu t_\nu = t_{\mu+\nu} = t_\nu t_\mu$ so the set $T_E = \{t_\lambda \mid \lambda \in V\}$ is an abelian group. For any point $x \in E$, the map $\lambda \mapsto x + \lambda$ is a bijection that identifies V and E but the isomorphism depends on this initial choice of a *basepoint* x in E . Lengths of vectors and angles between vectors are calculated using the usual formulas and distances and angles in E are defined by converting to vector-based calculations.

Definition 4.2 (Linear subspaces of V). A *linear subspace* of V is a subset closed under linear combination and every subset of V is contained in a unique minimal linear subspace called its *span*. Every subset U has an *orthogonal complement* U^\perp consisting of those vectors in V orthogonal to all the vectors in U . When U is a linear space there is a corresponding orthogonal decomposition $V = U \oplus U^\perp$ and the *codimension* of U is the dimension of U^\perp . For more general subsets $U^\perp = \text{SPAN}(U)^\perp$. The linear subspaces of V form a bounded graded self-dual complete lattice under inclusion that we call $\text{LIN}(V)$. The bounding elements are clear, the grading is by dimension (in that a k -dimensional subspace has rank k and corank $n - k$), the meet of a collection of subspaces is their intersection and their join is the span of their union. And finally, the map sending a linear subspace to its orthogonal complement is a bijection that establishes self-duality.

Definition 4.3 (Affine subspaces of E). An *affine subspace* of E is any subset B that contains every line determined by distinct points in B and every subset of E is contained in a unique minimal affine subspace called its *affine hull*. Associated with any affine subspace B is its (linear) *space of directions* $\text{DIR}(B) \subset V$ consisting of the collection of vectors connecting points in B . The *dimension* and *codimension* of B is that of its space of directions. The *affine subspaces* of E partially ordered by inclusion form a poset we call $\text{AFF}(E)$. It is a graded poset that is bounded above but not below since distinct points are distinct minimal elements. It is neither self-dual nor a lattice. There is, however, a well-defined rank-preserving poset map $\text{AFF}(E) \twoheadrightarrow \text{LIN}(V)$ sending each affine subspace B to its space of directions $\text{DIR}(B)$.

Definition 4.4 (Standard forms). An affine subspace of V is any subspace that corresponds to an affine subspace of E under an identification of V and E and the subspaces of this form are translations of linear subspaces. In particular, every affine subspace M in V can be written in the form $M = t_\mu(U) = U + \mu = \{\lambda + \mu \mid \lambda \in U\}$ where U is a linear subspace of V . This representation is not unique, since $U + \mu = U$ for all $\mu \in U$, but it can be made unique if we insist that μ to be of minimal length or, equivalently, that μ be a vector in U^\perp . In this case we say $U + \mu$ is the *standard form* of M .

The isometries of E form a group $\text{ISOM}(E)$ and every isometry has two basic invariants, one in V and the other in E .

Definition 4.5 (Basic invariants). Let w be an isometry of E . If λ is the vector from x to $w(x)$ then we say x is *moved by λ under w* . The collection $\text{MOV}(w) = \{\lambda \mid x + \lambda = w(x), x \in E\} \subset V$ of all such vectors is the *move-set* of w . The move-set is an affine subspace and thus has standard form $U + \mu$ where U is a linear subspace and μ is a vector in U^\perp . The points in E that are moved by μ under w are those that are moved the shortest distance. The collection $\text{MIN}(w)$ of all such points is an affine subspace called the *min-set* of w . The sets $\text{MOV}(w) \subset V$ and $\text{MIN}(w) \subset E$ are the *basic invariants* of w .

Definition 4.6 (Types of isometries). Let w be an isometry of E and let $U + \mu$ be the standard form of its move-set $\text{MOV}(w)$. There are points fixed by w iff μ is trivial iff $\text{MOV}(w)$ is a linear subspace. Under these conditions we say w is *elliptic* and the min-set $\text{MIN}(w)$ is just the *fix-set* $\text{FIX}(w)$ of points fixed by w . Similarly, w has no fixed points iff μ is nontrivial iff $\text{MOV}(w)$ a nonlinear affine subspace of V . Under these conditions we say w is *hyperbolic*.

The names elliptic and hyperbolic come from a tripartite classification of isometries of nonpositively curved spaces; the third type, parabolic, does not occur in this context [BH99]. The simplest examples of hyperbolic isometries are the nontrivial translations as defined in Definition 4.1. They can also be characterized as those isometries whose move-set is a single point or whose min-set is all of A . The simplest example of an elliptic isometry is a reflection.

Definition 4.7 (Reflections). A *hyperplane* H in E is an affine subspace of codimension 1 and there is a unique nontrivial isometry r that fixes H pointwise called a *reflection*. The space of directions $\text{DIR}(H)$ is a codimension 1 linear subspace in V and it has a 1-dimensional orthogonal complement L . The basic invariants of r are $\text{MOV}(r) = L$ and $\text{MIN}(r) = \text{FIX}(r) = H$. The set of all reflections is denoted R_E .

5. FACTORIZATIONS

This section reviews the structure of intervals in $\text{ISOM}(E)$ when viewed as a group generated by the set R_E of all reflections. In particular, it introduces the combinatorial models constructed in [BM] that encode the poset structure of these intervals. The first thing to note is that the length function with respect to all reflections is easy to compute using the basic invariants of isometries, a result known as Scherk's theorem [ST89].

Theorem 5.1 (Reflection length). *The reflection length of an isometry is determined by its basic invariants. More specifically, if w is an isometry of E whose move-set is k -dimensional, then $\ell_{R_E}(w) = k$ when w is elliptic, and $\ell_{R_E}(w) = k + 2$ when w is hyperbolic.*

Next, consider the following combinatorially defined posets.

Definition 5.2 (Model posets). We construct a global poset P from two types of elements. For each nonlinear affine subspace M in V , P contains a *hyperbolic* element h^M and for each affine subspace B in A , P contains an *elliptic* element e^B . We also define an *invariant map* $\text{inv}: \text{ISOM}(E) \rightarrow P$ that sends w to $h^{\text{Mov}(w)}$ when w is hyperbolic and to $e^{\text{Fix}(w)}$ when w is elliptic. This explains the names and the notation. The elements of P are ordered as follows. First, hyperbolic elements are ordered by inclusion and elliptic elements by reverse inclusion: $h^M \leq h^{M'}$ iff $M \subset M'$ and $e^B \leq e^{B'}$ iff $B \supset B'$. Next, no elliptic element is ever above a hyperbolic element. And finally, $e^B < h^M$ iff $M^\perp \subset \text{DIR}(B)$. Note, however, that since M is nonlinear, the vectors orthogonal to all of M are also orthogonal to its span, a linear subspace whose dimension is $\dim(M) + 1$. Transitivity is an easy exercise. It was shown in [BM] that when $\text{ISOM}(E)$ is viewed as a marked group generated by the set R_E of all reflections and viewed as a poset under the distance order, the invariant map is a rank-preserving order-preserving map from $\text{ISOM}(E)$ to P . As a consequence, for any isometry w the invariant map sends isometries in $[1, w]$ to elements less than or equal to $\text{inv}(w)$. Let $P(w)$ denote the subposet of P induced by restricting to those elements less than or equal to $\text{inv}(w)$ and call $P(w)$ the *model poset for w* .

The following is the main theorem proved in [BM].

Theorem 5.3 (Model posets). *For each isometry w , the invariant map establishes a poset isomorphism between the interval $[1, w]$ and the model poset $P(w)$. As a consequence, the minimum length reflection factorizations of w are in bijection with the maximal chains in $P(w)$.*

Theorem 5.3 is in sharp contrast with the non-injectivity of the invariant map in general. There are, for example, many different rotations that fix the same codimension 2 subspace. Since it is useful to have a notation for model subposets in the absence of an isometry, let P^M denote the subposet of P induced by restricting to those elements less than or equal to h^M for a nonlinear affine subspace $M \subset V$ and let P^B denote the subposet induced by restricting to those elements less than or equal to e^B for an affine subspace $B \subset E$. Thus

$P(w) = P^{\text{MOV}(w)}$ when w is hyperbolic and $P(w) = P^{\text{FIX}(w)}$ when w is elliptic. Note that this notation is not ambiguous because M and B are subsets of different spaces.

Remark 5.4 (Auxillary results). In the process of proving Theorem 5.3 many auxillary results are established in [BM] that are useful here. For example, the direction space of the min-set of an isometry is the orthogonal complement of the direction space of its move-set [BM, Lemma 5.3]. In symbols, when $\text{MOV}(w) = U + \mu$ in standard form, $\text{DIR}(\text{MOV}(w)) = U$ and $\text{DIR}(\text{MIN}(w)) = U^\perp$. Next, when w is an isometry and r is a reflection, the move-set of w and the move-set of rw are nested so that one is a codimension 1 subspace of the other [BM, Proposition 6.2]. As a consequence, the dimension of the min-set also changes by exactly one dimension in the opposite direction. A third useful result identifies the min-set of a hyperbolic isometry as the unique affine subspace B in E that is stabilized by w of the correct dimension and where all points undergo the same motion [BM, Proposition 3.5].

One final result that we need from [BM] is a characterization of exactly when hyperbolic posets are not lattices and the explicit locations of the bowties that bear witness to this fact.

Theorem 5.5 (Hyperbolic posets are not lattices). *Let M be a non-linear affine subspace of V . The poset P^M contains a bowtie and is not a lattice iff $\text{DIR}(M)$ contains a proper non-trivial linear subspace U , which is true iff the dimension of M is at least 2. More precisely, for every such subspace and for every choice of distinct elements h^{M_1} and h^{M_2} with $\text{DIR}(M_1) = \text{DIR}(M_2) = U$ and distinct elements e^{B_1} and e^{B_2} with $\text{DIR}(B_1) = \text{DIR}(B_2) = U^\perp$, these four elements form a bowtie. Conversely, all bowties in P^M are of this form.*

6. EUCLIDEAN COXETER GROUPS

The irreducible Coxeter groups of interest in this article are those that naturally act geometrically, i.e. properly discontinuously and co-compactly by isometries, on a euclidean space with its generators acting as reflections. Their classification is well-known and they are described by the Coxeter diagrams known as the *extended Dynkin diagrams*. There are four infinite families and five sporadic examples of such diagrams and they are displayed in Figure 2. In this restricted context, it is traditional to replace edges labeled 4 and 6 with double and triple edges, respectively. The white vertex and the orientations on the double and triple edges are explained below. This section records

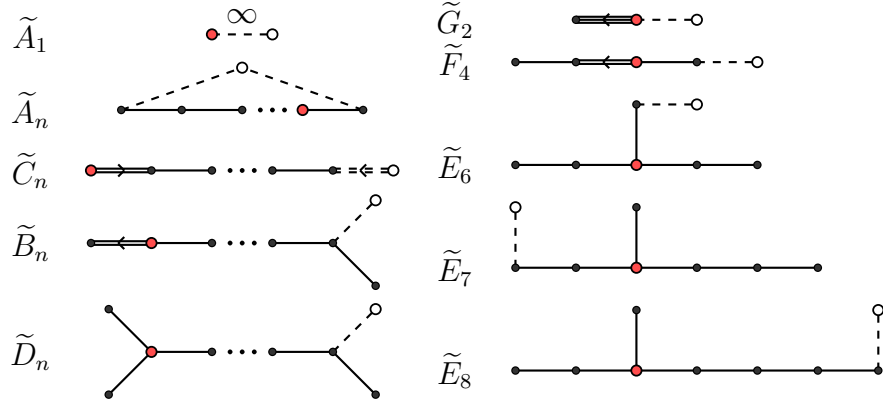


FIGURE 2. Four infinite families and five sporadic examples.

basic facts about these groups with the Coxeter group of type \tilde{G}_2 used to illustrate the concepts under discussion. For additional details see [Hum90].

Remark 6.1 (Diagrams and simplices). Each extended Dynkin diagram Γ is essentially a recipe that can be used to reconstruct a euclidean simplex σ with non-obtuse dihedral angles that are submultiples of π . The vertices of Γ index the outward pointing normal vectors α_s to the various facets of σ and the label $m = m(s, t)$ indicates that the angle between α_s and α_t is $\pi - \frac{\pi}{m}$ and thus the corresponding dihedral angle between their fixed facets is $\frac{\pi}{m}$. This is sufficient information to reconstruct a unique euclidean simplex up to similarity. The \tilde{G}_2 diagram in Figure 2, for example, leads to the construction of a 30-60-90 triangle.

Definition 6.2 (Coxeter complex). Let W be an irreducible euclidean Coxeter group with extended Dynkin diagram Γ and let σ be a corresponding euclidean simplex. If we embed σ in a euclidean space E so that E is the affine hull of σ and let S be the isometries of E that fix (the affine hull of) one of the facets of σ pointwise then these reflections generate a group of isometries naturally isomorphic to the Coxeter group W with standard generating set S . More precisely, using the orbit of σ under this group action, it is possible to give E the structure of a metric simplicial complex called the *Coxeter complex of W* . See Figure 3. The top-dimensional simplices are known as *chambers* and the one used to define the *simple system S* is the *fundamental chamber*. Other simple systems are obtained from other chambers. The resulting action of W on its Coxeter complex preserves the simplicial structure and is uniquely transitive on chambers. Using this action, the facets of

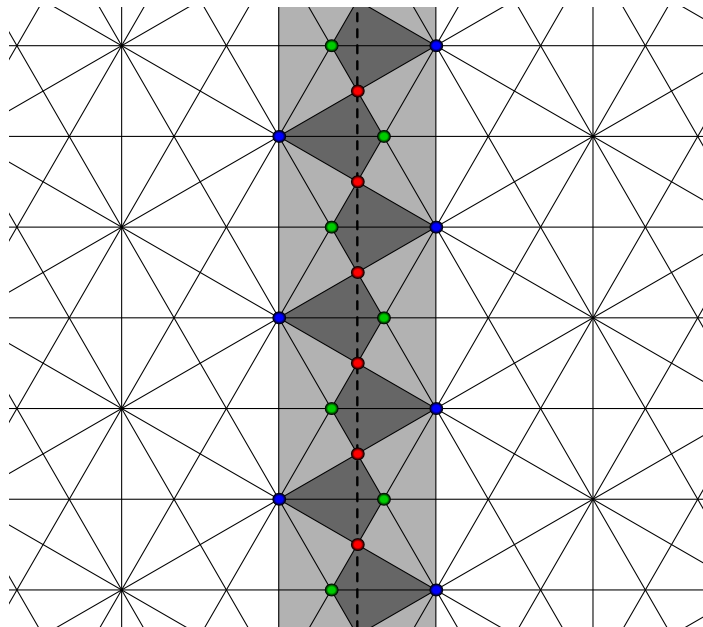


FIGURE 3. The \tilde{G}_2 tiling of the plane with annotations corresponding to a particular Coxeter element w . The dashed line is the glide axis of w , the heavily shaded triangles are those for which w is a bipartite Coxeter element, and the lightly shaded vertical strip is the convex hull of the vertices of these triangles.

any chamber, and the reflections that fix them, can be identified with the vertices of Γ in a canonical way. Also note that any proper subset of S is a collection of reflections whose hyperplanes intersect in a facet of σ and thus their product is elliptic.

There is an alternative encoding of the geometry of an irreducible euclidean Coxeter complex into a finite collection of vectors called roots.

Definition 6.3 (Roots). Let $W = \text{COX}(\tilde{X}_n)$ be an irreducible euclidean Coxeter group. If r is a reflection in W and α is any vector in V whose span is the line $L = \text{MOV}(r)$, then α is called a *root of r* . There is a finite collection of vectors $\Phi = \Phi_{X_n}$ called a *root system* that contains a pair of roots $\pm\alpha$ for each family of parallel hyperplanes defining reflections in W and the length of α encodes the minimal distance between these equally spaced parallel hyperplanes. The root system that encode the hyperplanes of the \tilde{G}_2 Coxeter group (shown in Figure 3) is illustrated in Figure 4. Note that longer roots correspond to hyperplanes with shorter distances between them. Root length is

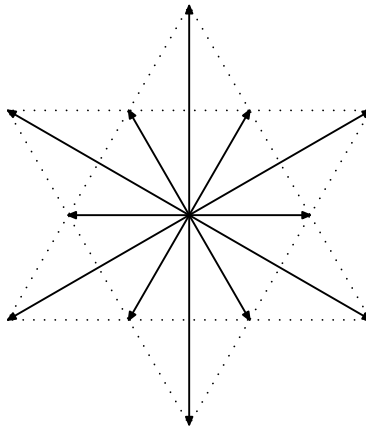


FIGURE 4. The root system Φ_{G_2} .

encoded in the extended Dynkin diagram as follows. When two vertices are connected by a single edge, the roots they represent are the same length, when they are connected by a double edge, one root is $\sqrt{2}$ times the length of the other and when they are connected by a triple edge, one root is $\sqrt{3}$ times the length of the other. The longer root is indicated by superimposing an inequality sign.

The Coxeter complex of an irreducible euclidean Coxeter group can be reconstructed from its root system because it always contains a point x with the property that every hyperplane of a reflection in W is parallel to a hyperplane of a reflection in W fixing x . In the \tilde{G}_2 example, x can be any corner of a triangle with a 30 degree angle. After identifying V and E using this special point as our origin, the reconstruction proceeds as follows.

Definition 6.4 (Reflections and hyperplanes). Let $\Phi = \Phi_{X_n}$ be a root system of an irreducible euclidean Coxeter group $W = \text{COX}(\tilde{X}_n)$. For each $\alpha \in \Phi$ and $i \in \mathbb{Z}$ let $H_{\alpha,i}$ denote the (affine) *hyperplane* in V of solutions to the equation $\langle x, \alpha \rangle = i$ where the brackets denote the standard inner product on V . The intersections of the hyperplanes give the simplicial structure. The unique nontrivial isometry of V that fixes $H_{\alpha,i}$ pointwise is a reflection that we call $r_{\alpha,i}$. The collection $R = \{r_{\alpha,i} \mid \alpha \in \Phi, i \in \mathbb{Z}\}$ generates a euclidean Coxeter group W and R is its set of reflections in the sense of Definition 1.1.

The reflections through the origin generate a finite Coxeter group W_0 related to W in two distinct ways.

Definition 6.5 (Dynkin diagrams). The hyperplanes $H_\alpha = H_{\alpha,0}$ are precisely the ones that contain the origin and the reflections $r_\alpha =$

$r_{\alpha,0}$ generate a finite Coxeter group W_0 that contains all elements of W fixing the origin. This embeds W_0 as a subgroup of W . There is also a well-defined group homomorphism $p: W \rightarrow W_0$ defined by sending each generating reflection $r_{\alpha,i}$ in W to r_α in W_0 . Choosing a fundamental chamber containing the origin shows that W_0 is a Coxeter group generated by all but one of the reflections in S . The vertex of the extended Dynkin diagram Γ corresponding to the missing reflection is shaded white. When Γ is an extended Dynkin diagram of type \tilde{X}_n , the subgraph without the white vertex is called a *Dynkin diagram of type X_n* . In particular, the finite Coxeter group $W_0 = \text{COX}(X_n)$.

In the notation of Definition 6.4, the white dot represents a reflection of the form $r_{\alpha,1}$ where $\alpha \in \Phi$ is a canonical vector of “highest weight”. The translations in W are described by coroots and the coroot lattice.

Definition 6.6 (Coroots). For each $\alpha \in \Phi$ consider the product $r_{\alpha,1}r_\alpha$, or equivalently $r_{\alpha,i+1}r_{\alpha,i}$. Reflecting through parallel hyperplanes produces a translation in the α direction and the exact translation is t_{α^\vee} where $\alpha^\vee = c\alpha$ is a *coroot* with $c = \frac{2}{\langle \alpha, \alpha \rangle}$. The collection of all coroots is denoted Φ^\vee and the integral linear combinations of vectors in Φ^\vee is a lattice $\mathbb{Z}(\Phi^\vee) \cong \mathbb{Z}^n$ called the *coroot lattice*. Because $t_\mu t_\nu = t_{\mu+\nu} = t_\nu t_\mu$, there is a translation of the form t_λ in W for each $\lambda \in \mathbb{Z}(\Phi^\vee)$ and the set $T = \{t_\lambda \mid \lambda \in \mathbb{Z}(\Phi^\vee)\}$ forms an abelian subgroup of W . In fact, these are the only translations that are contained in W (i.e. $T = T_E \cap W$), the subgroup T is the kernel of the map $p: W \rightarrow W_0$ and W is a semidirect product of W_0 and T .

7. COXETER ELEMENTS

This section coarsely classifies Coxeter elements in irreducible euclidean Coxeter groups. Recall that a *Coxeter element* in a Coxeter group W with standard generating set S is a product of the reflections of S in some linear order. We begin by determining the basic geometric invariants of a Coxeter element in an irreducible euclidean Coxeter group when viewed as a euclidean isometry. The key observation is that a collection of vectors normal to the facets of a euclidean simplex are almost linearly independent in the sense that every proper subset is linearly independent but the full set is not.

Proposition 7.1 (Simple systems and elliptic isometries). *If S is a simple system of an irreducible euclidean Coxeter group W corresponding to a chamber σ , then the product of any proper subset of the reflections in S is an elliptic element whose fix-set is the affine hull of*

the face of σ determined by the intersection of the corresponding hyperplanes. Moreover, this is a minimum length reflection factorization of the resulting elliptic isometry.

Proof. The hyperplanes corresponding to any proper subset of S have a face of σ in common and the product of the corresponding reflections fixes its affine hull. This shows that the product is elliptic. The fact that this is the full fix-set and that this product of reflections has minimum length follows immediately from [BM, Lemma 6.4] and the observation that the roots of these reflections are linearly independent. \square

Proposition 7.2 (Coxeter elements are hyperbolic isometries). *A Coxeter element for an irreducible euclidean Coxeter group is a hyperbolic isometry of E , its move-set is a nonlinear affine hyperplane in V and its min-set is a line in E . Moreover, any factorization of this element as a product of the elements in a simple system is a minimum length reflection factorization.*

Proof. Let $w \in W = \text{COX}(\tilde{X}_n)$ be the Coxeter element under discussion and let $w = r_0 r_1 \cdots r_n$ be a factorization of w as the product of the $n+1$ reflections in a simple system $S = \{r_0, \dots, r_n\}$ corresponding to a chamber σ in the Coxeter complex of W . By Proposition 7.1, the product $w_0 = r_1 \cdots r_n$ is an elliptic isometry that only fixes a single vertex of σ and consequently its move-set is all of V (Remark 5.4). The hyperplane of the reflection r_0 is determined by the facet through the other n vertices of σ and in particular, it does not contain the point $\text{FIX}(w_0)$. By [BM, Proposition 6.6], the product $w = r_0 w_0$ is a hyperbolic element that has the listed properties. \square

The line in E that is the min-set of a Coxeter element w in an irreducible euclidean Coxeter group is called its *axis*. The next step is to classify those Coxeter elements that are geometrically distinct. The first thing to note is that when standard generators commute, distinct orderings can produce the exact same element. In fact, the only critical information is the ordering of pairs of generators joined by an edge in the Coxeter diagram Γ . If we orient each edge of Γ according to the order in which the reflections corresponding to its endpoints occur in the fixed total order, the result is an acyclic orientation of Γ and it is easy to prove that two linear orderings of S that induce the same acyclic orientation of Γ produce the same element $w \in W$. On the hexagonal diagram for the \tilde{A}_5 Coxeter group, for example, there are $6! = 720$ different products of its 6 standard generators but at most $2^6 - 2 = 62$

distinct Coxeter elements produced since this is the number of acyclic orientations. There is also a coarser notion of geometric equivalence.

Definition 7.3 (Geometric equivalence). Call an automorphism $\psi : W \rightarrow W$ *geometric* if ψ sends reflections to reflections and simple systems to simple systems. When W is an irreducible euclidean Coxeter group this is equivalent to being induced by a metric-preserving simplicial automorphism of the Coxeter complex. The geometric automorphisms form a subgroup of the full automorphism group that contains the inner automorphisms and the automorphism induced by symmetries of the Coxeter diagram. In fact, every geometric automorphism is a composition of an inner automorphism and a diagram automorphism. Call two elements w and w' *geometrically equivalent* when there is a geometric automorphism ψ sending w to w' , the point being that geometrically equivalent elements have similar geometric properties.

Geometric equivalence is sufficient for our purposes since geometrically equivalent Coxeter elements produce intervals that are identical after a systematic relabeling of the edges. In particular, geometrically equivalent Coxeter elements produce isomorphic dual Artin groups. To help identify Coxeter elements that are geometrically equivalent, we use the following lemma with a complicated statement and an easy proof.

Lemma 7.4 (Sources and sinks). *Let W be a Coxeter group with fundamental chamber σ , let w be the Coxeter element of W produced by a fixed acyclic orientation of its diagram Γ and let r be a reflection with hyperplane H associated with a vertex $v \in \Gamma$ that is either a source or a sink in this orientation. Then the element w is also a product of the reflections associated with the chamber $r(\sigma)$ on the other side of H with respect to the orientation of Γ that agrees with the previous one except that the orientation is reversed for every edge incident with v . In particular, two acyclic orientations of Γ that differ by a single sink-source flip produce Coxeter elements that are geometrically equivalent.*

Proof. Pick a linear ordering of the vertices consistent with the orientation of Γ so that $w = r_0 r_1 r_2 \cdots r_n$ where the r_i are the reflections through the facets of the chamber σ and $r = r_0$ or $r = r_n$. If $r = r_0$ then the assertion is a consequence of the elementary observation that $w = r r_1 r_2 \cdots r_n = r_1^r r_2^r \cdots r_n^r r$ where a^b is shorthand for bab^{-1} . The reflections in the second factorization bound the simplex $r(\sigma)$ which shares a facet with σ and is the reflection of σ across H and the orientation of Γ induced by this factorization satisfies the given description. The case $r = r_n$ is similar. \square

It quickly follows that most irreducible euclidean Coxeter groups have only one Coxeter element up to geometric equivalence.

Proposition 7.5 (Geometrically equivalent). *If W is a Coxeter group whose diagram is a tree, then all of its Coxeter elements are geometrically equivalent. This holds, in particular, for every irreducible euclidean Coxeter group that is not type A .*

Proof. There is an easy induction argument using Lemma 7.4 which proves that any two acyclic orientations of a tree are geometrically equivalent. The rough idea is to remove a valence 1 vertex and the unique edge connected to it, apply the inductive hypothesis to this pruned tree and then use the resulting sequence of flips as a template for the original situation inserting flips of the removed valence 1 vertex as necessary in order to make sure the vertex at the other end of its unique edge is a sink/source when required. \square

When the Coxeter diagram is a tree, there are exactly two orientations under which every vertex is a source or a sink. The two Coxeter elements that result are inverses of each other and either one is called a *bipartite Coxeter element*. Because of the way in which Lemma 7.4 is proved, it is an immediate consequence of Proposition 7.5 that every Coxeter element of an irreducible euclidean Coxeter group that is not of type A can be viewed as a bipartite Coxeter element so long as the fundamental chamber is chosen carefully. The chambers which produce w as a bipartite Coxeter element have an elegant geometric characterization that is described in the next section.

Corollary 7.6 (Bipartite Coxeter elements). *If w is a Coxeter element of an irreducible euclidean Coxeter group W that is not of type A , then for each acyclic orientation of Γ there exists a chamber σ in its Coxeter complex which has w as the Coxeter element determined by this orientation. In particular, there is a chamber which produces w as its bipartite Coxeter element.*

The analysis of Coxeter elements in the group $W = \text{Cox}(\tilde{A}_n)$ up to geometric equivalent is slightly more delicate.

Definition 7.7 (Bigon Coxeter elements). When n is at least 2, the diagram \tilde{A}_n is a cycle and the two edges adjacent to a sink or a source point in opposite directions around the cycle. (The case $n = 1$ is covered by Proposition 7.5.) In particular, flipping sinks and sources does not change the number of edges pointing in each direction. Moreover, using flips and diagram symmetries it is clear that any Coxeter element produced by an acyclic orientation is geometrically equivalent to one

produced by an acyclic orientation in which there is a unique sink, a unique source, p consecutive edges pointing in the clockwise direction and q consecutive edges pointing in the counterclockwise direction with $p \geq q$ and $p + q = n + 1$. We call a Coxeter element w derived from such an orientation a (p, q) -bigon Coxeter element. We should also note that in many respects a Coxeter element of \tilde{A}_1 can be considered a $(1, 1)$ -bigon Coxeter element.

In the Coxeter group of type \tilde{A}_5 , the 62 acyclic orientations of its Coxeter diagram describe at most 3 geometrically distinct Coxeter elements since each is one is geometrically equivalent to a (p, q) -bigon Coxeter element where (p, q) is either $(5, 1)$, $(4, 2)$ or $(3, 3)$. More generally, the Coxeter group of type \tilde{A}_n has at most $\frac{n+1}{2}$ geometrically distinct Coxeter elements since this is an upper bound on q . The following is the type A analog of Corollary 7.6.

Corollary 7.8 (Bigon Coxeter elements). *If w is a Coxeter element of an irreducible euclidean Coxeter group of type \tilde{A}_n , then there is a chamber σ in its Coxeter complex which produces w as one of its bigon Coxeter elements.*

The upshot of this analysis is that there is exactly one dual Artin group up to isomorphism for each irreducible euclidean Artin group that is not of type A and when $W = \text{COX}(\tilde{A}_n)$ there are at most $\frac{n+1}{2}$ such dual groups.

8. BIPARTITE COXETER ELEMENTS

The next two sections are a slight digression into the geometry of bipartite Coxeter elements in irreducible euclidean Coxeter groups. They are not needed to prove Theorem A but this is a convenient location to establish various results for use in the next article in the series [MS]. Let W denote an irreducible euclidean Coxeter group that is not of type A , let w be one of its Coxeter elements and let σ be a chamber in its Coxeter complex. The goal in this section is to establish a close geometric relationship between the axis of w and the chambers σ that produce w as a bipartite Coxeter element. More precisely, we show that these chambers are exactly those whose interior intersects the axis of w (Theorem 8.10). We begin by focusing on the geometry of σ .

Definition 8.1 (Bipartite faces and subspaces). Since W is not of type A , its Coxeter diagram Γ is a tree with a unique bipartite structure. For any chamber σ in the Coxeter complex of W this leads to a pair of distinguished disjoint faces in σ . More explicitly, let $S_0 \sqcup S_1 = S$

be the bipartite partitioning of the reflections determined by the facets of σ corresponding to the unique bipartite structure on Γ , let F_i be the face of σ determined by the intersection of the hyperplanes of the reflections in S_i and let B_i be the affine hull of F_i . Note that since each hyperplane is determined by the vertex of σ that it does not contain, the face F_0 is the convex of the vertices not contained in the various reflections in S_1 and the F_1 is the convex hull of the vertices not contained in the various reflections in S_0 . The affine subspaces B_0 and B_1 are disjoint since the hyperplanes determined by the facets of σ have trivial intersection. We call F_0 and F_1 the *bipartite faces* of σ and B_0 and B_1 the *bipartite subspaces* of σ .

In the \tilde{G}_2 example F_0 and F_1 are a point and the hypotenuse of the 30-60-90 right triangle, respectively, and B_0 and B_1 are the point and line they determine. Before continuing we pause to record some elementary observations about euclidean simplices.

Remark 8.2 (Euclidean simplices). Consider the general situation where σ is a euclidean n -simplex embedded in a euclidean space equal to its affine hull and F_0 and F_1 are disjoint faces of σ that collectively contain all of its vertices (conditions satisfied by the bipartite faces defined above). If B_i is the affine hull of F_i , then B_0 and B_1 are disjoint, the linear subspaces $\text{DIR}(B_0)$ and $\text{DIR}(B_1)$ have trivial intersection and the subspace spanned by their union has codimension 1 in V . In particular, $(\text{DIR}(B_0) \cup \text{DIR}(B_1))^\perp = \text{DIR}(B_0)^\perp \cap \text{DIR}(B_1)^\perp$ is a line L . Next, there exist a unique pair of distinct points $x_i \in B_i$ that realize the minimal distance between B_0 and B_1 . The uniqueness of x_0 and x_1 follows from the properties of $\text{DIR}(B_0)$ and $\text{DIR}(B_1)$ mentioned above. Since the line segment connecting these distance minimizing points is necessarily in a direction orthogonal to both $\text{DIR}(B_0)$ and $\text{DIR}(B_1)$, its direction vector spans the line L .

Definition 8.3 (Bipartite lines and closest points). Let B_0 and B_1 be the bipartite subspaces of a chamber σ in the Coxeter complex of an irreducible euclidean Coxeter group that is not of type A . The unique pair of points $x_i \in B_i$ that realize the minimum distance between B_0 and B_1 are called *closest points* and the unique line they determine is the *bipartite line* of σ .

In $W = \text{COX}(\tilde{G}_2)$, the points x_0 and x_1 are the vertex with the right angle and the foot of the altitude dropped to the hypotenuse. In Figure 3 the dashed line is the bipartite line for each of the heavily shaded triangles through which it passes. In general, there is a practical method for finding the direction of the bipartite line which

is particularly useful once the Coxeter complex has more dimensions than can be easily visualized.

Remark 8.4 (Direction of the bipartite line). Let σ be a chamber in the Coxeter complex of an irreducible euclidean Coxeter group that is not of type A . The roots of the $n + 1$ reflections in the corresponding simple system S are linearly dependent and this essentially unique linear dependency must necessarily involve all $n + 1$ roots since any proper subset is linear independent. If we separate the terms of the equation according to the bipartite subdivision $S = S_0 \sqcup S_1$ so that the roots corresponding to the reflections in S_0 are on the left hand side and the roots corresponding to the reflections in S_1 are on the right hand side, then the vector λ described by either side of this equation is the direction of the bipartite line. To see this note that by construction λ is nontrivial (because of the linear independent of proper subsets of the roots) and it can be written as a linear combination of either the S_0 roots or the S_1 roots. In particular, λ is in $\text{DIR}(B_0)^\perp \cap \text{DIR}(B_1)^\perp = L$, where L is the direction of the bipartite line of σ . This procedure is used in Section 11.

Returning to the \tilde{G}_2 example, notice that x_i lies in the interior of face F_i rather than elsewhere in its affine hull B_i . This is, in fact, always the case.

Lemma 8.5 (Interior). *When σ , F_i , B_i and x_i are defined as above, the closest point $x_i \in B_i$ lies in the interior of the face F_i . In particular, the bipartite line of σ intersects the interior of σ .*

Proof. Let $y_i \in F_i$ be points that realize the minimum distance between the faces F_0 and F_1 . We first show that y_0 lies in the interior of F_0 . The key facts are that every dihedral angle in σ is non-obtuse and that the defining diagram Γ is connected. In particular, if y_0 is in the boundary of F_0 then there exist hyperplanes H_0 and H_1 determined by facets with $H_i \supset F_i$ and $y_0 \in H_1 \cap F_0$ such that the dihedral angle between H_0 and H_1 is acute. This means that the distance can be shrunk by moving y_0 into the interior of a higher dimensional face of F_0 , contradiction. Thus y_0 is in the interior of F_0 . After reversing the roles of 0 and 1, we see that y_1 is in the interior of F_1 . This means that the vector from y_0 to y_1 is orthogonal to both affine spans and is in the direction of the line L . In particular, the points x_0 , y_0 , y_1 and x_1 form a possibly degenerate rectangle where x_0 and x_1 are the unique points realizing the minimum distance between B_0 and B_1 . But because $\text{DIR}(B_0)$ and $\text{DIR}(B_1)$ have no nontrivial vector in common, $x_0 = y_0$ and $x_1 = y_1$. \square

The next step is to establish that the bipartite line of a chamber is the min-set of its bipartite Coxeter elements.

Definition 8.6 (Bipartite involutions). Let σ be a chamber in a Coxeter complex of W , let S be the corresponding simple system, and let $S = S_0 \sqcup S_1$ be its bipartite decomposition. Because the reflections in S_i pairwise commute, the product w_i of the reflections in S_i is independent of the order in which they are multiplied and it is an involution. We call w_0 and w_1 the *bipartite involutions* of σ and note that the two bipartite Coxeter elements of σ are $w = w_1w_0$ and $w^{-1} = w_0w_1$. Geometrically, w_i fixes B_i pointwise and (if we pick a point in B_i as the origin) it acts as the antipodal map on its orthogonal complement.

In the \tilde{G}_2 example, w_0 is a 180° rotation about x_0 and w_1 is a reflection fixing the horizontal line B_1 . The description of the action of w_i given above establishes the following lemma from which we conclude that the bipartite line of a chamber is the axis of its bipartite Coxeter elements.

Lemma 8.7 (Reflecting the bipartite line). *If σ is a chamber in the Coxeter complex of W with closest points x_i and bipartite line L , then its bipartite involution w_i restricts to a reflection on L fixing only x_i .*

Proposition 8.8 (Bipartite lines as axes). *The bipartite line of any chamber σ in the Coxeter complex of W is the axis of the bipartite Coxeter elements produced by σ .*

Proof. Let w_0 and w_1 be the bipartite involutions of σ . By Lemma 8.7 the product $w = w_1w_0$ stabilizes L and acts as a translation on L . By the characterization of min-sets quoted in Remark 5.4, L is the axis of w . The same reasoning show that L is the axis of $w^{-1} = w_0w_1$. \square

The bipartite involutions of σ can be used to extend our notation.

Definition 8.9 (Axial chambers). Let σ be a chamber in the Coxeter complex of W with bipartite faces F_0 and F_1 , bipartite subspaces B_0 and B_1 , and closest points x_0 and x_1 . The bipartite involutions w_0 and w_1 define an infinite dihedral group action on the line L through x_0 and x_1 . Using this action, we can extend the definitions of F_i , B_i , x_i and σ_i to arbitrary subscripts $i \in \mathbb{Z}$ by letting $F_{-i} / B_{-i} / x_{-i}$ denote the image of $F_i / B_i / x_i$ under w_0 and letting $F_{2-i} / B_{2-i} / x_{2-i}$ denote the image of $F_i / B_i / x_i$ under w_1 . The result is a sequence of equally spaced points x_i that occur in order along L , one for each $i \in \mathbb{Z}$. Finally, let σ_i denote the image of σ under this dihedral group action that contains x_i and x_{i+1} (so that $\sigma = \sigma_0$). We call these chambers *axial chambers* and their vertices are *axial vertices*.

The axial chambers in Figure 3 are the ones that are heavily shaded. One fact about the arrangement of the chambers σ_i along the axis L that is important to note is that every point of L that is not one of the points x_i lies in the interior of some chamber σ_i .

Theorem 8.10 (Axial chambers and Coxeter elements). *Let L be the axis of a Coxeter element w for an irreducible euclidean Coxeter group W that is not of type A and let σ be a chamber in its Coxeter complex. The chamber σ produces w as a bipartite Coxeter element iff the line L intersects the interior of σ .*

Proof. When σ produces w as a bipartite Coxeter element then by Proposition 8.8 the axis of w is the bipartite line of σ and by Lemma 8.5 this line intersects the interior of σ . In the other direction, we know that there is at least one chamber σ that produces w as a bipartite Coxeter element and this σ contains a portion of L in its interior. It is sufficient to note that interiors of chambers are disjoint open subsets of A and that the interiors of the axial chambers under the infinite dihedral group action generated by the bipartite involutions of σ contain all of L except the discrete set of points x_i with $i \in \mathbb{Z}$. \square

Another way to phrase this result is that a chamber σ produces w as a bipartite Coxeter element iff σ is a chamber in the smallest simplicial subcomplex containing the axis of w . We conclude this section with one final observation.

Corollary 8.11 (Hyperplanes crossing the axis). *Let L be the axis of a Coxeter element w in an irreducible euclidean Coxeter group W that is not of type A . If a hyperplane H of a reflection in W crosses the line L then H is determined by a facet of an axial simplex. More precisely, there is an index i such that H contains all of F_i , all but one vertex of F_{i-1} and all but one vertex of F_{i+1} .*

Proof. The intersection of H and L must occur at one of the points x_i since the remainder of L is covered by the interiors of axial simplices as noted after Definition 8.9. Moreover, since x_i lies in the interior of the face F_i , H contains all of F_i and thus all of its affine hull B_i . Next, recall that the facets of σ_i containing F_i determine hyperplanes representing pairwise commuting reflections, and as a result they intersect the orthogonal complement of B_i (based at x_i) in an arrangement that looks like the standard coordinate hyperplanes with the link of B_i in σ_i forming one of its orthants. As this orthant is a chamber of the finite reflection subgroup fixing B_i , it is clear that these are the only hyperplanes of W that contain B_i . In particular, H must itself be a

hyperplane determined by a facet of σ_i containing F_i and thus have the listed properties. \square

9. REFLECTIONS

Let w be a Coxeter element in an irreducible euclidean Coxeter group W that is not of type A . In this section we determine the set R_0 of reflections that occur in some minimal length reflection factorization of w . We call these the reflections below w .

Definition 9.1 (Reflections below w). It follows easily from Lemma 3.7 that for any reflection r in W the following conditions are equivalent: (1) $\ell_R(rw) < \ell_R(w)$ (2) r is the leftmost reflection in some minimal length factorization of w (3) r is a reflection in some minimal length factorization of w (4) r is the rightmost reflection in some minimal length factorization of w and (5) $\ell_R(wr) < \ell_R(w)$. When these conditions hold, we say that r is a *reflection below* w .

When analyzing the reflections below a Coxeter element in an irreducible euclidean Coxeter group it is useful to distinguish two types of reflections.

Definition 9.2 (Vertical and horizontal reflections). The Coxeter axis is a line in the euclidean space E and its space of directions is a line in the vector space V that has a hyperplane as its orthogonal complement. We call the vectors in this hyperplane *horizontal* and those in this line *vertical*. More generally, any vector with a nontrivial vertical component (i.e. any vector not in the hyperplane) is also called vertical. Using this distinction, we separate the reflections below w into two types based on the type of its roots. In other words, a reflection r is *horizontal* if its root is orthogonal to the direction of the Coxeter axis and *vertical* otherwise.

The main reason to distinguish vertical and horizontal reflections is that when a Coxeter element of an irreducible euclidean Coxeter group is multiplied by a vertical reflection, the result is elliptic and when it is multiplied by a horizontal reflection, the result is hyperbolic [BM].

Lemma 9.3 (Vertical reflections). *Let L be the axis of a Coxeter element w in an irreducible euclidean Coxeter group W that is not of type A and let H be the hyperplane of a vertical reflection in W that intersects L at the point x_i . If u and v are the unique vertices of F_{i-1} and F_{i+1} not contained in H , then w sends u to v , r swaps u and v , rw fixes u , and wr fixes v . Moreover, the elliptic isometry rw is a Coxeter element for the finite Coxeter subgroup of W that stabilizes u*

and the elliptic isometry wr is a Coxeter element for the finite Coxeter subgroup of W that stabilizes v .

Proof. Recall that w_i fixes F_i and acts as the antipodal map on its orthogonal complement sending σ_{i-1} to σ_i . In particular, it stabilizes any hyperplane that contains F_i and thus sends the vertices in $H \cap \sigma_{i-1}$ to the vertices in $H \cap \sigma_i$. This means that w_i must send the one remaining vertex u to the one remaining vertex v . Because $w = w_i w_{i-1}$ and w_{i-1} fixes all of F_{i-1} , we have that $w(u) = w_i w_{i-1}(u) = w_i(u) = v$. Next, because r is one of the commuting reflections whose product is w_i , rw_i is the same product with r deleted and $rw = rw_i w_{i-1}$ is the product of all the reflections defined by facets of σ_{i-1} except the reflection defined by the facet whose affine span is the hyperplane of r . In particular, all of these reflections contain the vertex u in their fixed hyperplanes, their product fixes u and they bound a spherical simplex formed by intersecting σ_{i-1} with a small sphere centered at u . Since these reflections bound a chamber of the corresponding spherical Coxeter complex, their product rw is a Coxeter element for this subgroup. Similarly, the factorization $w = w_{i+1} w_i$ shows that wr is a product of the reflections determined by the facets of σ_i with r removed and this product is a Coxeter element for the stabilizer of v . Finally r swaps u and v since w sends u to v , rw fixes u , and r has order 2. \square

From Lemma 9.3 the following is immediate.

Proposition 9.4 (Vertical reflections). *Let w be a Coxeter element in an irreducible euclidean Coxeter group W that is not of type A . If r is a reflection that is vertical with respect to axis of w then r contains many axial vertices in its fixed hyperplane, it is part of a bipartite factorization of w and it is contained in the set R_0 .*

For horizontal reflections, a more precise statement is necessary.

Proposition 9.5 (Horizontal reflections). *Let w be a Coxeter element in an irreducible euclidean Coxeter group W that is not of type A . If r is a reflection that is horizontal with respect to the axis of w then r in W is contained in a minimal length reflection factorization of w and thus in R_0 iff the hyperplane of r contains at least one axial vertex of w .*

Proof. If r is contained in R_0 then there is a factorization $r_0 r_1 \cdots r_n = w$ containing r and by Lemma 3.7 we can assume that $r = r_n$. Since every point under w is moved in the vertical direction, at least one of the reflections in this factorizations must be vertical and by Lemma 3.7 we can move this reflection to the r_0 position without altering $r = r_n$.

This shows that r is a reflection below $w' = r_0w = r_1r_2 \cdots r_n$ which is an elliptic isometry fixing an axial vertex v (Proposition 9.4). But this implies that $\text{FIX}(r) = H$ must contain $\text{FIX}(w') = v$ since $\text{FIX}(w')$ is the intersection of the hyperplanes of this minimal length reflection factorization of w' [BM, Lemma 6.4]. In particular, H contains an axial vertex.

In the other direction, let H be the hyperplane of r , let u be an axial vertex contained in H , and using the notation of Definition 8.9, let F_{i-1} be the face containing u . If we let r' be the reflection defined by the facet of σ_{i-1} not containing u then by Lemma 9.3 $r'w$ is a Coxeter element of the finite Coxeter group that stabilizes u , a group that contains r . Since it is well-known that every reflection in a finite Coxeter group occurs in some minimal length factorization of any of its Coxeter elements, r occurs in such a factorization $r_1r_2 \cdots r_n$ of $r'w$. The product $r'r_1r_2 \cdots r_n$ is then a minimal length reflection factorization of w that contains r and r belongs to R_0 . \square

Propositions 9.4 and 9.5 immediately establish the following.

Theorem 9.6 (Reflections). *Let w be a Coxeter element for an irreducible euclidean Coxeter group W that is not of type A and let R_0 be the set of reflections below w . A reflection r is in R_0 iff the hyperplane $H = \text{FIX}(r)$ contains an axial vertex.*

Although it requires a separate argument, we should note that Theorem 9.6 also holds when W is an irreducible euclidean Coxeter group of type \tilde{A}_n and w is any of its geometrically distinct Coxeter elements. Also, although every elliptic element in the interval $[1, w]$ has as its fixed-set an affine subspace of the Coxeter complex that contains at least one axial vertex, not all such subspaces occur. This should not be too surprising since this is similar to the situation in finite Coxeter groups where only certain “noncrossing” subspaces occur as fixed sets of elements below the Coxeter element.

Another important remark is that the length of an element with respect to the full set of all reflections R_E is, in general, quite different from its length with respect to the set of reflections in W . See [MP11] where this is discussed in detail. This distinction does not play too large of a role in the current context because the two length functions agree for a Coxeter element w and thus they agree for all of the elements in the interval $[1, w]$.

This section concludes with a discussion of the dual presentation of $\text{ART}(\tilde{G}_2)$ that is designed to make Theorem A more comprehensive. Craig Squier successfully analyzed its group structure in [Squ87] but

he did so prior to the development of the theory of Garside groups. The following theorem shows that its dual presentation is a Garside presentation in the sense of Digne.

Theorem 9.7 (Type G). *The interval used to define the unique dual presentation of the irreducible euclidean Artin group $\text{ART}(\tilde{G}_2)$ is a lattice and thus its dual presentation is a Garside presentation.*

Proof. Let w be a Coxeter element of $W = \text{COX}(\tilde{G}_2)$. Since it has a unique Coxeter element of W up to geometric equivalence, its axis and axial simplices can be arranged as in Figure 3 where the axis is dashed and the axial simplices are heavily shaded. By Theorem 9.6 the reflections that occur in minimal length factorizations of this glide reflection are the vertical reflections whose fixed lines cross the axis and the two horizontal reflections fixing one of the two vertical lines bounding the lightly shaded region in Figure 3. (Multiplying w by a reflection r fixing one of the other vertical lines results in a pure translation which has length 2 with respect to R_E but which does not have length 2 with respect to R since it translates in a direction that is not one of the root directions.) In this situation we can list the basic invariants of all of the isometries in the interval $[1, w]$. The only hyperbolic isometries strictly below w are the two translations that result when w is multiplied by one of the two horizontal reflections. Each translation is in a direction at a 30° angle with the horizontal, one to left and one to the right. The only length 2 factorizations of these translations are, of course, obtained by multiplying two parallel reflections whose root is in this direction. In addition there is one elliptic isometry below w for each axial vertex, one for each visible line in the Coxeter complex through an axial vertex and one for the entire plane. With these descriptions it is relatively straightforward to check that the poset is a lattice. Because the rank of the poset is so small any bowtie in the interval would be between two elements of rank 1 and two elements of rank 2. Consider two elements at rank 2, i.e. two elements that are either translations or rotations fixing a point. The unique meet of the two translation strictly below w is the identity since they have no common reflections in their possible factorizations. The unique meet of a translation and a rotation is either the reflection through the line perpendicular to the translation direction that contains the fixed point of the rotation if such a line exists, or the identity otherwise. And finally the meet of two rotations below w is the reflection that fixes the line through their fixed points, if it exists in the Coxeter complex, or the identity otherwise. Finally, since well-defined meets always exists

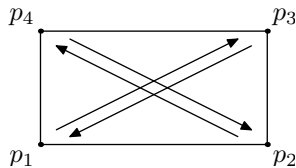


FIGURE 5. The rectangle with labeled corners used in Example 10.1.

between any two elements of rank 2 there are no bowties. As remarked in Definition 1.5 this means the interval $[1, w]$ is a lattice. \square

10. BOWTIES

In this section we establish a criterion which implies that a Coxeter interval in an irreducible euclidean Coxeter group is not a lattice. This turns out to be the key result needed to establish Theorem A. We begin with an elementary example in the euclidean plane.

Example 10.1 (Reflections and translations). Fix a rectangle in the euclidean plane with horizontal and vertical sides and label its corners as in Figure 5. Next, consider the group G generated by the following eight elements: the four reflections that fix one of the four sides and the four translations that send a corner of this rectangle to the opposite corner. The reflection fixing p_i and p_j is denoted r_{ij} and the translation sending p_i to p_j is t_{ij} . Note that the subscripts of r_{ij} are unordered and the subscripts of t_{ij} are ordered. In this notation r_{12} and r_{34} are reflections fixing horizontal lines, r_{23} and r_{41} are reflections fixing vertical lines and the four translations are t_{13} , t_{31} , t_{24} and t_{42} . The group G contains the isometry w that rotates the rectangle 180° about its center. For example, w can be factored as $w = t_{13}r_{12}r_{41}$. That this factorization is as short as possible over this generating set follows from the fact that every factorization must contain at least one horizontal reflection, one vertical reflection and one translation. In fact, it is straightforward to check that there are exactly 24 such factorizations of length 3. The interval $[1, w]$ is shown in Figure 6. The dashed lines represent translations and the solid lines represent reflections with the thickness distinguishing horizontal and vertical reflections. Two final notes. The interval $[1, w]$ is not a lattice because there is a bowtie connecting the two leftmost vertices in each of the two middle rows. Concretely, the four factorizations of w involved are $r_{34}t_{13}r_{41}$, $r_{34}t_{24}r_{23}$, $r_{12}t_{31}r_{23}$ and $r_{12}t_{42}r_{41}$. And finally, these 24 factorizations of w form a single closed orbit under the Hurwitz action.

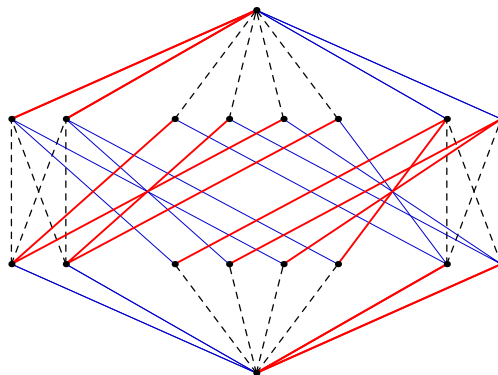


FIGURE 6. The poset of minimal factorizations described in Example 10.1.

The 2-dimensional configuration described in Example 10.1 captures the essential reason why many of the intervals that define dual euclidean Artin groups fail to be lattices.

Proposition 10.2 (Bowties). *Let r and r' be orthogonal reflections in an irreducible euclidean Coxeter group W and let $t = t_\lambda$ be a translation in W in a root direction. If t does not commute with either reflection and λ is not in the plane spanned by the roots of r and r' , then the interval $[1, w]$ below the element $w = trr'$ contains a bowtie and is not a lattice.*

Proof. The first step is to show that the interval $[1, w]$ contains a copy of Figure 6 as an induced subposet. Because t_λ is a translation in a root direction it is a product of two parallel reflections in W . This means that w has reflection length at most 4. Next, note that rr' is an elliptic element with a 2-dimensional move-set U spanned by the roots of r and r' . The hypothesis that λ does not lie in U means that the move-set of trr' is not through the origin and thus is a nonlinear 2-dimensional affine subspace. Thus w is a hyperbolic isometry with reflection length at least 4. Combining these facts shows that $\ell_R(w) = 4$ and, after factoring t into two parallel reflections in W , we have one of its minimal length factorizations and a corresponding maximal chain in $[1, w]$.

Next, let B be any 2-dimensional affine subset of the euclidean space E with $\text{DIR}(B) = U$ and note that the hyperplanes fixed by the reflections r and r' intersect B in orthogonal lines ℓ and ℓ' and that both r and r' stabilize B . If we uniquely decompose λ as a vector λ_1 in U plus a vector λ_2 in U^\perp then we can factor t into a pair of translations t_{λ_1} and t_{λ_2} , one with a translation direction in U and one with a

translation direction in U^\perp . The translation t_{λ_2} commutes with r , r' and t_{λ_1} and the hypotheses on t ensure that t_{λ_1} is nontrivial and not a direction vector of ℓ or ℓ' . In particular, the way r , r' and t_{λ_1} act on U can be essentially identified with the action of r_{12} , r_{41} and t_{13} of Example 10.1.

As we use the Hurwitz orbit in the example to alter the factorization there, we can mimic that action on the minimal length reflection factorizations of w , treating the translation as a product of two parallel reflections that always stay together and where both get conjugated simultaneously when necessary. Under this action, the translation t_{λ_2} simply follows the conjugates of the translation t_{λ_1} around by virtue of the fact that it commutes with all three actions on B . The result is a copy of Figure 6 as an induced subposet of $[1, w]$. The final step is to note that the bowtie visible on the left of Figure 6 remains a bowtie in the larger poset $[1, w]$ because $[1, w]$ is, in turn, an induced subposet of the hyperbolic poset $P^{\text{Mov}(w)}$ where these same four elements, two of rank 1 and two of rank 3 are four elements forming one of the known bowties in $P^{\text{Mov}(w)}$ as described in Theorem 5.5. \square

Translations and reflections satisfying the hypotheses of Proposition 10.2 can be found below a Coxeter element in an irreducible euclidean Coxeter group whenever the roots orthogonal to the direction of its Coxeter axis form a reducible root system.

Theorem 10.3 (Reducibility and bowties). *Let L be the axis of a Coxeter element w in an irreducible euclidean Coxeter group $W = \text{COX}(\Gamma)$. If the root system $\Phi_\Gamma \cap \text{DIR}(L)^\perp$ of horizontal roots is reducible, then the interval $[1, w]$ contains a bowtie, it is not a lattice and the dual Artin group $\text{ART}^*(\Gamma, w)$ is not a Garside group.*

Proof. Let σ be a chamber where the reflections defined by its facets can be multiplied in an appropriate order to produce $w = r_0 r_1 \cdots r_n$. By repeatedly applying Lemma 7.4 and replacing σ with another chamber if necessary, we may assume without loss of generality that the reflection corresponding to the white dot is the reflection r_0 . When this is the case, the remaining reflections in the factorization fix a vertex x of σ with the property that every hyperplane of a reflection in W is parallel to a hyperplane of a reflection in W fixing x . In other words, we can identify x as our origin as in Definition 6.4, the group generated by r_1 through r_n is the group W_0 , and the product $r_1 r_2 \cdots r_n$ is a Coxeter element for W_0 . Because every reflection through x occurs in some minimal length factorization of this Coxeter element, we may modify

the product $r_1 r_2 \cdots r_n$ so that r_1 is a reflection with a hyperplane parallel to the hyperplane of r_0 . As a consequence $r_0 r_1$ is a translation in a root direction and concretely a translation in the direction λ that is the root of highest weight relative to this simple system.

Next, consider the element $r_2 r_3 \cdots r_n$. This is an elliptic element fixing a line L' and since it differs from w by a translation, L' is parallel to L , the Coxeter axis of w . Moreover, L' must lie in the hyperplane fixed by r_i for each $i \geq 2$, so the roots of these reflections belong to the root system $\Phi_{\text{hor}} = \Phi_{\Gamma} \cap \text{DIR}(L)^\perp$. We should note that the common root λ of r_0 and r_1 is not in this root system because they are now the only roots capable of moving points in a direction that includes motion in the direction of L .

By hypothesis this system of horizontal roots is reducible, say $\Phi_{\text{hor}} = \Phi_1 \cup \cdots \cup \Phi_j$ for some $j > 1$ with each Φ_i an irreducible root system that spans a subspace V_i and $V_1 \oplus \cdots \oplus V_j$ is an orthogonal decomposition of $\text{DIR}(L)^\perp$. Because the reflections r_i , $2 \leq i \leq n$ form a minimal length factorization of an elliptic isometry, their roots are linearly independent [BM, Lemma 6.4]. Thus the number of roots in each Φ_i is bounded above by the dimension of the corresponding V_i . Moreover, since the number of reflections equals the dimension of $\text{DIR}(L)^\perp$, the roots of these reflections that lie in each Φ_i form a basis for V_i .

Finally, note that because the reflections r_0, r_1, \dots, r_n generate all of W , their roots must generate the entire irreducible root system Φ_{Γ} . In particular, the vector λ is not in V_i^\perp for any i since this would lead to an obvious decomposition. More to the point, there must be an r_i with a root in Φ_1 is not orthogonal to λ since these roots form a basis of V_1 , and the same is true for the other components as well. In particular, we can select two reflections from our given factorization of w with roots from distinct irreducible components of the horizontal root system $\Phi_{\Gamma} \cap \text{DIR}(L)^\perp$ and that neither one commutes with the translation $t = r_0 r_1$ and by Lemma 3.7 we may assume that they are r_2 and r_3 . At this point, we can apply Proposition 10.2 with $t = r_0 r_1$, $r = r_2$ and $r' = r_3$ to conclude that the interval from 1 to $r_0 r_1 r_2 r_3$ contains a bowtie $(a, b : c, d)$. But this interval is contained inside $[1, w]$ and the additional elements cannot resolve the bowtie since any element below a and b was already contained in $[1, r_0 r_1 r_2 r_3]$. Thus $[1, w]$ contains a bowtie and is not a lattice. \square

Consider the case of the \tilde{G}_2 Coxeter group. By comparing Figures 3 and 4 it is clear that there are exactly 2 horizontal roots forming a Φ_{A_1} root system. Since this is irreducible, Theorem 10.3 does not apply. This is consistent with Theorem 9.7 where we showed that the interval

used to define the dual presentation of $\text{ART}(\tilde{G}_2)$ is a lattice and thus the dual presentation is a Garside presentation.

11. COMPUTATIONS AND REMARKS

In this final section we use Theorem 10.3 to complete the proof of our main theorem, Theorem A. At this point it is relatively straightforward. For each type and representative Coxeter element we compute the direction of the Coxeter axis using Remark 8.4 and then we compute the system of horizontal roots. In each case not covered by [Dig06], [Dig12], or Theorem 9.7, the system of horizontal roots is reducible. We begin by introducing the relevant root systems using a slightly idiosyncratic notation that John Crisp and I have found to be quite useful when performing explicit root computations by hand.

Definition 11.1 (Root notation). In almost every standard root system for a euclidean Coxeter group, a root is completely specified by indicating the location and the sign of its nonzero entries. This is because all nonzero entries has the same absolute value and this common value only depends on the number of nonzero entries. We list the locations of the nonzero entries in the subscript together with a slash “/”. The locations of the positive entries occur before the slash and the locations of the negative entries occur afterwards. For example, $r_{ij/}$, $r_{i/j}$ and $r_{/ij}$ denote the vectors $e_i + e_j$, $e_i - e_j$ and $-e_i - e_j$, respectively and in the E_8 root system the vector $\frac{1}{2}(1, -1, 1, -1, 1, 1, -1, -1)$ is written $r_{1356/2478}$.

Definition 11.2 (Root systems). Let $\Phi_k^{(n)}$ be the collection of $2^k \binom{n}{k}$ vectors of the form:

$$\Phi_k^{(n)} = \{\pm e_{i_1} \pm e_{i_2} \pm \cdots \pm e_{i_k} \mid 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}$$

and let $\Phi_{k,\text{even}}^{(n)}$ be the subset of these vectors with an even number of minus signs. The standard root systems of types B_n , C_n , D_n , F_4 and E_8 are very easy to describe using this notation. See Table 1. The others involve slight modifications. The roots in Φ_{C_n} that are orthogonal to the vector (1^n) , i.e. the vector with all n coordinates equal to 1, form the standard A_{n-1} root system. The roots in Φ_{E_8} orthogonal to $r_{7/8}$, i.e. the roots with $x_7 = x_8$, form the standard E_7 root system. And the roots in Φ_{E_8} orthogonal to $r_{6/7}$ and $r_{7/8}$, i.e. the roots with $x_6 = x_7 = x_8$, form the standard E_6 root system.

We begin with the four classical families. The computations for type C are done in greater detail because they are straightforward and

Type	Roots
B_n	$\Phi_2^{(n)} \cup \Phi_1^{(n)}$
C_n	$\Phi_2^{(n)} \cup 2\Phi_1^{(n)}$
D_n	$\Phi_2^{(n)}$
F_4	$\Phi_2^{(4)} \cup 2\Phi_1^{(4)} \cup \Phi_4^{(4)}$
E_8	$\Phi_2^{(8)} \cup \frac{1}{2}\Phi_{8,\text{even}}^{(8)}$

TABLE 1. Roots systems with simple descriptions.

indicate the kinds of computations that are merely sketched for the remaining types.

Example 11.3 (Type C). We begin by selecting vectors from the Φ_{C_n} root system whose nonpositive dot products are encoded in the \tilde{C}_n diagram of Figure 2. From left to right we use $2e_1$, $-e_1 - e_2$, $e_2 + e_3$, $\dots (-1)^{n-1}(e_{n-1} + e_n)$ and $(-1)^n 2e_n$. In the notation of Definition 11.1 these are the vectors $r_{1/}$, $(-1)^i r_{i(i+1)/}$ and $(-1)^n r_{n/}$. Our choice of vectors is nonstandard, but it has the advantage of producing a bipartite Coxeter element whose axis is in a direction that makes computing and identifying the horizontal root system trivial. We compute the direction of the Coxeter axis following the procedure outlined in Remark 8.4. The unique linear dependency among these vectors involves adding the first and last vectors to two times each of the remaining vectors. The bipartite structure separates them based on parity in the list and the sum of the odd terms is the vector $(2, 2, \dots, 2)$, or (2^n) in Conway's shorthand notation. The roots orthogonal to this direction are those of the form $r_{i/j} = e_i - e_j$ and these form the irreducible root system $\Phi_{A_{n-1}}$. Note that Theorem 10.3 is not applicable and that this is consistent with the results in [Dig12] where Digne established that the interval that defines the dual euclidean Artin group of type \tilde{C}_n is, in fact, a lattice and the dual presentation of $\text{ART}(\tilde{C}_n)$ is a Garside presentation.

Type B is very similar but has a reducible horizontal root system.

Example 11.4 (Type B). Next consider the root system of type B_n and select the vectors e_1 , $-e_1 - e_2$, $e_2 + e_3$, \dots , $(-1)^{n-2}(e_{n-2} + e_{n-1})$, $(-1)^{n-1}(e_{n-1} + e_n)$ and $(-1)^{n-1}(e_{n-1} - e_n)$ to represent the vertices of the \tilde{B}_n diagram of Figure 2 from left to right. In shorthand notation, these are the vectors $r_{1/}$, r_{12} , r_{23} , $\dots (-1)^{n-2} r_{(n-2)(n-1)/}$, $(-1)^{n-1} r_{(n-1)n/}$ and $(-1)^{n-1} r_{(n-1)/n}$, keeping in mind that $r_{1/}$ denotes

the vector e_1 in the B_n root system and not $2e_1$ as in the C_n root system. The unique linear dependency among these vectors is obtained by adding the two final vectors to two times each of the remaining vectors. In \tilde{B}_4 , for example, the linear dependency is $2r_{1/} + 2r_{/12} + 2r_{23/} + r_{/34} + r_{4/3} = (0, 0, 0, 0)$. Separating the terms based on the bipartite structure shows that the direction of the Coxeter axis is $(2, 2, \dots, 2, 0) = (2^{n-1}0)$. The horizontal roots are those of the form $r_{i/j} = e_i - e_j$ with $i, j < n$ and the two roots $\pm r_{n/} = \pm e_n$. This is clearly a reducible horizontal root system with the components isomorphic to $\Phi_{A_{n-2}}$ and Φ_{A_1} . In particular, Theorem 10.3 applies, the interval used to define the dual euclidean Artin group of type \tilde{B}_n is not a lattice and the dual presentation of $\text{ART}(\tilde{B}_n)$ is not a Garside presentation.

Type D is another slight variation.

Example 11.5 (Type D). Consider the root system of type D_n and select the vectors $e_1 - e_2, -e_1 - e_2, e_2 + e_3, \dots, (-1)^{n-2}(e_{n-2} + e_{n-1}), (-1)^{n-1}(e_{n-1} + e_n)$ and $(-1)^{n-1}(e_{n-1} - e_n)$ to represent the vertices of the \tilde{D}_n diagram of Figure 2 from left to right. The unique linear dependency among these vectors is obtained by adding the first two vectors, the last two vectors, and two times each of the remaining vectors. In \tilde{D}_5 , for example, the linear dependency is $r_{1/2} + r_{/12} + 2r_{23/} + 2r_{/34} + r_{45/} + r_{4/5} = (0, 0, 0, 0, 0)$. Separating the terms based on the bipartite structure shows that the direction of the Coxeter axis is $(0, 2, 2, \dots, 2, 0) = (02^{n-2}0)$. The horizontal roots are those of the form $r_{i/j} = e_i - e_j$ with $1 < i, j < n$ and the four roots $\pm r_{1n/}$ and $\pm r_{/1n}$. This is a reducible horizontal root system with three irreducible factors (since $r_{1n/}$ and $r_{/1n}$ are orthogonal) that are isomorphic to $\Phi_{A_{n-3}}, \Phi_{A_1}$ and Φ_{A_1} . In particular, Theorem 10.3 applies, the interval used to define the dual euclidean Artin group of type \tilde{D}_n is not a lattice and the dual presentation of $\text{ART}(\tilde{D}_n)$ is not a Garside presentation.

The final classical family is type A .

Example 11.6 (Type A). Let W be the Coxeter group $\text{Cox}(\tilde{A}_n)$. For each (p, q) with $p + q = n + 1$ and $p \geq q \geq 1$ we can construct a (p, q) -bigon Coxeter element w as follows. First let W act on \mathbb{R}^{n+1} in the natural way, permuting coordinates and translating along vectors orthogonal to (1^{n+1}) . Next label these $n + 1$ coordinates $(x_1, x_2, \dots, x_p, y_1, y_2, \dots, y_q)$. Let the unique source in the acyclic orientation be the reflection that swaps coordinates x_1 and y_1 , let the vertices along one side of the bigon represent reflections that swap x_i and x_{i+1} in ascending order and let the vertices along the other

side represent reflections that swap y_i and y_{i+1} in ascending order. Finally, let the unique sink represent the reflection that sends the coordinates (x_p, y_q) to $(y_q - 1, x_p + 1)$. This is a reflection fixing the hyperplane $y_q = x_p + 1$. The product of these reflections in this order is an isometry of \mathbb{R}^{n+1} that sends $(x_1, x_2, \dots, x_p, y_1, y_2, \dots, y_q)$ to $(x_p + 1, x_1, x_2, \dots, x_{p-1}, y_q - 1, y_1, y_2, \dots, y_{q-1})$. Although we cannot use the method of Remark 8.4 to calculate the direction of the Coxeter axis, it is easy enough to compute that its pq -th power of this motion is a pure translation in the direction $(p^q, -q^p)$, i.e. the vector with its first q coordinates equal to p and its next p coordinates equal to $-q$. Thus this is the direction of the axis of this Coxeter element.

As a consequence, the horizontal roots are those of the form $r_{i/j} = e_i - e_j$ with $i, j \leq p$ or with $i, j > p$. So long as q (and therefore p) is at least 2, there is at least one horizontal root of each type and the horizontal root system is reducible with one component isomorphic to a $\Phi_{A_{p-1}}$ and the other component isomorphic to a $\Phi_{A_{q-1}}$. In particular, Theorem 10.3 applies to any (p, q) -bigon Coxeter element with $p \geq q \geq 2$, the interval used to define the dual euclidean Artin group of type \tilde{A}_n with respect to this Coxeter element is not a lattice and the corresponding dual presentation of $\text{ART}(\tilde{A}_n)$ is not a Garside presentation. In the remaining case where $q = 1$ and $p = n$, the horizontal roots for an irreducible $\Phi_{A_{n-1}}$ root system and Theorem 10.3 is not applicable. This is consistent with the results in [Dig06] where Digne established that the interval that defines the dual euclidean Artin group of type \tilde{A}_n with respect to an $(n, 1)$ -bigon Coxeter element is, in fact, a lattice and the corresponding dual presentation is a Garside presentation.

And finally we shift our attention to the exceptional types. Since type G is covered by Theorem 9.7, we only need to discuss the four remaining examples. In each case we list the vectors chosen to as our simple system, the resulting direction of the Coxeter axis and the vectors of the horizontal root system grouped into irreducible components. These computations were initially carried out by hand and then a few lines of code in `GAP` were used to doublecheck and validate these results.

Example 11.7 (Type F). Consider the root system of type F_4 . If we select the vectors $r_{1/2}$, $r_{23/}$, $r_{/1234}$, $r_{4/}$, and $r_{12/34}$ from the Φ_{F_4} root system as the vectors represented in the extended Dynkin diagram of type \tilde{F}_4 , then the direction of the axis of the corresponding bipartite Coxeter element is $(0, 1, 1, 2)$. There are 8 roots that are horizontal with respect to this axis and they split into two irreducible factors. There is a Φ_{A_1} root system formed by the roots $\pm\{r_{2/3}\}$ and a Φ_{A_2}

root system formed by the roots $\pm\{r_1, r_{123/4}, r_{23/14}\}$. As a consequence Theorem 10.3 applies, the interval used to define the dual euclidean Artin group of type \tilde{F}_4 is not a lattice and the dual presentation of $\text{ART}(\tilde{F}_4)$ is not a Garside presentation.

Example 11.8 (Type E). Consider the root system of type E_6 . If we select the vectors $r_{12/}$, $r_{5/2}$, $r_{/45}$, $r_{4/3}$, $r_{235678/14}$, $r_{2345/1678}$ and $r_{134678/25}$ from the Φ_{E_6} root system as the vectors represented in the extended Dynkin diagram of type \tilde{E}_6 , then the direction of the axis of the corresponding bipartite Coxeter element is $(1, 1, 1, -3, -3, 1, 1, 1)$. There are 14 roots that are horizontal with respect to this axis and they split into three irreducible factors. There is a Φ_{A_2} root system formed by the roots $\pm\{r_{1/2}, r_{2/3}, r_{1/3}\}$, another Φ_{A_2} root system formed by the roots $\pm\{r_{4/5}, r_{1234/5678}, r_{1235/4678}\}$, and a Φ_{A_1} root system formed by the roots $\pm\{r_{12345678/}\}$.

Next consider the root system of type E_7 . If we select the vectors $r_{/15}$, $r_{12/}$, $r_{/27}$, $r_{78/}$, $r_{/38}$, $r_{34/}$, $r_{/46}$ and $r_{2356/1478}$ from the Φ_{E_7} root system as the vectors represented in the extended Dynkin diagram of type \tilde{E}_7 , then the direction of the axis of the corresponding bipartite Coxeter element is $(1, 1, 1, 1, 0, 0, 2, 2)$. There are 20 roots that are horizontal with respect to this axis and they split into three irreducible factors. There is a Φ_{A_3} root system formed by the roots $\pm\{r_{1/2}, r_{1/3}, r_{1/4}, r_{2/3}, r_{2/4}, r_{3/4}\}$, a Φ_{A_2} root system formed by the roots $\pm\{r_{56/}, r_{1234/5678}, r_{12345/78}\}$, and a Φ_{A_1} root system formed by the roots $\pm\{r_{5/6}\}$.

Finally consider the root system of type E_8 . If we select the vectors $r_{12/}$, $r_{/25}$, $r_{5/6}$, $r_{6/7}$, $r_{78/}$, $r_{/38}$, $r_{34/}$, $r_{28/134567}$ and $r_{2367/1458}$ from the Φ_{E_8} root system as the vectors represented in the extended Dynkin diagram of type \tilde{E}_8 , then the direction of the axis of the corresponding bipartite Coxeter element is $(1, 1, 1, 1, 3, -3, 2, 2)$. There are 28 roots that are horizontal with respect to this axis and they split into three irreducible factors. There is a Φ_{A_4} root system formed by the roots $r_{1/2}$, $r_{1/3}$, $r_{1/4}$, $r_{2/3}$, $r_{2/4}$, $r_{3/4}$, $r_{15/234678}$, $r_{25/134678}$, $r_{35/124678}$, $r_{45/123678}$ and their negatives, a Φ_{A_2} root system formed by the roots $\pm\{r_{56/}, r_{1234/5678}, r_{123456/78}\}$ and a Φ_{A_1} root system formed by the roots $\pm\{r_{7/8}\}$.

In each case, Theorem 10.3 applies, the interval used to define the dual euclidean Artin group of type \tilde{E}_n for $n = 6, 7$ or 8 is not a lattice and the dual presentation of $\text{ART}(\tilde{E}_n)$ is not a Garside presentation.

At this point we have shown that for each type and for each geometric equivalence class of Coxeter elements not covered by earlier results,

the resulting interval is not a lattice and the corresponding dual presentation is not a Garside presentation. This completes the proof of Theorem A. As a final comment we note that the existence of a uniform reason for the failure of the lattice property in all of these cases (i.e. reducibility of the horizontal root system) leads one to hope for the existence of a uniform way to work around the problem. This indeed turns out to be the case as Robert Sulway and I show in [MS] where we clarifying the basic structural properties of all euclidean Artin groups.

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