NORMAL FORMS FOR FREE APERIODIC SEMIGROUPS

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Abstract. The implicit operation ω is the unary operation which sends each element of a finite semigroup to the unique idempotent contained in the subsemigroup it generates. Using ω there is a well-defined algebra which is known as the free aperiodic semigroup. In this article we show that for each n, the n-generated free aperiodic semigroup is defined by a finite list of pseudoidentities and has a decidable word problem. In the language of implicit operations, this shows that the pseudovariety of finite aperiodic semigroups is κ-recursive. This completes a crucial step towards showing that the Krohn-Rhodes complexity of every finite semigroup is decidable.

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1. Introduction

The implicit operation ω is the unary operation which sends each element of a finite semigroup to the unique idempotent contained in the subsemigroup it generates. Using ω there is a well-defined algebra which is known as the free aperiodic semigroup (see Definition 3.1). In this article we show that the n-generated, free aperiodic semigroup, usually denoted $\Omega_n^\kappa \mathbf{A}$, is defined by a finite list of pseudoidentities and has a decidable word problem. In the language of implicit operations, this shows that the pseudovariety of finite aperiodic semigroups is κ-recursive. This completes a crucial step towards showing that the Krohn-Rhodes complexity of every finite semigroup is decidable [1].

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1.1. **Overview of the article.** The article consists of two parts. In sections 2 through 7 we show that some particular n-generated κ-semigroups F_n defined by a finite list of identities have a decidable term problem. In the second half of the article, we use the structure of the Burnside semigroups to show that these κ-semigroups are in fact the free aperiodic semigroups known as $\Omega_n^\kappa \mathbf{A}$. The combination of these two lines of reasoning yields the main results.

2. Implicit operations

In this section we review the necessary background definitions and results about relatively free profinite semigroups. Detailed descriptions of implicit operations and the profinite topology can be found in [1], [6], and [7]. For background on term algebras see [2].

Definition 2.1 (Implicit Operations)**.** Recall that a pseudovariety is any collection of finite semigroups which is closed under subsemigroups, homomorphic image, and finite direct products. If $\mathbf{V}$ is a pseudovariety, an *n-ary implicit operation on* $\mathbf{V}$ is an operation π which consistently assigns an element of S to every n-tuple of elements of $S \in \mathbf{V}$. In symbols we have functions $\pi_S(s_1, \ldots, s_n) = s$ for each S in $\mathbf{V}$. The consistency requirement is that for every morphism $\phi : S \to T$, $\phi(\pi_S(s_1, \ldots, s_n)) = \pi_T(\phi(s_1), \ldots, \phi(s_n))$. The set of all n-ary implicit operations on $\mathbf{V}$ is denoted $\overline{\Omega_n}\mathbf{V}$. The bar is meant to suggest the topological closure operator, and in fact there is a way of defining a metric and a topology on the projective limit of $\mathbf{V}$ – viewed as a category – with the consequence that the set of all n-ary implicit operations is indeed the topological closure of the so-called explicit operations. Since this viewpoint will not be needed here, we refer the interested reader to [7] for further details.

Example 2.2. The operation ω described above is an example of a unary implicit operation. The usual binary multiplication is an example of a binary implicit operation. These two implicit operations (and their compositions) will be the only implicit operations which arise in this article. We will denote the two-element set consisting of just these two implicit operations by κ. Technically, κ is usually reserved for the set containing binary multiplication and another implicit operation denoted $\omega - 1$. Since ω is derivable from these two operations and since $\omega - 1$ will not be needed in the sequel, this slight deviation from the standard notation will be of no consequence.

Definition 2.3 (κ-semigroups)**.** An n-generated κ-semigroup is an algebra with n distinct variables, say $x_1, \ldots, x_n$, and the two implicit operations in κ. The terms of the algebra are formed from the variables and the operations via composition. Let T_n denote the set of all such terms. The terms are then subject to various identities, including, of course, the associativity of the binary multiplication. Whenever fewer than 3 variables are involved we will use x, y, and z in place of the x_i.

Remark 2.4 (Notation)**.** The variable x_i can itself be viewed as the n-ary implicit operation corresponding to the projection onto the i-th coordinate, so that $(x_i)_S(s_1, \ldots, s_n) = s_i$ for all semigroups S and for all elements $s_1, \ldots, s_n \in S$. Notice that since the composition of implicit operations is another implicit operation, all of the terms in T_n are well-defined n-ary implicit operations on the collection of finite semigroups. The notation typically used for the implicit operation which results from the composition of ω with the term α is $(\alpha)^\omega$. Since we will be working with κ-semigroups exclusively, we will simplify the notation. Since the binary multiplication is associative, parentheses can be used for convenience to suggest a grouping of repeated binary multiplications, but such a grouping will not change the resulting implicit operation. We choose instead never to use parentheses in this manner so that parentheses can be reserved for the purpose of indicating composition of ω with a term. Thus the operation ω applied to the term α will be indicated by (α). If ω is applied to (α), the result will be $((\alpha))$. This economical notation will help to facilitate the proofs. Terms of the form (α) will be called *limit terms*, and α will be called its *content*.

Definition 2.5 (Subexpressions)**.** If $\alpha = \beta\gamma$ for possibly empty terms β and γ, then β is called an *initial segment* of α and γ is called a *final segment* of α. If $\alpha = \beta\delta\gamma$ where δ is a term, but β and γ are not necessarily terms, then δ is called a *subexpression* of α. The reason why β and γ may not be terms in this instance is that β may contain an open parenthesis which is closed only inside γ. For example, in the term $xx(xy)xz$, xx is an initial segment, $(xy)xz$ is a final segment, and xx, $(xy)xz$, and xy are subexpressions. On the other hand, the sequence of symbols $xx(x$ is neither an initial segment nor a subexpression of the term.

Definition 2.6 (Length)**.** The length of a term α can be defined inductively. The length of each x_i is 1, the length of $\alpha\beta$ is the sum of the lengths of α and β and the length of (α) is two more than the length of α. Alternatively, each term in T_n can be viewed as an element of the free semigroup generated by the characters "x_i", the open parenthesis "(" and the close parenthesis ")". The length of α defined above is thus the number of characters needed to write it as an element of this semigroup. As an example, the length of $x(x(y))z$ is 8.

Definition 2.7 (Simple Terms and Proper Powers)**.** A term α will be called *simple* if there does not exist a term β and an integer $k > 1$ such that $\alpha = \beta^k$. Terms which are not simple will be called *proper powers*. If $\alpha = \beta^k$, where β is simple and k is a positive integer, then β will be called the *root* of α, and k will be called its *exponent*. A term α is said to be *periodic in* β if α is a subexpression of β^k for some k. Finally, if $\alpha = \gamma_1\gamma_2$ and $\beta = \gamma_2\gamma_1$, then α and β are called *cyclic conjugates*. We should note that the notion of cyclic conjugation in κ-semigroups is slightly different from the corresponding notion in semigroups without additional operations. In the latter situation, any string of symbols can be moved from the end of the word to the beginning, while in the former, this can only be done when the string of symbols form a term in their own right. Thus $(xz)xy$ is a cyclic conjugate of $xy(xz)$, but $z)xy(x$ is not.

As with words in the free semigroup, every proper power can be written uniquely as a power of a simple term, and there are bounds on the length of terms which are periodic in two non-conjugate simple terms, or periodic in a single simple term in two distinct ways.

Lemma 2.8. *For every term α in T_n there is a unique simple term β and unique integer k such that $\alpha = \beta^k$.*

Proof. When we view α as a word in the free semigroup generated by the "x_i", ")", and "(", the result follows immediately from the corresponding result in that context. ☐

Lemma 2.9. *Let α and β be simple terms which are not cyclic conjugates of each other. If γ is a term which is periodic in both α and β, then the length of γ is less than $|\alpha| + |\beta|$.*

Proof. Once again, when we view these terms as words in the free semigroup, the result follows immediately from the corresponding theorem in that context. See [4] for further details. □

Lemma 2.10. *Let α and β be distinct simple terms which are cyclic conjugate to each other. If γ is an initial segment of both α^k and β^k for some k, then the length of γ is less than $|\alpha| = |\beta|$.*

Proof. This follows from the observation that all of the proper cyclic conjugates of simple words in the free semigroup are distinct. □

Definition 2.11 (Preferred Cyclic Conjugates)**.** Let α be a simple term which is not a limit term. Since α is simple, there is a unique cyclic conjugate which occurs first in the dictionary ordering. The usual dictionary ordering will be arbitrarily extended so that an open parenthesis is alphabetized before each of the x_i and the closed parenthesis is alphabetized after each of the x_i. The cyclic conjugate which occurs first in this ordering will be called the *preferred cyclic conjugate* of α. If, on the other hand, α is a proper power, say $\alpha = \beta^k$ where β is simple, then the preferred cyclic conjugate of α is the one which is the k-th power of the preferred cyclic conjugate of β.

3. Free aperiodic semigroups

The identities satisfied by a κ-semigroup can either be given explicitly, as a possibly recursive list, or implicitly, as the list of all identities which are valid in some particular class of semigroups. The following definitions define our main objects of study and they illustrate both of these possibilities.

Definition 3.1 (Free Aperiodic Semigroups)**.** The n-generated free aperiodic semigroup, usually denoted $\Omega_n^\kappa \mathbf{A}$, is the n-generated κ-semigroup which is subject to exactly those identities which are valid in every finite aperiodic semigroup. In particular, if α and β are two terms in T_n, then $\alpha = \beta$ is an identity in $\Omega_n^\kappa \mathbf{A}$ if and only if the element of S selected by the n-ary implicit operation α is the same as the element of S selected by the n-ary implicit operation β, for every semigroup $S \in \mathbf{A}$ and for every n-tuple of elements in S. In symbols, this requires $\alpha_S(s_1, \ldots, s_n) = \beta_S(s_1, \ldots, s_n)$ for every $S \in \mathbf{A}$ and for all $s_1, \ldots, s_n \in S$.

Rather than study $\Omega_n^\kappa \mathbf{A}$ directly, we begin by studying the following κ-semigroup which is defined by a finite list of pseudoidentities.

Definition 3.2. Let F_n be the κ-semigroup subject only to the following identities:

1. $((\alpha)) = (\alpha)$
2. $(\alpha^k) = (\alpha)$
3. $(\alpha)(\alpha) = (\alpha)$
4. $(\alpha)\alpha = \alpha(\alpha) = (\alpha)$
5. $(\alpha\beta)\alpha = \alpha(\beta\alpha)$

where α and β represent arbitrary terms in T_n, k is a positive integer, and the notational conventions are those of Remark 2.4.

Since all of these identities are true in every finite aperiodic semigroup, it is clear that there is a homomorphism of κ-semigroups, $\phi : F_n \to \Omega_n^\kappa \mathbf{A}$. The main goal of the next several sections will be to establish that the term problem for F_n is algorithmically decidable using a set of normal forms defined below (Definition 6.1). To do so, we need to introduce a number of additional concepts and notations.

Definition 3.3 (Elementary Changes)**.** Instead of viewing the equations in Definition 3.2 as identities, we can also view them as rules for rewriting expressions for the term algebra T_n. The various ways of applying the rules can then be described according to their effects. For example, if we start with an expression which contains a subexpression of the form (α) and we rewrite this subexpression to look like the left-hand side of one of the first four rules, then this will be called an *elementary expansion* since (α) will be a subexpression of the subexpression which results. The reverse operation will be called an *elementary contraction*. An application of the fifth rule, in either direction, will be called an *elementary shift*.

We will distinguish between a shift left and a shift right by the direction the parentheses are moving. When we wish to clarify which expansion rule is being applied we will refer to an expansion of the form $(\alpha) \to ((\alpha))$ as an expansion of type 1, etc. For expansions of type 4 we will distinguish between 4_L and 4_R depending on the side on which the new material appears. Thus $(\alpha) \to \alpha(\alpha)$ is an expansion of type 4_L. Contractions will be similarly denoted.

Definition 3.4 (Derivations)**.** A *derivation from α to β* is a finite sequence of elementary expansions, contractions, and shifts which start at α and end at β. Terms in T_n will be called *equivalent in F_n* (and will represent the same element of F_n) if and only if there is a derivation from one to the other.

Example 3.5. By Definition 3.2, the term $(xy)x(yx)$ can be rewritten as follows:

$$\underline{(xy)}x(yx) \Rightarrow (xy)x\underline{yx(yx)} \Rightarrow \underline{(xy)x}(yx) \Rightarrow x\underline{(yx)(yx)} \Rightarrow x(yx)$$

The first step in this derivation is an expansion of type 4_R, the second is a contraction of type 4_L, the third step is a shift, and the final step is a contraction of type 3. In each case, we have underlined the subexpression to which the rule is about to be applied.

Definition 3.6 (Rank of an Expression)**.** The *rank of an expression* α in T_n is defined inductively. The variables $\{x_1, \ldots, x_n\}$ have rank 0. If α has rank i and β has rank j, then $\alpha\beta$ has rank $\max i, j$. If α has rank i, then (α) has rank $i+1$. In essence, the rank is the maximum number of nested parentheses which occur in the expression. Rank provides a convenient way to assign a grading to the terms in T_n. We let T_n^i be the set of all terms of rank i, and let $T_n^{\leq i}$ be the union of T_n^j with $j \leq i$. We will also extend this notion of rank to the parentheses. The *rank of a pair of parentheses* which enclose the term α is the rank of the term (α). Thus $(x(y))(z)$ has two pairs of parentheses of rank 1 and one pair of rank 2.

Definition 3.7 (Rank of an Identity)**.** The *rank of a identity* is the maximum of the ranks of the two expressions used to define it. In most cases, these ranks are identical, the only exception being expansions and contractions of type 1. As an example, the rule $xy(xy) \Rightarrow (xy)$ is a rank 1 contraction of type 4_L, while the rule $(x(y)) \Rightarrow ((x(y)))$ is a rank 3 expansion of type 1. We will call two terms *rank i equivalent* if there is a derivation of one from the other using only identities of rank at most i. Similarly, we define two terms to be *conjugate in rank i* if there is a finite sequence of terms which start at one term, end at the other term, and if every term in the sequence is derived from the previous one by either an identity of rank at most i, or a cyclic conjugation.

In order to describe the normal form algorithm more efficiently, we introduce the following terminology.

Definition 3.8 (Portions)**.** Let α be a rank i term. The smallest initial segment of α which contains a limit term of rank i will be called the *initial portion* of α. Similarly, the smallest final segment of α which contains a limit term of rank i will be called the *final portion* of α. A minimal subexpression of α which contains exactly two limit terms of rank i will be called a *crucial portion* of α.

Example 3.9. If α is the rank 2 term

$$x(x)(y(yyy))yxy((x)y)x((y))x(x)$$

then $x(x)(y(yyy))$ is the initial portion of α, $((y))x(x)$ is the final portion of α, and there are two crucial portions: $(y(yyy))yxy((x)y)$ and $((x)y)x((y))$. Notice that initial portions and crucial portions always have a rank i limit term as a final segment and that final portions and crucial portions always have a rank i limit term as an initial segment.

4. The normal form algorithm in rank 1

In this section we show how to reduce every term of rank 1 to a unique normal form. Although these procedures will be incorporated in the general, inductively defined normal form reduction algorithm (Definition 6.1), the detailed description of this special case should help to orient the reader to the inductive situation. Before continuing with our description in rank 1, we will briefly review the situation for terms with no parentheses at all.

Definition 4.1 (Rank 0 Normal Forms)**.** If α is a rank 0 term then none of the identities can be applied to α and thus the class of terms equivalent to α contains only α itself. As a result we define all rank 0 terms to already be in *normal form* and in *cyclic normal form*. If the term α happens to be a simple term and it occurs first in the dictionary ordering among all of its cyclic conjugates, then α will be said to be in *preferred cyclic normal form*. Finally, notice that since multiplication of rank 0 terms is simply concatenation, there are no idempotents in rank 0. Also, for the purposes of the inductive definitions, a rank 0 limit term will simply be another name for a single variable (in other words, for one of the x_i).

Definition 4.2 (Rank 1 Normal Form Algorithm)**.** The rank i normal form reduction algorithm contains four major steps. The procedure for each step will be given separately, followed by a series of lemmas outlining its major properties. Roughly speaking, the algorithm consists of the following steps: (1) apply all possible rank i contractions of type 1, (2) apply all possible rank i contractions of type 2, (3) apply all possible rank i contractions of type 3, and (4) standardize the result. As each step is described it will become clear what we mean by the phrases "all possible" and "standardize."

Throughout the remainder of this section let α be a rank 1 term to which the normal form reduction algorithm is being applied. We will denote the term which emerges after the j-th step as α_j.

Step 1: The first step, in rank 1 at least, is trivial. This is because step one only applies to idempotents in the previous rank and there are no idempotents in rank 0. Thus $\alpha_1 = \alpha$. The full description of this step will be given in the inductive context.

Step 2: In the second step, we rewrite each rank 1 limit term in α in a normal form. In particular, if (β) is a limit term in α, then there is a well-defined term of the form $\epsilon_1(\gamma)\epsilon_2$ with the following properties:

1. γ is a simple rank 0 term in preferred cyclic normal form
2. the term $\epsilon_1(\gamma)\epsilon_2$ is rank 1 equivalent to (β)
3. the term $(\gamma)\epsilon_2\epsilon_1(\gamma)$ is rank 1 equivalent to (γ)

This term will be called the *standard form for* (β). The procedure which produces this standard form goes as follows. If β is not a simple term, then by Lemma 2.8 we can write it as a proper power of a simple rank 0 term, and a contraction of type 2 can be applied to the limit term (β). As a result, we might as well assume that β is already a simple term. Next, let γ denote the preferred cyclic conjugate of β. If $\beta = \gamma$, then we set $\epsilon_1 = \epsilon_2 = \gamma$ and we are done. Otherwise, we use two expansions of type 4 and a shift to produce a term of the form $\epsilon_1(\gamma)\epsilon_2$ which is rank 1 equivalent to (β) for which ϵ_1 and ϵ_2 are not empty. Although there may be more than one way to arrange the parentheses to satisfy these conditions, all that matters is that at least one way exists and that every time the second step of the reduction algorithm is applied to (β) the same result be obtained. For concreteness, we will stipulate that the parentheses be shifted to the leftmost position satisfying these conditions.

We observe that $\beta^2 = \epsilon_1\epsilon_2$ and $\gamma^2 = \epsilon_2\epsilon_1$. Thus the term $(\gamma)\epsilon_2\epsilon_1(\gamma)$ is identical to $(\gamma)\gamma^2(\gamma)$ and two contractions of type 4 and another of type 3 show that it is rank 1 equivalent to (γ).

Given a rank 1 term α_1, the term α_2 is formed by replacing limit term (β) with the equivalent term $\epsilon_1(\gamma)\epsilon_2$ which results from the procedure described above. Notice that we have not claimed (or shown) that γ, ϵ_1, and ϵ_2 are the unique terms which satisfy these conditions, only that these terms are well-defined by the procedure described.

The procedure for placing rank 1 limit terms in their normal forms has the following properties:

Lemma 4.3. *The normal form for* (β) *is identical to the normal form for* (β^k).

Proof. If β is a power of a simple term β', then β^k will also be a power of the same simple term. Thus these procedures merge as soon as the contents of the parentheses are made to be simple terms. □

Lemma 4.4. *If* β *is a simple rank* 1 *term and the normal form for* (β) *is* $\epsilon_1(\gamma)\epsilon_2$, *then the second step applied to the term* $\beta^j(\beta^k)\beta^l$ *results in the term* $\epsilon_1\gamma^j(\gamma)\gamma^l\epsilon_2$.

Proof. The result is immediate since by the construction $\beta\epsilon_1 = \epsilon_1\gamma$ and $\epsilon_2\beta = \epsilon_2\gamma$. □

Lemma 4.5. *Let* β *be a term of rank* 0 *and let* η *be a single limit term of rank* 0 *(i.e. a variable such as* x*). If the second step is applied to the terms* $(\eta\beta)\eta$ *and to* $\eta(\beta\eta)$, *then the results will differ at most by an expansion and a contraction of type* 4.

Proof. We may assume $\eta\beta$ is already simple. When the expansions of type 4 are applied to each term, the results are $\eta\beta(\eta\beta)\eta\beta\eta$ in the first case and $\eta\beta\eta(\beta\eta)\beta\eta$ in the second. The same shifts can be applied to both. Let γ denote the preferred cyclic normal form of $\beta\eta$ and $\eta\beta$. Since the first pair of parentheses originally extended one limit term further to the left, this may be enough of a change to cause the rank 1 parentheses for the first term to end up one copy of term γ further to the left. This can be counteracted by a contraction of type 4_R followed by an expansion of type 4_L. □

Lemma 4.6. *If α and β differ by a single shift, then α_2 and β_2 differ by contractions and expansions of type* 4.

Proof. Since every shift can be decomposed into shifts which only shift a single letter, the result follows from Lemma 4.5. □

Step 3: In the third step we eliminate crucial portions of the form $(\gamma)\gamma^j(\gamma)$ for some integer $j \geq 0$. This can be done by applying j contractions of type 4 followed by one contraction of type 3. Notice that doing this does not change the initial portion, the final portion, or any of the other crucial portions. It simply eliminates one of the crucial portions. Thus the order in which these eliminations take place is irrelevant and the final result is the same. We say that (γ) is the *normal form* for crucial portions of the form $(\gamma)\gamma^j(\gamma)$ where γ is a simple rank 0 term in preferred cyclic normal form.

Step 4: The final step is to standardize the results, which we do one portion at a time. In particular, we apply contractions of type 4_L to the initial portion of α_3 and contractions of type 4_R to the final portion of α_3 until this is no longer possible. For the crucial portions, a more complicated procedure is necessary.

If γ and δ are simple rank 0 terms which are in preferred cyclic normal form, ϵ is a rank 0 term, and $(\gamma)\epsilon(\delta)$ is a crucial portion of α_3, then there is another rank 0 term ϵ' with the following properties:

1. $(\gamma)\epsilon(\delta)$ is rank 1 equivalent to $(\gamma)\epsilon'(\delta)$
2. ϵ' is not an initial segment of γ^j for any integer j
3. ϵ' is not a final segment of δ^j for any integer j
4. any contractions of type 4 applied to $(\gamma)\epsilon'(\delta)$ will result in a term which fails to have these properties

The procedure for producing such an ϵ' goes as follows. First choose j to be the smallest integer with $|\gamma^j| \geq |\gamma| + |\delta|$, and then apply j expansions of type 4_R to the limit term (γ) so that $(\gamma)\gamma^j$ is an initial segment of the result. Similarly, choose k to be the smallest integer with $|\delta^k| \geq |\gamma| + |\delta|$, and apply k expansions of type 4_L to the limit term (δ) so that $\delta^k(\delta)$ is a final segment of the result. Let ϵ' denote the term between the rank 1 limit terms at this point. This ϵ' will satisfy the first three of the four desired properties.

To see this, note that since this crucial portion remains in α_3, either Lemma 2.9 or Lemma 2.10 can be applied. According to these lemmas, the largest initial segment of ϵ' which is also an initial segment of γ^l for some large l cannot be all of ϵ'. Similarly, the largest final segment of ϵ' which is also a final segment of δ^l for some large integer l cannot be all of ϵ'.

We now apply any contractions of type 4 which will preserve these properties. We note that since a contraction of type 4 on an initial segment cannot affect whether the third condition is satisfied and a contraction of type 4 on a final segment cannot affect whether the second condition is satisfied, the resulting term is independent of the order in which these contractions are performed. The expression $(\gamma)\epsilon'(\delta)$ which results will be called the *normal form* of $(\gamma)\epsilon(\delta)$.

The term α_4 is formed from α_3 by replacing each portion with its normal form. Since none of these procedures change any of the limit terms at either end, they can be carried out independently of each other. This completes the rank 1 normal form reduction algorithm. The final result, α_4, will be called the rank 1 *normal form for* α.

When α is identical to its normal form we will say that α is in *normal form*. Subexpressions of normal forms will be called *subnormal forms*. Before describing the properties of these normal and subnormal forms in greater detail, we record a few properties of this fourth step in the algorithm, followed by an example and a few remarks.

Lemma 4.7. *Let γ be a simple rank 0 term in preferred cyclic normal form. If ϵ is any term of rank 0, then the normal forms for $\epsilon(\gamma)$ and $\epsilon\gamma^j(\gamma)$ are identical. Similarly, the normal forms for $(\gamma)\epsilon$ and $(\gamma)\gamma^j\epsilon$ are identical.*

Lemma 4.8. *Let γ and δ be simple rank 0 terms in preferred cyclic normal form. If ϵ is any term of rank 0 then the normal forms for $(\gamma)\epsilon(\delta)$ and $(\gamma)\gamma^j\epsilon\delta^k(\delta)$ are identical.*

Example 4.9. If $\alpha = \alpha_1 = y(xy)x(xyxy)x(yx)yxyx(yxyxyx)(zzzz)$, then α_2 will be $y(xy)x(xy)xy(xy)xyxyxy(xy)x(z)$. Among the four crucial portions, the first is not removable even though it starts and ends with the same limit term. The second and third crucial portions are removable, while the fourth is not. Thus α_3 is the term $y(xy)x(xy)x(z)$, and finally, the normal form will be $y(xy)xxy(xy)xz(z)$.

Remark 4.10 (Overlaps)**.** Notice that the term ϵ' produced by the procedure in the fourth step may need to contain more than one copy of γ or δ in order to satisfy the second and third conditions. For example, the normal form for the term $(xxxxy)(x)$ is $(xxxxy)xxxxyxxxxx(x)$. Five copies of x need to appear before the limit term (x) in order to ensure that the term between these two pairs of parentheses is not periodic in $xxxxy$.

Remark 4.11. Now that the algorithm has been completely described, it should be clear that calling the reduction of a limit term in the first step its normal form is indeed justified. If the term α had consisted solely of the limit term (β), then the term α_2 would be unchanged by the remaining two steps. The other uses of the phrase normal form can be similarly justified.

By far the most important property of rank 1 normal forms is that they can be used to decide whether two rank 1 terms are rank 1 equivalent. Similar results will be shown for terms in rank i (Theorem 6.9) and in general (Theorem 7.2).

Theorem 4.12. *Let α and β be rank 1 terms. The terms α and β are equivalent in rank 1 if and only if α and β have the same normal form. In particular, the normal form reduction algorithm can be used to decide whether rank 1 terms are rank 1 equivalent.*

Proof. Since terms are rank 1 equivalent with their normal forms, it is clear that rank 1 terms with the same rank 1 normal form are themselves rank 1 equivalent.

Thus we only need to show the other implication. Suppose that α and β are rank 1 terms which are rank 1 equivalent but whose normal forms are distinct. By considering the normal forms for each intermediate stage in a rank 1 derivation from α to β, we may in fact assume that α and β differ by the application of a single rule. For concreteness, assume that β is obtained from α by a contraction or a shift. We will consider each type of rule in turn and show that their normal forms cannot in fact be distinct.

If α and β differ by a contraction of type 2, then by Lemma 4.3, α_2 and β_2 will already be identical and their eventual normal forms will thus be the same.

If α and β differ by a contraction of type 3, then α contains a pair of adjacent limit terms, say $(\delta)(\delta)$. When these are placed in normal form in the second step, the result is $\epsilon_1(\gamma)\gamma^2(\gamma)\epsilon_2$. Thus α_2 and β_2 will differ by two contractions of type 4 and a contraction of type 3. More importantly, the crucial portion of α_2 bracketed by (γ) will be removed in the third step and α_3 and β_3 will be identical, and their eventual normal forms will be the same.

If α and β differ by a contraction of type 4, then by Lemma 4.4, α_2 and β_2 differ by a series of contractions of type 4. There are now three possibilities. If the contractions involve a crucial portion of α_2 which is removed in the third step, it will also remove this crucial portion in β_2. Thus α_3 and β_3 will be identical and so will their normal forms. If the contractions affect a crucial portion which is not removed in the third step, then α_3 and β_3 will still differ by a series of contractions of type 4, but by Lemma 4.8 the resulting normal forms will be the same. Similarly, if the contractions affect an initial portion or a final portion then α_3 and β_3 will differ by a series of contractions of type 4, but by Lemma 4.7 the normal forms will be identical.

Finally, if α and β differ by a shift, then by Lemma 4.6 α_2 and β_2 will differ by contractions and expansions of type 4 and we have already shown that these types of changes yield the same normal forms. In all cases, α and β must have the same normal form and we have contradicted our assumption. This completes the proof. □

5. Consequences of the normal form algorithm

We will now begin the inductive step in the argument. Most of the definitions and theorems in Section 5 and Section 6 are defined inductively. To begin we will assume that all of these results have been shown in the previous ranks, and we will now show that they are true in rank i. The first time through, i will have a value of 1 and the only results which will be needed either were established in the previous section or are trivial.

At this point in the argument we have shown that every term of rank at most i has a normal form and that two terms of rank at most i are equivalent in rank i if and only if they have the same normal form (Theorem 4.12 or Theorem 6.9). We will now make a number of observations which will allow us to draw the same conclusion about the words of rank $i+1$. In particular, we begin with a number of fairly immediate consequences of normal form algorithm.

As can be seen from the description of the algorithm, certain types of subexpressions are more important than others for determining whether a term is in normal or subnormal form.

Lemma 5.1. *A rank i term α is in normal form if and only if its initial portion, its final portion, and all of its crucial portions are in normal form.*

Proof. The result of the rank i normal form algorithm clearly produces these types of words, and conversely, any word of this type will pass unchanged through every step of the normal form procedure. □

Lemma 5.2. *A rank i term α is in subnormal form if and only if all of its crucial portions are in normal form and its initial and final portions are in subnormal form.*

Proof. Using Lemma 5.1 is it possible to mix and match the normal forms which contain each of these pieces to create a new normal form which contains the entire term. □

The most important thing to note is that each of these conditions is checked locally. In particular, the following assertions are now immediate. They describe replacements in terms which are in normal and subnormal form, respectively.

Lemma 5.3. *Let α be a rank i term in normal form which contains a subexpression β which starts and ends with limit terms of rank i. If β' is another expression in normal form which starts and ends with the same limit terms, then the term α' formed by replacing the expression β in α with β' will still be in normal form. Similarly, if β is an initial segment (final segment) of α which ends (starts) with a rank i limit term and β' is another expression in normal form which ends (starts) with the same limit term, then the term obtained by replacing the expression β with the expression β' will still be in normal form.*

Lemma 5.4. *Let α be a rank i term in subnormal form which contains a expression β which starts and ends with limit terms of rank i. If β' is another expression in subnormal form which starts and ends with the same limit terms, then the term α' formed by replacing the expression β in α with β' will still be in subnormal form. Similarly, if β is an initial segment (final segment) of α which ends (starts) with a rank i limit term and β' is another expression in subnormal form which ends (starts) with the same limit term, then the term obtained by replacing the expression β with the expression β' will still be in subnormal form.*

In order to extend our knowledge of rank i normal form to terms in rank $i+1$, we will need a number of results about products and idempotents.

Definition 5.5 (i-length)**.** The *i-length* of a rank i term α is the number of rank i limit terms which are contained in the normal form for α. Specifically, every rank i term α in normal form can be written as

$$\alpha = \epsilon_0(\beta_1)\epsilon_1(\beta_2)\epsilon_2 \dots (\beta_N)\epsilon_N \tag{1}$$

where each β_j is a rank $i-1$ term in preferred cyclic normal form and each ϵ_j is a possibly empty term of rank at most $i-1$. The i-length of α in this case is N.

Definition 5.6 (Cyclic normal forms)**.** Let α be a rank i term in normal form with i-length N and let ϵ_j and β_j be subexpressions of α as shown in Equation 1. The normal form of the term

$$\alpha' = (\beta_1)\epsilon_1(\beta_2)\epsilon_2 \dots \epsilon_{N-1}(\beta_N)\epsilon_N\epsilon_0(\beta_1)$$

is a term which is conjugate to α in rank i. This is because it can be obtained from α by an expansion of type 3 applied to (β_1), followed by a cyclic conjugation and a reduction. A slightly simpler form α'' is obtained if we then cyclically conjugate the initial segment (β_1) to the end of the term, followed by a contraction of type 3. The reason for the expansion of type 3 followed by a contraction of type 3 is to ensure that this procedure will yield the correct result even in the case where the i-length of α is 1. The final result α'' will be called a *cyclic normal form for* α. Notice that in order to reduce α' to its normal form, by Lemma 5.1 it is sufficient to reduce the crucial portion $(\beta_N)\epsilon_N\epsilon_0(\beta_1)$ to its normal form. In particular, the i-length of α' will either be N or $N+1$ depending on whether the expression $(\beta_N)\epsilon_N\epsilon_0(\beta_1)$ is equivalent to a single limit term or not, and the i-length of α'' will be either $N-1$ or N. The cyclic conjugate of α'' which is preferred by dictionary ordering (as extended in Definition 2.11) will be called the *preferred cyclic normal form* for α.

Lemma 5.7. *If α is a rank i term in preferred cyclic normal form, then α begins with a rank i limit term and i open parentheses in a row. More generally, each limit term of rank j in α begins with j open parentheses in a row for $j \leq i$.*

Proof. Since open parentheses are ordered prior to all of the x_i and the closed parenthesis, it is clear that the preferred cyclic normal form will begin with a limit term of some rank. Since the lemma in the previous rank shows that the content of each limit term of rank i begins with $i-1$ open parentheses, each rank i limit term will itself begin with i open parentheses. Given this, it is clear that the preferred cyclic normal form will always have one of the maximal rank limit terms as an initial segment. ☐

Lemma 5.8. *Let α and β be two rank i terms in normal form. When the crucial portion of the product $\alpha\beta$ containing the transition between α and β is reduced to its normal form, the result is the normal form for $\alpha\beta$. In particular, if α and β have i-length of k and l respectively, then the i-length of $\alpha\beta$ will be either $k+l$ or $k+l-1$. Moreover, the latter case occurs only when the final portion of α followed by the initial portion of β is a crucial portion which reduces to a single rank i limit term.*

Proof. This follows immediately from Lemma 5.1 since reducing this crucial portion to its normal form does not alter the rank i limit terms at either end and every portion of the resulting term will be in normal form. ☐

Lemma 5.8 allows us to classify exactly which rank i terms are idempotents.

Definition 5.9 (Idempotent)**.** If α is a rank i term in normal form, and α is equivalent in rank i to $\alpha\alpha$, then α will be called an *idempotent in rank i*.

Lemma 5.10. *Let α be a rank i term in normal form. The term α is an idempotent in rank i if and only if the i length of α is 1 and its preferred cyclic normal form is a rank i limit term. In other words, the normal form for α must have the form $\epsilon_0(\beta_1)\epsilon_1$ and the term $(\beta_1)\epsilon_2\epsilon_1(\beta_1)$ must be rank i equivalent to (β_1). As a consequence, it can be algorithmically determined whether a rank i term in normal form is in fact an idempotent in rank i.*

Proof. Let k denote the i-length of α. Since α is rank i, k is at least 1. On the other hand, since α and $\alpha\alpha$ are equivalent in rank i, by Theorem 6.9 they must have the same normal form and the i length of $\alpha\alpha$ must also be k. By Lemma 5.8 this implies that k is at most 1 and that the term $(\beta_1)\epsilon_2\epsilon_1(\beta_1)$ reduces to (β_1). In this case, (β_1) is the cyclic normal form for α and since it has no non-trivial cyclic conjugations, it is also the preferred cyclic normal form. The reverse implication is immediate since the conditions listed are enough to show that α and $\alpha\alpha$ are equivalent in rank i. Finally, note that both of the conditions listed can be checked using the normal form reduction algorithm and Theorem 6.9. □

Lemma 5.11. *If α is a rank i term in normal form which is not an idempotent in rank i, then the i-length of the normal form of $\alpha\alpha$ is strictly greater than that of α.*

Proof. This is immediate from the proof of Lemma 5.10. □

Lemma 5.12. *If α is an idempotent in rank i then (α) is equivalent in rank $i+1$ to the term α.*

Proof. By Lemma 5.10, we can assume that $\alpha = \epsilon_0(\beta_1)\epsilon_1$. The remainder of the proof is a derivation.

$$\begin{aligned}(\alpha) = (\epsilon_0(\beta_1)\epsilon_1) &\Rightarrow (\epsilon_0(\beta_1)(\beta_1)\epsilon_1) \Rightarrow (\epsilon_0(\beta_1)(\beta_1)\epsilon_1)\epsilon_0(\beta_1)(\beta_1)\epsilon_1\\ &\Rightarrow \epsilon_0(\beta_1)((\beta_1)\epsilon_1\epsilon_0(\beta_1))(\beta_1)\epsilon_1 \Rightarrow \epsilon_0(\beta_1)((\beta_1))(\beta_1)\epsilon_1\\ &\Rightarrow \epsilon_0(\beta_1)(\beta_1)(\beta_1)\epsilon_1 \Rightarrow \epsilon_0(\beta_1)\epsilon_1 = \alpha\end{aligned}$$

Each of these steps is an elementary operation of rank at most $i+1$ except for the derivation on the second line. This particular equivalence is justified by the second condition of Lemma 5.10. □

Lemma 5.8 also gives us a great deal of information about the structure of proper powers. For example, the following lemma is now immediate.

Lemma 5.13. *Let α be a rank i term in normal form which is not an idempotent. Even though the normal form for α^j need not be a proper power, the cyclic normal form for α^j will be a j-th power of the cyclic normal form for α.*

Lemma 5.14. *If α is a rank i term which is not an idempotent, then the i length of α^j is the i-length of α plus $j-1$ times the i-length of its cyclic normal form.*

Proof. Let k be the i length of α. In the case where the final portion of α followed by the initial portion of α is a crucial portion does not reduce to a single rank i limit term, the i length of α^j is $kj = k + k(j-1)$. If this crucial portion does reduce to a single limit term as its normal form, then the i length of α^j is $kj-(j-1) = k+(k-1)(j-1)$. When α is an idempotent, it falls into this latter category, but the verbal description will be inaccurate in that case since the cyclic normal form still has one rank i limit term despite the reduction. □

Definition 5.15 (Simple in Rank i)**.** If β is a rank i term in normal form which is not an idempotent, then β will be called *simple in rank i* if its cyclic conjugate is a simple term. If β is a simple term in rank i and β^j is equivalent in rank i to α then β will be called the *rank i root of α* and k will be called the *rank i exponent of α*.

We will now show that the rank i exponent and the rank i root of a rank i term in normal form are unique.

Lemma 5.16. *For each rank i term α in normal form which is not an idempotent, there is a unique simple term β which has rank i and is in normal form for which α and β^j are rank i equivalent for some integer j. Moreover, the term β and the integer j can be algorithmically derived from α.*

Proof. If β is a simple term in rank i and β^j is rank i equivalent to α, then by Lemma 5.13 the cyclic normal form of α will be a j-th power of the cyclic normal form for β. Moreover, since the cyclic normal form of β is itself a simple term in the sense of Definition 2.7, by the definition of rank i simple terms, it is the unique root of the cyclic normal form for α.

Conversely, by Lemma 2.8 the cyclic normal form for α has a unique root which we can use to construct a root in rank i for α. Let β'' be this unique root of the cyclic normal form for α and let j be its exponent. By the first part of this argument, if β is to be a simple term with a power equivalent in rank i to α, then β must have β'' as its cyclic normal form and the power of β rank i equivalent to α must be j. Moreover, since the initial portion of the normal form for β^j will be the same as the initial portion of β and must be the same as the initial portion of α in order for them to be rank i equivalent, β must has the same initial portion as α. Similarly, β must have the same final portion as α.

At this point we have determined the initial portion, the final portion, and all of the crucial portions that any potential simple term β must have. Using Lemma 5.1, combining these portions yields a term in normal form which will satisfy all of our requirements, and since each step was uniquely determined, this is the only term which meets these conditions. Finally, notice that the proof also gives a procedure for deriving the term β and the integer j from the term α. □

In addition to being able to determine which rank i terms are simple in rank i, we can also determine which are idempotents.

6. The normal form algorithm in rank $i+1$

We are now ready to describe the normal form algorithm for terms of rank $i+1$.

Definition 6.1 (Normal Form Algorithm in Rank $i+1$)**.** Let α be a rank $i+1$ term. The *normal form reduction algorithm* applied to α proceeds through the following four steps: (1) apply all possible rank $i+1$ contractions of type 1, (2) apply all possible rank $i+1$ contractions of type 2, (3) apply all possible rank $i+1$ contractions of type 3, and (4) standardize the result. We will describe each step in detail. Let α_j denote the word which emerges after the j-th step.

Step 1: In order to apply all possible rank $i+1$ contractions of type 1, first, we use the rank i normal form reduction algorithm to ensure that the content of each rank $i+1$ limit term is in normal form. Some of the limit terms which began as rank $i+1$ limit terms may cease to be rank $i+1$ limit terms at this point. Next, we use Lemma 5.10 to determine whether the content of any of the rank $i+1$ limit terms is an idempotent in rank i. If it is, then by Lemma 5.12 the term obtained by simply removing the rank $i+1$ pair of parentheses surrounding this idempotent is rank $i+1$ equivalent to the original term. Clearly, the order in which the parentheses are tested is irrelevant to the final outcome. The term which has all possible pairs of rank $i+1$ parentheses removed will be called α_1. If α_1 is no longer of rank $i+1$ (i.e. if all of the rank $i+1$ parentheses have been removed or there were none to begin with), then we apply the rank i normal form reduction algorithm instead to α_1 and the result of this process will be called the *normal form for* α. If rank $i+1$ limit terms still exist, then we continue on with the second step.

Step 2: In the second step, we write each rank $i+1$ limit term in a normal form. In particular, if (β) is a rank $i+1$ limit term in α, then there is a well-defined term of the form $\epsilon_1(\gamma)\epsilon_2$ with the following properties:

1. γ is a simple rank i term in preferred cyclic normal form
2. the term $\epsilon_1(\gamma)\epsilon_2$ is rank $i+1$ equivalent to (β)
3. $\epsilon_1\gamma\epsilon_2$ is the rank i normal form for β^3
4. the term $\epsilon_1\epsilon_2$ is the rank i normal form for β^2
5. the term $\epsilon_2\epsilon_1$ is rank i equivalent to γ^2 when its crucial portions are reduced to normal form and as a result $(\gamma)\epsilon_2\epsilon_1(\gamma)$ is rank $i+1$ equivalent to (γ)
6. ϵ_1 and ϵ_2 each contain at least one limit term of rank i

This term will be called the *standard form for* (β). The procedure which produces this standard form goes as follows. First, recall that β was already reduced to normal form in the first step. Next, if β is not a simple term in rank i, then by Lemma 5.16 we can rewrite it as a proper power of a simple rank i term, and apply a contraction of type 2 to the resulting limit term which contains a proper power. As a result, we might as well assume that β is already a simple term in rank i. Next, let γ denote the preferred cyclic normal form for β. If $\beta = \gamma$, then we declare ϵ_1 and ϵ_2 to be equal to γ and we are done. Otherwise, let

$$\beta = \eta_0(\delta_1)\eta_1 \ldots \eta_{N_1}(\delta)\eta_N$$

be the rank i normal form for β. The normal form of $(\delta_N)\eta_N\eta_0(\delta_1)$ be either (δ_1) or $(\delta_N)\eta(\delta_1)$ for some term η. If β has an i length of at least 2, then after applying an expansion of type 4_R and another of type 4_L to (β) we can shift the parentheses so that neither of the transitions from (δ_N) to (δ_0) is split by either the open or the closed parenthesis of rank $i+1$. At this point we can reduce both copies of the crucial portion $(\delta_N)\eta_N\eta_0(\delta_1)$ to its normal form. The rank $i+1$ limit term in the result will now contain a cyclic conjugate of the cyclic normal form for β. Furthermore, by shifting the parentheses it will be possible to make the content of the rank $i+1$ limit term be the preferred cyclic normal form subject even to the additional restriction that there be at least one limit term of rank i before and after the unique rank $i+1$ limit term. Although there may be more than one way to arrange the parentheses to satisfy these conditions, all that matters is that at least one exists and that the same result be obtained every time the second step of the reduction algorithm is applied to (β). For concreteness, we will stipulate that the parentheses be shifted to the leftmost position satisfying these conditions. Once the rank $i+1$ parentheses have been placed we define γ, ϵ_1 and ϵ_2 appropriately.

The only remaining case is where the i length of β is 1. The normal form for β will be $\eta_0(\delta_1)\eta_1$ and since parentheses surrounding idempotents were removed in the first step we can assume that the normal form for $(\delta_1)\eta_1\eta_0(\delta_1)$ is $(\delta_1)\eta(\delta_1)$ for some term η of rank at most i. Thus the cyclic normal form for β will be $(\delta_1)\eta$ and this will also be the preferred cyclic normal form by Lemma 5.7. We have the following derivation:

$$\begin{aligned}(\beta) = (\eta_0(\delta_1)\eta_1) &\Rightarrow (\eta_0(\delta_1)\eta_1)\eta_0(\delta_1)\eta_1 \Rightarrow (\eta_0(\delta_1)(\delta_1)\eta_1)\eta_0(\delta_1)\eta_1 \\ &\Rightarrow \eta_0(\delta_1)((\delta_1)\eta_1\eta_0(\delta_1))\eta_1 \Rightarrow \eta_0(\delta_1)((\delta_1)\eta(\delta_1))\eta_1 \\ &\Rightarrow \eta_0((\delta_1)(\delta_1)\eta)(\delta_1)\eta_1 \Rightarrow \eta_0((\delta_1)\eta)(\delta_1)\eta_1 \Rightarrow \eta_0(\delta_1)\eta((\delta_1)\eta)(\delta_1)\eta_1\end{aligned}$$

Each step is either an elementary rule of rank at most $i+1$ or the equivalence assumed at the beginning of this case. In the final result, the content of the rank

$i+1$ limit term is the preferred cyclic normal form for β. We now define γ, ϵ_1 and ϵ_2 appropriately.

We will now show that this procedure produces results which have the desired properties. From the explicit forms given above, it is easy to check that in all cases $\epsilon_1\epsilon_2$ is the normal form for β^2 and that the term $\epsilon_2\epsilon_1$ is rank i equivalent to γ^2. The latter statement does, however, rely on the fact that both ϵ_1 and ϵ_2 contain at least one limit term of rank i. Finally, the derivations given above have shown that $\epsilon_1(\gamma)\epsilon_2$ is rank $i+1$ equivalent to (β).

Given a rank $i+1$ term α_1, the term α_2 is formed by replacing each limit term (β) with the equivalent term $\epsilon_1(\gamma)\epsilon_2$ which results from the procedure described above, and then reducing all of the rank i crucial portions which fall between the rank $i+1$ limit terms. This last step is meant to ensure that the terms between the limit terms are at least in subnormal form. Notice that we have not claimed (or shown) that γ, ϵ_1, and ϵ_2 are the unique terms which satisfy these conditions, only that these terms are well-defined by the procedure described. Since the form which results from this procedure is not quite the eventual normal form in all cases, we will call this the *standard form for the limit term* (β).

The procedure for placing rank i limit terms in their standard forms has the following properties:

Lemma 6.2. *The standard form for* (β) *is identical to the standard form for* (β^k).

Proof. If β is a power of a simple term β' in rank i, then β^k will also a power of the same simple term in rank i. Thus these procedures merge as soon as the contents of the parentheses are made to be simple terms in rank i. □

Lemma 6.3. *If* β *is a simple term in rank* i *and the standard form for* (β) *is* $\epsilon_1(\gamma)\epsilon_2$*, then the second step applied to the term* $\beta^j(\beta^k)\beta^l$ *results in the term* $\epsilon_1\gamma^j(\gamma)\gamma^l\epsilon_2$.

Proof. The result follows from the observation that the normal form for β^j is $\epsilon_1\gamma^{j-2}\epsilon_2$. Thus as we proceed through the second step, $\beta^j(\beta^k)\beta^l$ becomes first $\beta^j\epsilon_1(\gamma)\epsilon_2\beta^k$, and then as the crucial terms not contained in the rank $i+1$ limit terms are reduced to their normal form, it becomes $\epsilon_1\gamma^{j-2}\epsilon_2\epsilon_1(\gamma)\epsilon_2\epsilon_1\gamma^{l-2}\epsilon_2$ and finally $\epsilon_1\gamma^j(\gamma)\gamma^l\epsilon_2$. For this last reduction we have used the property that $\epsilon_2\epsilon_1$ is equivalent to γ^2 when its crucial portions are reduced to normal form. □

Lemma 6.4. *Let* β *be a term of rank* i *and let* η *be a limit term of rank at most* i. *If the second step is applied to the terms* $(\eta\beta)\eta$ *and to* $\eta(\beta\eta)$*, then the results will differ at most by an expansion or a contraction of type* 4 *and rank* $i+1$.

Proof. We may assume $\eta\beta$ is already simple. When the expansions of type 4 are applied to each term, the results are $\eta\beta(\eta\beta)\eta\beta\eta$ in the first case and $\eta\beta\eta(\beta\eta)\beta\eta$ in the second. The shift and reductions which take place are the same. Let γ denote the preferred cyclic normal form of $\beta\eta$ and $\eta\beta$. Since the first pair of parentheses originally extended one limit term further to the left, this may be enough of a change to cause the rank $i+1$ parentheses for the first term to end up one copy of term γ further to the left. This can be counteracted by a rank $i+1$ contraction of type 4_R followed by a rank $i+1$ expansion of type 4_L. □

Lemma 6.5. *If* α *and* β *differ by a single shift, then* α_2 *and* β_2 *differ by rank* $i+1$ *contractions and expansions of type* 4.

Proof. Since every shift can be decomposed into shifts which only shift a single variable or a single limit term, the result follows from Lemma 6.4. □

Step 3: In the third step we eliminate crucial portions of the form $(\gamma)\gamma^j(\gamma)$ for some integer $j \geq 0$. This can be done by applying j contractions of type 4 followed by one contraction of type 3. Notice that doing this does not change the initial portion, the final portion, or any of the other crucial portions. It simply eliminates one of the crucial portions. Thus the order in which these eliminations take place is irrelevant and the final result is the same. We say that (γ) is the *normal form* for crucial portions of the form $(\gamma)\gamma^j(\gamma)$ where γ is a simple rank i term in preferred cyclic normal form.

Step 4: The final step is to standardize the results, which we do one portion at a time. We begin with the crucial portions.

If γ and δ are simple rank i terms which are in preferred cyclic normal form, ϵ is a term of rank at most i, and $(\gamma)\epsilon(\delta)$ is a crucial portion of α_3, then there is another rank i term ϵ' with the following properties:

1. $(\gamma)\epsilon(\delta)$ is rank $i+1$ equivalent to $(\gamma)\epsilon'(\delta)$
2. The term $\gamma\epsilon'\delta$ is in rank i normal form
3. ϵ' is not an initial segment of γ^j for any integer j
4. ϵ' is not a final segment of δ^j for any integer j
5. any contractions of type 4 applied to $(\gamma)\epsilon'(\delta)$ will result in a term which fails to have these properties

The procedure for producing such an ϵ' goes as follows. First choose j to be the smallest integer with $|\gamma^j| \geq |\gamma| + |\delta|$, and then apply j expansions of type 4_R to the limit term (γ) so that $(\gamma)\gamma^j$ is an initial segment of the result. Similarly, choose k to be the smallest integer with $|\delta^k| \geq |\gamma| + |\delta|$, and apply k expansions of type 4_L to the limit term (δ) so that $\delta^k(\delta)$ is a final segment of the result. Let ϵ' denote the term between the rank 1 limit terms at this point. This ϵ' will satisfy the first three of the four desired properties.

To see this, note that since this crucial portion remains in α_3, either Lemma 2.9 or Lemma 2.10 can be applied. According to these lemmas, the largest initial segment of ϵ' which is also an initial segment of γ^l for some large l cannot be all of ϵ'. Similarly, the largest final segment of ϵ' which is also a final segment of δ^l for some large integer l cannot be all of ϵ'.

We now apply any rank $i+1$ contractions of type 4 which will preserve these properties. We note that since a rank $i+1$ contraction of type 4 on an initial segment cannot affect whether the third condition is satisfied and a rank $i+1$ contraction of type 4 on a final segment cannot affect whether the second condition is satisfied, the resulting term is independent of the order in which these contractions are performed. The expression $(\gamma)\epsilon'(\delta)$ which results will be called the *normal form* of $(\gamma)\epsilon(\delta)$.

A similar procedure is used for the initial and final portions. If γ is a simple rank i term which is in preferred cyclic normal form, ϵ is a term of rank at most i, and $(\gamma)\epsilon$ is a final portion of α_3, then there is another rank i term ϵ' with the following properties:

1. $(\gamma)\epsilon$ is rank $i+1$ equivalent to $(\gamma)\epsilon'$
2. The term $\gamma\epsilon'$ is in rank i normal form

3. any contractions of type 4 applied to $(\gamma)\epsilon'$ will result in a term which fails to have these properties

The procedure for producing ϵ' goes as follows. First apply an expansion of type 4_R to produce $(\gamma)\gamma\epsilon$, and then reduce all of the rank i portions of $\gamma\epsilon$ (except the initial one) to their normal forms. Let $(\gamma)\epsilon'$ denote the result. At this point the first two properties are satisfied. Finally, we apply as many rank $i+1$ contractions of type 4 to $(\gamma)\epsilon'$ as possible. This final alteration cannot disrupt the second property since the term which results is a subexpression of a rank i term we know to be in normal form. The final result is called the *normal form* of $(\gamma)\epsilon$. The procedure for the initial portion is analogous.

The term α_4 is formed from α_3 by replacing each portion with its normal form. Since none of these procedures change any of the limit terms at either end, they can be carried out independently of each other. This completes the rank $i+1$ normal form reduction algorithm. The final result (α_4) will be called the rank $i+1$ *normal form for* α.

Before describing the properties of these normal forms in greater detail, we record a few properties of this fourth step in the algorithm, followed by a remark.

Lemma 6.6. *Let γ be a simple rank i term in preferred cyclic normal form. If ϵ is any term of rank i, then the normal forms for $\epsilon(\gamma)$ and $\epsilon\gamma^j(\gamma)$ are identical. Similarly, the normal forms for $(\gamma)\epsilon$ and $(\gamma)\gamma^j\epsilon$ are identical.*

Proof. The reason the extra γ's do not change the final normal form is that in the second step a copy of γ is added to each side of the rank $i+1$ limit term and the resulting word $\gamma\epsilon$ is reduced in rank i. When the extra copies of γ are present, the normal form for $\gamma^{j+1}\epsilon$ appears instead. Since γ is cyclically reduced the difference between these two forms is j exact copies of γ which are removed as the standardization proceeds. □

Lemma 6.7. *Let γ and δ be simple rank 0 terms in preferred cyclic normal form. If ϵ is any term of rank 0 then the normal forms for $(\gamma)\epsilon(\delta)$ and $(\gamma)\gamma^j\epsilon\delta^k(\delta)$ are identical.*

Proof. The proof is similar to the proof of Lemma 6.6. In the second step, extra copies of γ and δ are added to the beginning and the end. Any modifications which occur during the reduction part of the second step will not change the fact that the results differ by j exact copies of γ occurring at the beginning of the term between the rank $i+1$ limit terms and k exact copies of δ occurring at the end. At this point it should be clear that the result of the fourth step applied to one will be the same as the fourth step applied to the other. □

Remark 6.8. Notice that if the rank $i+1$ normal form reduction algorithm is applied to a term α of rank $j \leq i$, then the result is the same as the normal form produced by the rank j normal form reduction algorithm applied to α. Thus the normal form reduction algorithms in the various ranks agree whenever several of them can be applied to the same term.

We will now conclude the inductive step by showing that rank $i+1$ equivalence between words of rank at most $i+1$ can be determined based on whether their rank $i+1$ normal forms are identical.

Theorem 6.9. *Let α and β be terms of rank at most $i+1$. The terms α and β are equivalent in rank $i+1$ if and only if α and β have the same normal form. In* *particular, the normal form reduction algorithm can be used to decide whether rank $i+1$ terms are rank $i+1$ equivalent.*

Proof. Since terms are rank $i+1$ equivalent with their normal forms, it is clear that rank $i+1$ terms with the same rank $i+1$ normal form are themselves rank $i+1$ equivalent. Thus we only need to show the other implication. Suppose that α and β are terms of rank at most $i+1$which are rank $i+1$ equivalent but whose normal forms are distinct. By considering the normal forms for each intermediate stage in a rank $i+1$ derivation from α to β, we may in fact assume that α and β differ by the application of a single rule. For concreteness, assume that β is obtained from α by a contraction or a shift either of rank $i+1$ or of lower rank. We will consider each type of rule in turn and show that their normal forms cannot in fact be distinct.

If α and β differ by either a contraction or a shift of rank lower than $i+1$, then this rule must be applied either inside one of the rank $i+1$ limit terms or in one of the words between the rank i limit terms. In the former case, the difference between α and β is eliminated at the beginning of the first step. In the latter it is eliminated at the end of the second step. In both cases, the eventual normal form will be identical.

If α and β differ by a rank $i+1$ contraction of type 1, then this extra rank $i+1$ parenthesis is removed during the first step. Thus the eventual normal forms will be the same. If α and β differ by a rank $i+1$ contraction or shift and the pair of rank $i+1$ parentheses used in the relation is removed during the first step, then the content of the limit term is an idempotent, and the difference now occurs in a portion of the terms which is outside of the rank $i+1$ limit terms. As such this difference is eliminated at the end of the second step. Thus from now on we may assume that our rank $i+1$ contractions and shifts emerge virtually unchanged from the first step (other than having the contents of the rank $i+1$ limit terms be reduced to normal forms).

If α and β differ by a rank $i+1$ contraction of type 2, then by Lemma 6.2, α_2 and β_2 will already be identical and their eventual normal forms will thus be the same.

If α and β differ by a contraction of type 3, then α contains a pair of adjacent limit terms, say $(\delta)(\delta)$. When these are placed in normal form in the second step, the resulting α_2 contains $(\gamma)\gamma^2(\gamma)$ where β_2 contains only (γ). Thus α_2 and β_2 will differ by at most a contraction of type 4 and a contraction of type 3. More importantly, the crucial portion of α_2 bracketed by two limit terms (γ) will be removed in the third step, α_3 and β_3 will be identical, and their eventual normal forms will again be the same.

If α and β differ by a contraction of type 4, then by Lemma 6.3, α_2 and β_2 differ by a series of contractions of type 4. There are now three possibilities. If the contractions involve a crucial portion of α_2 which is removed in the third step it will also remove this crucial portion in β_2. Thus α_3 and β_3 will be identical and so will their normal forms. If the contractions affect a crucial portion which is not removed in the third step, then α_3 and β_3 will still differ by a series of contractions of type 4, but by Lemma 6.7 the resulting normal forms will be the same. Similarly, if the contractions affect an initial portion or a final portion then α_3 and β_3 will differ by a series of contractions of type 4, but by Lemma 6.6 the normal forms will again be identical.

Finally, if α and β differ by a shift, then by Lemma 6.5 α_2 and β_2 will differ by contractions and expansions of type 4 and we have already shown that these types of changes yield the same normal forms. In all cases, α and β must have the same normal form and we have contradicted our assumption. This completes the proof. □

7. The term problem for F_n

Now that the inductive step is complete, it is easy to show that every element of F_n has a unique normal form and that the term problem for F_n is decidable.

Lemma 7.1. *If α is a term of rank i, then α is equivalent to a unique term in normal form. These terms are moreover equivalent in rank i.*

Proof. Applying the normal form reduction algorithm (Definition 6.1) in rank i produces a well-defined normal form for α. Moreover, by Remark 6.8, applying the normal form reduction algorithm in rank j for any $j \geq i$ would produce the identical normal form. □

Theorem 7.2. *Two terms α and β are equivalent in F_n if and only if they have the same normal form. In particular, the normal form reduction algorithm can be used to decide whether terms in T_n are equivalent in F_n, and thus the term problem is decidable for the κ-semigroup F_n.*

Proof. If α and β have the same normal form, then since both α and β are equivalent to this normal form, they are equivalent to each other in F_n. Conversely, if α and β are equivalent in F_n, then there is a derivation from α to β, this derivation has an upper bound on the ranks of the rules invoked, and thus α and β are also equivalent in rank i for some i. By Theorem 6.9, α and β must then have the same normal form. Finally, notice that since the normal form reduction algorithm is an algorithm which stops in finite time, this procedure is effective and the term problem is decidable. □

8. Burnside semigroups

In the second half of the article we will show that F_n and $\Omega_n^\kappa \mathbf{A}$ are in fact identical. The proof will rely heavily on the structure of the Burnside semigroups of large exponent. In this section and the next we will review this structure using the notation and citing the results from [5]. Similar results were obtained independently by de Luca and Varricchio ([3]). None of the results stated in these two sections are new and the reader is referred to [5] for detailed proofs.

Definition 8.1 (Burnside Semigroups)**.** A Burnside semigroup is a relatively free semigroup satisfying a single identity of the form $T^r = T^{r+s}$ for every word T. More explicitly, the n-generated aperiodic Burnside semigroup of exponent m has the presentation

$$\mathcal{B}_n(m) = \langle x_1, \dots x_n | T^m = T^{m+1} \ \forall T \in \{x_1, \dots, x_n\}^* \rangle$$

For our purposes, two of the most important properties of the Burnside semigroups are that they are finite $\mathcal{J}$-above, and that the language of words equivalent to a given element can be expressed as either a Kleene expression without unions or as the language accepted by a loop automaton.

Definition 8.2 (Finite $\mathcal{J}$-Above)**.** Let S be a fixed semigroup and let x and y be elements of S. If there are elements u and v in S such that $uxv = y$, then x is said to be *$\mathcal{J}$-above y*. If for every element y in S there are only a finite number of elements which are $\mathcal{J}$-above y, then S is said to be *finite $\mathcal{J}$-above*.

The property of being finite $\mathcal{J}$-above plays the role in semigroup theory that residual finiteness plays in group theory. In particular, if S is a semigroup which is finite $\mathcal{J}$-above, then for every pair of distinct elements, there is a homomorphism onto a finite semigroup in which their images remain distinct. Explicitly, if x and y are the two elements and I is the ideal of all elements which are not $\mathcal{J}$-above the product xy, then S/I is finite and x and y remain distinct in this quotient.

Definition 8.3 (Languages)**.** Let $X = \{x_1, \dots, x_n\}$, and let X^* denote the set of all finite strings of variables from X. The elements of X^* are typically called *words*. A subset of X^* is called a *language*.

Definition 8.4 (Automata)**.** Let X be a set and let ϵ be a symbol which is not contained in X. A *finite state automaton* over X is a finite directed graph with an element of $X \cup \epsilon$ assigned to each edge together with a unique vertex known as the *start state* and a subset of vertices known as the *end states*. The automaton is *deterministic* if for every vertex and choice of label, there is at most one directed edge starting at that vertex which has that particular label. The *language accepted* by such an automaton is the set of words which are formed by concatenating the labels on the edges of a path which starts at the start state and ends at one of the end states. The ϵ label corresponds to the empty word and contributes nothing to the concatenation.

Definition 8.5 (Regular Languages)**.** A *regular language* is a subset of X^* which is the language accepted by a finite state automaton over X. The *rational languages* are the smallest collection of languages which contain the singleton sets and are closed under union, product and monoid closure.

Remark 8.6. By Kleene's theorem, the set of rational languages and the set of regular languages coincide. By the Kleene-Rabin-Scott theorem, the automata used to define regular languages can be required to be deterministic and to exclude edges whose label ϵ represents the empty word without altering the collection of languages defined.

We will mainly be interested in a third representation of these languages, namely, by Kleene expressions. Of particular interest will be the Kleene expressions defined without using unions and the corresponding loop automata which accept them.

Definition 8.7 (Kleene Expressions)**.** A *Kleene expression* is a compact way to describe a regular language. Kleene expressions can be defined inductively as follows. A single variable x_j is a Kleene expression and the regular language associated to this expression is the language consisting of one word, namely the singleton x_j. If α and β are Kleene expressions, then the concatenation $\alpha\beta$, the union $(\alpha\cup\beta)$ and Kleene star $(\alpha)^*$ are also Kleene expressions. The corresponding regular languages are defined as follows. Let A and B be the regular languages which correspond to α and β, respectively. The language corresponding to the concatenation $\alpha\beta$ is the set of words formed by concatenating a word from A with a word from B. The language corresponding to $(\alpha \cup \beta)$ is the union of A and B. The language corresponding to $(\alpha)^*$ is the set of words formed by concatenating a finite sequence of words from

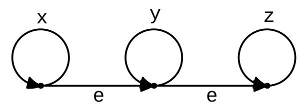

FIGURE 1. A loop automaton which requires ϵ edges

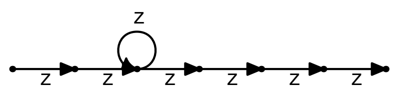

FIGURE 2. A rank 1 straightline automaton

A. This finite sequence can also be empty so that the empty word is always an element in the language of $(\alpha)^*$ for any α. Alternatively, the language of $(\alpha)^*$ can be described as the union of the languages of the expressions $\{\}, \alpha, \alpha\alpha, \alpha\alpha\alpha, \ldots$.

In this article, all of our Kleene expressions will be defined using only concatenation and the Kleene star. Unions will not be used. One consequence of not allowing unions is that the automata which accept the languages of these Kleene expressions can be chosen to be of a very restricted type. Moreover, these types of Kleene expressions closely parallel the normal forms defined for the elements of F_n.

Definition 8.8 (Loop Automata)**.** A *loop automaton* is a finite state automaton which can be constructed inductively as follows. Start with an automaton which is a sequence of directed edges, directed from left to right, with the leftmost vertex as its unique start state and the rightmost vertex as its unique end state. Then repeatedly attach sequences of edges which form a directed loop with exactly one of its vertices in the pre-existing construction. Examples can be seen in Figures 1 through 4.

Kleene expressions which do not contain unions and loop automata are essentially equivalent. The following lemma is, in fact, nearly immediate.

Lemma 8.9. *A regular language can be defined by a unionless Kleene expression if and only if it is the language accepted by a loop automaton.*

Example 8.10. Let α be the Kleene expression $zz(z)^*zzzz$. The loop automaton for this expression is shown in Figure 2. A loop automaton for the expression $xyxy(xy)^*xyxyxyxyx$ can be found in Figure 3. In general, these loop automata require ϵ edges in their construction. For example, consider the Kleene expression $(x)^*(y)^*(z)^*$. There is an easily constructed loop automaton which accepts this language, as shown in Figure 1. It is not hard to show that a loop automaton for this language cannot be constructed without using ϵ edges.

We can now summarize the basic properties of the Burnside semigroups of sufficiently large exponent.

Theorem 8.11. *For $m \geq 6$, the aperiodic Burnside semigroup $\mathcal{B}_n(m)$ is finite $\mathcal{J}$-above and has a decidable word problem. Moreover, for each element α of $\mathcal{B}_n(m)$, the language of words which represent this element is regular and can be described by either a single Kleene expression without unions or as the language accepted by a loop automaton without ϵ edges.*

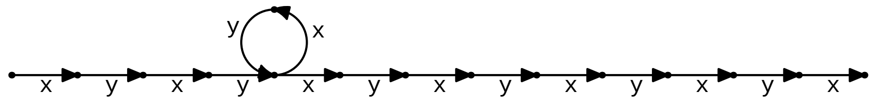

Figure 3. Another rank 1 straightline automaton

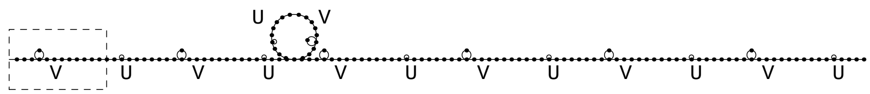

Figure 4. A rank 2 straightline automaton

Example 8.12. Let U be the word z^{20}, and let α be the element of $\mathcal{B}_2(6)$ which is represented by U. The language of words which represent α can be described by the Kleene expression $zz(z)^*zzzz$ or by the loop automaton shown in Figure 2. Similarly, let V be the word $(xy)^{10}x$, and let β be the element it represents. The language of words which represent β is described by the Kleene expression $xyxy(xy)^*xyxyxyxyx$ or by the loop automaton shown in Figure 3. For a more complicated example, consider the word $W = (VU)^{105}$, where U and V are as above. The loop automaton for the language of words equivalent to this particular word in $\mathcal{B}_2(6)$ is shown in Figure 4. The portions of the automaton labeled by α and β correspond to copies of the automata in Figure 2 and Figure 3, respectively. In particular, the portion contained in the dashed box is the same as that shown in Figure 3.

9. Normal forms for Burnside semigroups

In addition to showing that the word problem is decidable, [5] provides an explicit algorithm for constructing the normal form of an element, a Kleene expression for the language of equivalent words, and a loop automaton which accepts this language. At first glance, it might seem like the set of rules which replace T^{m+1} with T^m for each word T should be sufficient to replace every word with its shortest possible form. The reason why this does not quite work is illustrated by the following example.

Example 9.1. Let T be the word $x(yx)^m z$ and let W be the word $(yx)^m zT^m$. Even though W does not contain an $(m+1)$-st power, it is equivalent to a strictly shorter word. In particular,

$$(yx)^m zT^m \Rightarrow yx(yx)^m zT^m = yT^{m+1} \Rightarrow yT^m = yx(yx)^m zT^{m-1} \Rightarrow (yx)^m zT^{m-1}$$

Notice that the first step in this derivation is an expansion. Despite this example, there is a modified set of rules which allow every word to be reduced to its shortest possible form using only rules which strictly reduce its length. This leads to an explicit procedure for reducing words in Burnside semigroups to their normal forms. In order to fully describe the procedure, we will need to review a few more definitions from [5].

First note that not all of the identities of the form $T^{m+1} = T^m$ are needed in order to define a presentation for the Burnside semigroups. It is sufficient to consider only words T which are simple and which do not already contain $(m+1)$-st powers of shorter words. More generally, we only need consider identities of the form $T^{m+1} = T^m$ where T is a simple, cir-reduced word.

Definition 9.2 (Simple in $\mathcal{B}_n(m)$)**.** An element of $\mathcal{B}_n(m)$ $(m \geq 6)$ is called a *proper power* if it is equal to a nontrivial power of another element. An element which is not a proper power of any element is called *simple*. A word representing an element of $\mathcal{B}_n(m)$ is called *simple* if it represents a simple element.

When W is a proper power, the unique simple word T of which W is a power is called the *simple root* of W.

Lemma 9.3. *Every word W representing an element of $\mathcal{B}_n(m)$ $(m \geq 6)$ satisfies exactly one of the following conditions:*

1. *it is a simple word*
2. *it is equivalent to T^j where T is a simple word, j is number less than m and T and j are uniquely defined*
3. *it is equivalent to T^j for all $j \geq m$ where T is a uniquely defined simple word.*

In addition W represents an idempotent in $\mathcal{B}_n(m)$ if and only if it satisfies the third condition.

Definition 9.4 (cir-reduced words)**.** A word is called *str-reduced* if it does not contain subwords of a particular type. We will postpone the precise definition of str-reduced words until after these particular types of subwords have been defined. A simple word T is called *cir-reduced* if and only if T^3 is str-reduced. One consequence of this definition is that every simple, cir-reduced word is itself str-reduced. In most cases str-reduced words are also cir-reduced. The exceptions occur when the prohibited subwords exist in T^2 or T^3, but not in T itself as in Example 9.14. Words which are proper powers will be called cir-reduced only if they are equal to (not just equivalent to) a proper power of simple, cir-reduced word.

Definition 9.5 (Minimal Supports)**.** If T^i is a power of a simple, cir-reduced word with $i \geq 3$, then there is a unique minimal subword U of T^i such that T^i is $\mathcal{J}$-above U in $\mathcal{B}_n(m)$. This minimal subword is called the *minimal support of T^i* and it will be denoted by $\text{MinSup}(T^i)$. Minimal supports could also be defined for arbitrary str-reduced words, but in that generality they are not unique. They *are* unique for high powers of simple, cir-reduced words.

Definition 9.6 (str-reduced words)**.** A word T is called *str-reduced* if it does not contain $\text{MinSup}(U^{m+1})$ for any simple, cir-reduced word U. Since minimal supports of this type will contain U^{m-1} as a subword (see Lemma 9.9 below), there are only a very limited number of minimal supports of this type in any particular word T and it is easy to identify where they might occur.

Although these definitions may appear circular, they can be rigorously defined by inducting on the length of the words under consideration.

Lemma 9.7. *Let $\mathcal{B}_n(m)$ be fixed with $m \geq 6$. Every element in $\mathcal{B}_n(m)$ is represented by a unique str-reduced word and every conjugacy class is represented by a cir-reduced word which is unique up to cyclic conjugation. In particular, every word T in $\mathcal{B}_n(m)$ is equivalent to a unique word which is str-reduced and it is conjugate to a cir-reduced word which is unique up to cyclic conjugation.*

Lemma 9.8. *An element of $\mathcal{B}_n(m)$, $m \geq 6$, is simple in the sense of Definition 9.2 if and only if the unique cir-reduced word in its conjugacy class is a simple word in the sense of Definition 2.7.*

Lemma 9.9. *If T is a simple, cir-reduced word, then the subword MinSup(T^j) is formed by removing fewer than $|T|$ letters from the beginning of T^j and fewer than $|T|$ letters from the end of T^j. In particular, the minimal support for T^j will contain T^{j-2} as a subword. Moreover, the exact number of letters removed is independent of j so that the minimal supports of T^i and T^j, $i, j \geq 3$, differ by the insertion or deletion of an integral number of copies of T in the interior of the word. Finally, the words MinSup(T^{m+1}) and MinSup(T^m) are equivalent in $\mathcal{B}_n(m)$.*

The easiest way to organize a study of minimal supports is to arrange them by their rank.

Definition 9.10 (str-rank)**.** Let $\mathcal{B}_n(m)$ be a Burnside semigroup with $m \geq 6$. The *str-rank* of a str-reduced word T is the length of the longest chain of nested minimal supports for m-th powers of simple, cir-reduced words. If, for example, a word contains no minimal supports for m-th powers at all, then its str-rank is 0. If it contains a minimal support for U^m which in turn strictly contains a minimal support for V^m, then it has a str-rank of at least 2. If T is not str-reduced, then the str-rank of T is defined to be the str-rank of its unique str-reduction.

Definition 9.11 (cir-rank)**.** The cir-rank of a simple, cir-reduced word W in $\mathcal{B}_n(m)$ is the str-rank of W^3. If W is a cir-reduced word which is a proper power, then its cir-rank is the cir-rank of the cir-reduction of its unique simple root.

Example 9.12. The normal forms of the words U, V and W described in Example 8.12 have str-ranks of 1, 1 and 2, respectively. As can be seen from the corresponding figures, the rank of a word also corresponds to the number of loops within loops within loops which need to be attached to the initial base in order to form the corresponding loop automaton.

For our purposes we only need the following result.

Lemma 9.13. *Let W be a word in $\mathcal{B}_n(m)$ with $m \geq 6$ and let i and j denote the str-rank and the cir-rank of W, respectively. If W is an idempotent in $\mathcal{B}_n(m)$ then $j = i - 1$. If W is not an idempotent in $\mathcal{B}_n(m)$ then either $j = i$ or $j = i + 1$. The latter case only occurs in the situation described in Example 9.14.*

Example 9.14. Let W be the simple word $yz(xyzxy)^{m-1}x$. The str-rank of W is 0 since it does not contain any m-th powers. The str-rank of W^2 is also 0 for the same reason. The word W^3, however, contains an m-th power of $xyzxy$. Thus W^3 has str-rank 1 and W has cir-rank 1. If some high power of W contains a minimal support of a rank i base word U whose length is less than that of W, then by Lemma 2.9 the entire periodic word has length less than $|U| + |W| < |W^2|$. This does not imply that the U-periodic section occurs in W^2, as we saw, but this subword does occur somewhere in W^3. Thus the str-rank of W^i and W^j is the same for all i and j between 3 and $m - 2$. This is the reason why W^3 is used to define the cir-rank of simple words.

Definition 9.15 (Normal Forms)**.** Let $\mathcal{B}_n(m)$ be a fixed Burnside semigroup with $m \geq 6$, let T be a word and let Γ be the loop automaton which accepts the language of words equivalent to T in $\mathcal{B}_n(m)$. The word T will be str-reduced with respect to $\mathcal{B}_n(m)$ if and only if it is the unique shortest word accepted by Γ. In particular, every word T is equivalent to a unique str-reduced word which is called its *normal form*.

Definition 9.16 (Reduction Algorithm)**.** Let $\mathcal{B}_n(m)$ be a fixed Burnside semigroup with $m \geq 6$. Every word T can be reduced to its str-reduced normal form by repeatedly replacing subwords of the form $\text{MinSup}(U^{m+1})$ with the equivalent subword $\text{MinSup}(U^m)$. More specifically, these substitutions can be carried out by rank and only for minimal supports of powers of simple, cir-reduced words. If U is a simple, cir-reduced, cir-rank i word, then U will be called a *rank i base*. In this language, the basic type of reduction replaces a minimal support of an $(m+1)$ power of a rank i base with a minimal support of an m-th power of the same rank i base.

The complete algorithm goes as follows. It is sufficient to first reduce all minimal supports of $(m+1)$ powers of a rank 0 base. When no more reductions of this type are possible, the uniquely defined result is called the *str_1-reduction of T*. Then all minimal supports of $(m+1)$ powers of a rank 1 base are reduced. When no more reductions of this type are possible, the uniquely defined result is called the *str_2-reduction of T*. This process continues up through the ranks. The word which results from reducing all of the minimal supports of $(m+1)$ powers of a rank $k-1$ base is called the *str_k-reduction of T*. If the str-rank of T is k, then this process only needs to continue up to rank k. In other words, the str_k-reduction of T is the unique str-reduced word which is equivalent to T.

Definition 9.17 (Incompatible Subwords)**.** Let W be a word and let U and V be rank i bases such that W contains a minimal support of U^m and a minimal support of V^m. If the minimal support for V^m is contained in the maximal periodic subword of W which contains the minimal support for U^m, then since they have the same rank, U and V must be cyclic conjugates of each other and the minimal support for U^m and the minimal support for V^m will be called *compatible*. Notice that this definition is symmetric with respect to U and V. The minimal supports for U^m and V^m will be called *incompatible* in all other cases.

Definition 9.18 (Domain)**.** Let W be a word which represents an element of $\mathcal{B}_n(m)$ with $m \geq 6$ and let U be a rank i base word. If W contains a minimal support for U^m, then the largest U-periodic subword of W which contains this minimal support will be called the *domain of this minimal support*.

Lemma 9.19. *Let U be a rank i base word which represents an element of $\mathcal{B}_n(m)$ with $m \geq 6$. If V is a rank j base word with $j < i$ and a minimal support of V^m is contained in U^m, then the domain of this particular minimal support has length at most $|U^2|$ and it will mostly survive in the minimal support for U^m. In particular, the portion of the domain of V^m which survives will still contain a minimal support for a cyclic conjugate of V^m.*

It is in fact this feature of minimal supports which allows us to establish most of the properties listed in Lemma 9.9.

Lemma 9.20. *If S is a word with cir-rank at least i, then there exist words T, P, and Q such that for all $j \geq 4$, the str_i-reduction of S^j is PT^jQ, where T is a cir_i-reduced word. That is, the words T, P, and Q can be chosen independently of j. Moreover, the cir-rank of T is the same as that of S.*

Proof. This is Lemma 4.8 in [5]. □

Lemma 9.21. *Let S be a subword of T. If the str_i-reduction of S (in $\mathcal{B}_n(m)$, $m \geq 6$) contains a minimal support for U^m where U is a rank j base, then the str_i-reduction of T will also contain this particular minimal support. In particular, the str-rank of S in $\mathcal{B}_n(m)$ is at most that of T.*

Proof. This is the essential content of Lemma 2.8 in [5]. □

Lemma 9.22. *Let $f_i(m)$ be the length of the shortest word of cir-rank i in $\mathcal{B}_n(m)$ for $m \geq 6$. For every i, f_i is bounded below by a polynomial in m of degree i. In particular, f_i is bounded below by $(m-3)^i$. As a consequence the length of the shortest str-rank i word is bounded below by $(m-2)(m-3)^i$.*

Proof. The argument is by induction. If U is cir-reduced and has cir-rank 1, then U^3 contains a minimal support for an m-th power of a rank 0 base V. By Lemma 9.9 this minimal support contains V^{m-2}. Moreover, since U and V have different cir-ranks they can not be cyclic conjugates. Thus, by Lemma 2.9 $|V^m - 2| < |U| + |V|$. This shows that $|U| > (m-3)|V|$. Since the length of V is at least 1, $|U| > (m-3)$. Next, suppose that the result is true for some i, and let U be a cir-reduced word with cir-rank $i+1$. By definition U^3 contains a minimal support for an m-th power of a rank i base word V. The same argument shows that $|U| > (m-3)|V|$. Since by assumption $|V| > (m-3)^i$, the length of U is greater than $(m-3)^{i+1}$. For the final statement simply note that every str-reduced word of str-rank i contains a subword of the form V^{m-2} where V is a cir-reduced word of cir-rank $i-1$. □

10. Operations on Terms

In this section we present a number of general definitions about manipulating terms in T_n which will be needed below. There are two major processes: expansions and reconstructions.

Definition 10.1 (The Map ϕ_m)**.** Let $\phi_m : T_n \to X^*$ be the map which sends x_i to x_i, which sends the implicit operation $\omega : \alpha \mapsto (\alpha)$ to the explicit operation $\alpha \mapsto \alpha^m$, and which extends to the other elements in T_n by composition. The term $(x_1(x_2))x_3 \in T_n$, for example, will be sent by ϕ_m to the word $(x_1(x_2)^m)^m x_3$. Since pairs of terms defining an identity of F_n are sent under ϕ_m to pairs of words which are equivalent in the Burnside semigroups, there is an induced map, which we will also call ϕ_m from F_n to $\mathcal{B}_n(m)$.

The following lemma is immediate.

Lemma 10.2. *Let α be a term of rank i and let $g(m)$ denote the length of $\phi_m(\alpha)$. The function $g(m)$ is an i-th degree polynomial in m. In particular, if c_j is the number of variables in α of rank j, then $g(m)$ is $c_0 + c_1 m + c_2 m^2 + \cdots + c_i m^i$.*

As a corollary we can immediately bound the eventual rank of $\phi_m(\alpha)$ as m gets large. Since this type of statement will occur frequently below, we will adopt the following convention: whenever we state that something happens *for sufficiently large m*, this means that there is a constant M such that for all $m \geq M$, the statement is true.

Corollary 10.3. *If α is a term of rank i, then for sufficiently large m, the str-rank of $\phi_m(\alpha)$ is at most i. As a consequence, the cir-rank of $\phi_m(\alpha)$ is also at most i.*

Proof. The result is immediate from the growth rates shown in Lemma 9.22 and Lemma 10.2. Specifically, since $g(m)$ is only a polynomial of degree i, it will eventually be overtaken by each of the polynomials $f_j(m)$ with $j > i$. This shows that $\phi_m(\alpha)$ cannot continue to be a word of str-rank greater than i as m gets large. The second statement follows from the first by applying it to α^3. □

Lemma 10.4. *If α and β are terms and α is a subexpression of β, then $\phi_m(\alpha)$ will be a subword of $\phi_m(\beta)$ and consequently, the str-rank of the word $\phi_m(\alpha)$ in $\mathcal{B}_n(m)$ ($m \geq 6$) is at most the str-rank of $\phi_m(\beta)$.*

Proof. The first part of the statement is immediate from the definition of ϕ_m. The second part follows from Lemma 9.21. □

In order to make this map easier to work with we will break it down into simpler steps.

Definition 10.5 (Rank i Expansions)**.** Let $\alpha = \epsilon_0(\gamma_1)\epsilon_1(\gamma_2)\ldots(\gamma_N)\epsilon_N$ be a rank $i+1$ term in normal form, where each ϵ_j is a term with rank at most i and each (γ_j) is a simple rank $i+1$ limit term in preferred cyclic normal form. Let β be a rank i term formed by replacing the limit term (γ_j) with r_j copies of γ_j. Thus $\beta = \epsilon_0\gamma_1^{r_1}\epsilon_1\gamma_2^{r_2}\ldots\gamma_N^{r_N}\epsilon_N$. If each r_j is at least 1, then β is called a *rank i expansion of α*. If all of the r_j are the same, then the expansion will be called *uniform*. The subexpressions of the form $\gamma_j^{r_j}$ will be called *expanded limit terms*.

Remark 10.6. Notice that if α has rank 1 and $m \geq 2$, then $\phi_m(\alpha)$ is an example of a uniform rank 0 expansion of α. More generally, if α has rank $i+1$ and $m \geq 2$, then the word $\phi_m(\alpha)$ can be formed by first performing a uniform rank i expansion of α, followed by a uniform rank $i-1$ expansion of the result, and continuing in this way until in the final step a uniform rank 0 expansion is performed. These expansions along the way will be the main objects under investigation below.

Lemma 10.7. *If α is a rank $i+1$ term in normal form and β is a rank i expansion of α, then β is a rank i term in normal form.*

Proof. Let $\alpha = \epsilon_0(\gamma_1)\epsilon_1\ldots(\gamma_N)\epsilon_N$ where each (γ_j) is a rank $i+1$ limit term in normal form and each ϵ_j has rank at most i, and let β_1 be the rank i expansion of α which replaces each rank $i+1$ limit term (γ_j) with γ_j. In other words β_1 is the term obtained from α by simply removing all of the parentheses of rank $i+1$. Since by definition each subexpression γ_j contains a rank i limit term, each rank i portion of β_1 will be contained in one of the expressions $\epsilon_0\gamma_1$, $\gamma_N\epsilon_N$, or $\gamma_j\epsilon_j\gamma_{j+1}$. By the stipulations in Step 4 of the normal form algorithm, each of the expressions in this list are in normal form. Thus by Lemma 5.1 all of the rank i portions of β_1 are in normal form and hence, β_1 itself is in normal form.

Next consider an arbitrary rank i expansion, β. Notice that all of the rank i portions of β either occur already in β_1, or else they are formed by concatenating the final rank i portion of some γ_j with its initial rank i portion. Since by the definition of normal form, the term γ_j is in cyclic normal form, this new crucial portion is also in normal form. Thus, by Lemma 5.1 again, every rank i expansion of α will be in normal form. □

Definition 10.8 (Associated Limit Terms)**.** Let α be a rank $i+1$ term in normal form and let β be a rank i expansion of α. In particular, let β be the term obtained from α by replacing each rank $i+1$ limit term (γ_j) with r_j copies of γ_j. A subexpression of the form η^k, $k \geq 2$, is said to be *associated with the limit term* (γ_j) if η is a cyclic conjugate of γ_j and η^k is contained in the maximal γ_j-periodic subexpression of β which contains the expanded limit term $\gamma_j^{r_j}$. This maximal γ_j-periodic subexpression of β will be called the *domain of the limit term* (γ_j), and it will typically be denoted Γ_j. There is a rough correspondence between the domain of a minimal support and the domain of a limit term, hence the similar terminology.

Definition 10.9 (Portion Length)**.** Let α be a rank $i+1$ term in normal form and let β be one of the portions of α (i.e. its initial portion, its final portion, or one of its crucial portions). The *portion length* of β is defined as the i-length of β plus the $i+1$-length of β. Recall that the i-length of a term is the number of limit terms of rank i which the normal form contains. The 0-length of a term is simply the length of the word which remains once all of the parentheses of all ranks have been removed. For example, the portion length of the rank 1 final portion $(xy)z$ would be 4.

Definition 10.10 (Portion Bound)**.** Let α be a rank $i+1$ term in normal form. The least upper bound on the portion length of the portions of α will be called the *portion bound of* α.

Example 10.11. Let α be the rank 1 term $(xy)x$ in T_n. The image of α under ϕ_6 is $(xy)^6x$ and Figure 3 shows the loop automaton which accepts the language of words equivalent in $\mathcal{B}_2(6)$ to this particular word. Similarly, Figure 4 shows the loop automaton which accepts the language of words equivalent in $\mathcal{B}_2(6)$ to the image of the term $\beta = ((xy)xz)$ under the map ϕ_6.

11. Recovery in rank 1

The philosophy of the proof is as follows: if W is the str-reduction of $\phi_m(\alpha)$ for some very large value of m, then starting in rank 1 and working up through the ranks, the rank i portions of W will allow us to reconstruct the rank i parentheses of α. In this section we begin by showing that these ideas can be carried out for terms of rank 0 and rank 1. In the next section, we will provide an inductive step which shows that this can be carried out for terms of higher rank.

Remark 11.1 (Rank 0 Terms)**.** Let α be a rank 0 term. The effect of the maps ϕ_m on α is particularly easy to describe since for m greater than the length of α they leave it unchanged. In F_n the terms that are rank 0 are the sole members of their equivalence classes and thus are in normal form. Similarly, for large m, $\phi_m(\alpha)$ will contain no m-th powers. Thus no identities can be applied and $\phi_m(\alpha)$ will also be the sole member of its equivalence class.

Lemma 11.2. *If α and β are two distinct rank 0 terms such that $\phi_m(\alpha)$ and $\phi_m(\beta)$ are equivalent for all m, then α and β are identical.*

Proof. Let m be any number larger than the length of α and the length of β. For this m, $\alpha = \phi_m(\alpha) = \phi_m(\beta) = \beta$. □

Lemma 11.3. *If α is a rank 0 term in normal form, then for sufficiently large m, $\phi_m(\alpha)$ is not an idempotent in $\mathcal{B}_n(m)$, and $\phi_m(\alpha)$ has str-rank 0 and cir-rank 0. Moreover, if β is the simple root and j is the exponent of α, then for sufficiently large m, $\phi_m(\alpha)$ is a j-th power of the simple word $\phi_m(\beta)$.*

Proof. Immediate. □

Lemma 11.4. *If $\alpha = (\gamma)$ is a rank 1 limit term in normal form, then for sufficiently large m, the word $\phi_m(\gamma)$ is a rank 0 base.*

Proof. By Lemma 11.3, for sufficiently large m, γ is a simple word which has cir-rank 0. Moreover, by applying Lemma 11.3 to γ^3, we can conclude that for these large values of m, $\phi_m(\gamma)$ is also cir-reduced, and thus has all three properties of a rank 0 base. □

Lemma 11.5. *If α is a rank 1 term in normal form, then for sufficiently large m, the str-rank of $\phi_m(\alpha)$ in $\mathcal{B}_n(m)$ will be 1. Moreover, the same is true for all terms β which are equivalent to α in F_n.*

Proof. By Corollary 10.3, for sufficiently large m, the str-rank of $\phi_m(\alpha)$ is at most 1. By Lemma 11.4, for sufficiently large m, $\phi_m(\alpha)$ contains an m-th power of a rank 0 base. This shows that $\phi_m(\alpha)$ has str-rank at least 1 and completes the proof. The final assertion merely reflects the fact that ϕ_m is a homomorphism from F_n to $\mathcal{B}_n(m)$. □

Lemma 11.6. *If α is a rank 1 term in normal form which is not an idempotent, then for sufficiently large m the cir-rank of $\phi_m(\alpha)$ in $\mathcal{B}_n(m)$ will be 1. If α is a rank 1 idempotent in normal form then for sufficiently large m, the cir-rank of $\phi_m(\alpha)$ will be 0. Moreover, the same is true for all terms β which are equivalent to α in F_n.*

Proof. The first statement follows from applying Lemma 11.5 to α^3. If α is a rank 1 term which is an idempotent then by Lemma 11.5, the str-rank of $\phi_m(\alpha)$ in $\mathcal{B}_n(m)$ will be 1 for all sufficiently large m. Since ϕ_m is a homomorphism, $\phi_m(\alpha)$ is also an idempotent and so by Lemma 9.13, the cir-rank of $\phi(\alpha)$ will be 0 for these values of m. □

We can now show that rank 0 terms and rank 1 terms are eventually distinguished by their images in various Burnside semigroups.

Lemma 11.7. *If α and β are two terms in normal form of rank 0 and 1 respectively, then there exists an m such that $\phi_m(\alpha)$ and $\phi_m(\beta)$ are not equivalent in $\mathcal{B}_n(m)$.*

Proof. By Remark 11.1 and Lemma 11.5, for sufficiently large values of m, $\phi_m(\alpha)$ will be an element in $\mathcal{B}_n(m)$ of rank 0, and $\phi_m(\beta)$ will be an element in $\mathcal{B}_n(m)$ of rank 1. Thus, the elements represented by $\phi_m(\alpha)$ and by $\phi_m(\beta)$ cannot possibly be equal in $\mathcal{B}_n(m)$. □

Definition 11.8 (Rank 1 Structures)**.** Let W be a word which represents an element of $\mathcal{B}_n(m)$. If the str-rank of W is at least 1, then W will contain a minimal support of an m-th power of a rank 0 base as a subword. Consider all of the subwords of W which are the domain of a minimal support of this type. Since they are all domains of minimal supports of m-th powers of words of the same rank, none of these subwords can properly contain any of the others. Thus it makes sense to list them as they appear from left to right. Let Γ_1, Γ_2, etc., denote the domains in the order they appear, and let γ_j be a rank 0 base word in which Γ_j is periodic. The word γ_j may not uniquely defined, but by Lemma 2.9 and Lemma 2.10 it is at least well-defined up to cyclic conjugate. For each Γ_j place one of the copies of γ_j inside a pair of parentheses. The result is a term of rank 1 which we will call the *intermediate step*. Applying the normal form algorithm for F_n to this term results in a new term which will be called the *rank* 1 *structure* for W.

Lemma 11.9. *If W is any word in any Burnside semigroup $\mathcal{B}_n(m)$ with $m \geq 6$, then the rank* 1 *structure of W is well-defined.*

Proof. The only ambiguous point in the definition is where to place the parentheses. In the language of the earlier sections, all of the possibilities differ by rank 1 shifts. Since terms which differ by shifts are equivalent, and since by Theorem 7.2 every equivalence class of terms has a unique normal form, the final result will be uniquely defined, even though the intermediate step is not. □

Lemma 11.10. *Let W be a word in $\mathcal{B}_n(m)$ with $m \geq 6$. If W' is the str_1-reduction of W, then the rank* 1 *structure of W and of W' will be identical.*

Proof. Without loss of generality assume that W' is obtained from W by the application of a single reduction of rank 1. In other words, assume that there is a rank 0 base word U and that W' is obtained from W by replacing a subword which is the minimal support for U^{m+1} with a minimal support for U^m. This operation cannot change the list of domains of minimal supports of m-th powers of rank 0 base words since the one which contains the minimal support of U^m certainly remains and none of the others are affected. More specifically, by Lemma 2.9 and Lemma 2.10 they overlap so little with the domain of $\text{MinSup}(U^m)$ as to have their domains unchanged. Finally, note that by choosing the placement of the parentheses appropriately, the intermediate stage of W and of W' will differ by a single contraction of type 4. That they are equivalent in F_n at this stage guarantees that the final results will be identical. The full strength of the statement is obtained by repeating this procedure. □

Lemma 11.11. *Let $\alpha = (\gamma)\epsilon$ be a rank* 1 *final portion in normal form, let l be the portion length of α, and let β be a rank* 0 *expansion of α. If $j > l$, then every j-th power of a rank* 0 *term which is contained in β will be associated with the rank* 1 *limit term (γ). A similar result holds for initial portions.*

Proof. Let Γ denote the domain of (γ) in $\beta = \gamma^r\epsilon$. If there were a subexpression η^j in $\gamma^r\epsilon$ which was not contained in Γ, then by Lemma 2.9 or Lemma 2.10, the overlap between η^j and γ^r would be at most $|\gamma| + |\eta|$. In particular, the final segment $\gamma\epsilon$ would contain η^{j-1}, but this is not possible given the choice of j. The proof of the second statement is analogous. □

Lemma 11.12. *Let $\alpha = (\gamma)\epsilon(\delta)$ be a rank* 1 *crucial portion in normal form, let l denote the portion length of α, and let β be a rank* 0 *expansion of α. If $j > l$, then every j-th power of a rank* 0 *term which is contained in β will be associated with one of the two limit terms of α.*

Proof. Let Γ denote the domain of (γ) and let Δ denote the domain of (δ) in $\beta = \gamma^r\epsilon\delta^s$. If there were a subexpression η^j in $\gamma^r\epsilon\delta^s$ which was not contained in Γ or Δ, then by Lemma 2.9 or Lemma 2.10, the overlap between η^j and γ^r would be at most $|\gamma| + |\eta|$ and the overlap between η^j and δ^s would be at most $|\delta| + |\eta|$. In particular, the segment $\gamma\epsilon\delta$ would contain η^{j-2}, but this is not possible given the choice of j. □

Lemma 11.13. *Let α be a rank 1 term in normal form, let l be a portion bound of α, and let β be a rank 0 expansion of α. If $j > l$, then every j-th power in β which does not strictly contain the domain of one of the limit terms will be associated with one of the limit terms of α.*

Proof. If there were a subexpression η^j in β which did not strictly contain the domain of one of the limit terms, then η^j would be a subexpression of a rank 0 expansion of the initial portion, of the final portion, or of one of the crucial portions of α. By Lemma 11.11 and Lemma 11.12 the proof is complete. □

Lemma 11.14. *If α is a rank 1 term in normal form, then for sufficiently large m, α is the rank 1 structure for $\phi_m(\alpha)$. Moreover, α will also be the rank 1 structure for the str_1-reduction of $\phi_m(\alpha)$. In other words, for large values of m, α can be reconstructed from its image in $\mathcal{B}_n(m)$.*

Proof. Let β_m be the rank 0 expansion of α formed by replacing each rank 1 limit term with m copies of its contents. Since each β_m has rank 0, it is automatically in normal form when viewed as a rank 0 term in T_n. Let (γ_j) denote the the rank 1 limit term in α which is the j-th one from the left and let Γ_j be the domain of this limit term in β_m. By Lemma 11.4 for each γ_j, a sufficiently large m will guarantee that $\phi_m(\gamma_j)$ is a rank 0 base word in $\mathcal{B}_n(m)$. Since there are only a finite number of such limit terms, there are only a finite number of lower bounds on m. Thus for sufficiently large m, each Γ_j in β_m will contain an m-th power of a rank 0 base.

Let S be a subword of β_m which is a minimal support for U^m where U is a rank 0 base. By Lemma 9.9, S contains U^{m-2}. Let l be the portion bound for α. If M is also chosen so that $M \geq l+2$, then by Lemma 11.13 all of the subwords of this form are associated with one of the limit terms (γ_j). Thus exactly one pair of parentheses needs to be added to β_m for each of the domains Γ_j.

Let α' be the term which results when, starting from α, $m-1$ expansions of type 4 have been applied to each rank 1 limit term (γ_j). Notice that α' is equivalent in F_n to α, and that α' is a possible intermediate step in the creation of the rank 1 structure for $\phi_m(\alpha)$. Lemma 11.9 shows that all possible intermediate steps lead to the same rank 1 structure which is in normal form. Since α' is equivalent to α and α is the unique normal form in this equivalence class, the rank 1 structure for $\phi_m(\alpha)$ must be α. This completes the proof of the first assertion. The second one now follows from Lemma 11.10. □

Lemma 11.15. *If α and β are rank 1 terms in normal form, and for all m, $\phi_m(\alpha)$ and $\phi_m(\beta)$ are equivalent in $\mathcal{B}_n(m)$, then α and β are in fact identical.*

Proof. By Lemma 11.5, for all sufficiently large m, the words $\phi_m(\alpha)$ and $\phi_m(\beta)$ have str-rank 1 in $\mathcal{B}_n(m)$. Similarly, by Lemma 11.14, for all sufficiently large m, α is the rank 1 structure of the str_1-reduction of $\phi_m(\alpha)$ and β is the rank 1 structure of the str_1-reduction of $\phi_m(\beta)$. Let m be chosen large enough to make all four of these facts true. For this m, $\phi_m(\alpha)$ and $\phi_m(\beta)$ have str-rank 1 and thus their str_1-reductions are in fact their normal forms. Moreover, since $\phi_m(\alpha)$ and $\phi_m(\beta)$ are equivalent in $\mathcal{B}_n(m)$, they have the same normal form. Finally, since the rank 1 structure for this common normal form is both α and β, α and β must be identical. □

12. Recovery in rank $i+1$

We will now begin the inductive step in the argument. Most of the lemmas in this section parallel fairly closely those in the previous one. To begin we will assume that we have shown that for every term α of rank i, there is an M such that the element of $\mathcal{B}_n(m)$ represented by $\phi_m(\alpha)$ has rank i for all $m \geq M$ (Lemma 11.5 or Lemma 12.3), and that for every pair of terms α and β of rank at most i, the words $\phi_m(\alpha)$ and $\phi_m(\beta)$ will be equivalent in $\mathcal{B}_n(m)$ for all m if and only if α and β are equivalent in F_n (Lemma 11.7 and Lemma 11.15 or Lemma 12.5 and Lemma 12.16). We will now show that these same assertions are true in rank $i+1$. The first time through, i will have a value of 1 and the only results which will be needed either were established in the previous section or are trivial.

Lemma 12.1. *If α is an idempotent in rank i, then $\phi_m(\alpha)$ is an idempotent in $\mathcal{B}_n(m)$. Conversely, if α is a rank i term in normal form which is not an idempotent in rank i, then for sufficiently large m, $\phi_m(\alpha)$ is not an idempotent in $\mathcal{B}_n(m)$.*

Proof. The first statement is true because $\phi_m : F_n \to \mathcal{B}_n(m)$ is a homomorphism, so consider the converse. By Lemma 12.15 (in the previous rank), for sufficiently large m, α is the rank i structure for $\phi_m(\alpha)$ and the normal form of $\alpha\alpha$ is the rank i structure for $\phi_m(\alpha\alpha)$. Since α is not an idempotent in rank i, α and $\alpha\alpha$ are not equivalent in F_n. One way to see this is that their normal forms have different i-lengths (Lemma 5.11). This shows that for these m, the normal forms of $\phi_m(\alpha)$ and of $\phi_m(\alpha)^2$ cannot be identical. In particular, $\phi_m(\alpha)$ is not an idempotent in $\mathcal{B}_n(m)$. □

Lemma 12.2. *If $\alpha = (\gamma)$ is a rank $i+1$ limit term in normal form, then for sufficiently large m, $\phi_m(\alpha)$ has str-rank $i+1$ in $\mathcal{B}_n(m)$.*

Proof. By Corollary 10.3, for sufficiently large m, the str-rank of $\phi_m(\alpha)$ in $\mathcal{B}_n(m)$ is at most $i+1$. By applying Lemma 12.3 and Lemma 12.4 in the previous rank, we also know that for sufficiently large m, the str-rank and the cir-rank of $S = \phi_m(\gamma)$ are i. The word $\phi_m(\alpha) = S^m$ is an m-th power of a word of cir-rank i. By Lemma 9.20, there are words P, Q and T such that the str$_i$ reduction of S^m will be PT^mQ where the cir-rank of T is i and T is cir-reduced. If T is simple, then the str-reduction of S^m contains an m-th power of a simple, cir-reduced, cir-rank i word. If it represents a proper power in $\mathcal{B}_n(m)$, since T is cir-reduced, by Lemma 9.8, T will be a proper power of a simple word representing a simple element of $\mathcal{B}_n(m)$. In this case, the normal form for S^m contains an even higher power of a simple, cir-reduced, cir-rank i word. In either case, $\phi_m(\alpha)$ has rank at least $i+1$. □

Lemma 12.3. *If α is a rank j term in normal form where $j \leq i+1$, then for sufficiently large m, $\phi_m(\alpha)$ represents an element of $\mathcal{B}_n(m)$ whose str-rank is precisely j. More generally, if β is any term whose normal form is α and α is not an idempotent, then $\phi_m(\beta)$ will also have cir-rank j.*

Proof. If α has rank at most i, then the result has already been shown in previous ranks, so assume that α has rank $i+1$. By Corollary 10.3, for sufficiently large m, the rank of $\phi_m(\alpha)$ is at most $i+1$. On the other hand, since it is a rank $i+1$ term, it contains a rank $i+1$ limit term (γ) in normal form as a subexpression. By Lemma 12.2, for sufficiently large m, the str-rank of $\phi_m((\gamma))$ is exactly $i+1$. Finally, by Lemma 10.4, the str-rank of $\phi_m(\alpha)$ is also at least $i+1$. The final statement merely reflects the fact that ϕ_m induces a homomorphism from F_n to $\mathcal{B}_n(m)$. □

Lemma 12.4. *If α is a rank $i+1$ term in normal form which is not an idempotent in rank $i+1$, then for sufficiently large m the cir-rank of $\phi_m(\alpha)$ in $\mathcal{B}_n(m)$ will be $i+1$. If α is a rank $i+1$ idempotent in normal form then for sufficiently large m, the cir-rank of $\phi_m(\alpha)$ will be i. Moreover, the same is true for all terms β which are equivalent to α in F_n.*

Proof. The first statement follows from applying Lemma 12.3 to α^3. If α is a rank 1 term which is an idempotent then by Lemma 12.3, the str-rank of $\phi_m(\alpha)$ in $\mathcal{B}_n(m)$ will be $i+1$ for all sufficiently large m. Since ϕ_m is a homomorphism, $\phi_m(\alpha)$ is also an idempotent and so by Lemma 9.13, the cir-rank of $\phi(\alpha)$ will be i for these values of m. □

We can now show that terms of different ranks (bounded above by i) can be distinguished by their images under the maps ϕ_m.

Lemma 12.5. *Let α and β be two terms in normal form of rank i and j respectively. If $j < i$ then there exists an m such that $\phi_m(\alpha)$ and $\phi_m(\beta)$ are not equivalent in $\mathcal{B}_n(m)$.*

Proof. By Lemma 12.3, for sufficiently large values of m, $\phi_m(\alpha)$ will represent an element in $\mathcal{B}_n(m)$ of rank i, and $\phi_m(\beta)$ will represent an element in $\mathcal{B}_n(m)$ of rank j. Since they represent elements of different ranks in $\mathcal{B}_n(m)$, they cannot possibly represent the same element. □

Definition 12.6 (Rank $i+1$ Structures)**.** Let W be a str$_i$-reduced word which represents an element of $\mathcal{B}_n(m)$ of str-rank at least $i+1$. By definition, the word W will contain a minimal support for U^m where U is a rank i base. Consider all of the subwords of W which are the domain of a minimal support of this type. Since they are all domains of minimal supports of m-th powers of words of the same rank, none of these subwords can properly contain any of the others. Thus it makes sense to list them in the order they appear from left to right. Let Υ_1, Υ_2, etc., denote the domains as they appear, and let U_j be a rank i base word in which Υ_j is periodic. The word U_j is not uniquely defined, but by Lemma 2.9 and Lemma 2.10 it is at least well-defined up to cyclic conjugate.

Let α be the rank i structure for W. By Lemma 12.8, each Υ_j domain in W corresponds to an $(m-3)$ power of a rank i term in α. Let γ_j be the rank i term of which it is a power and let Γ_j be the maximal γ_j-periodic subexpression of α which contains this γ_j^{m-3}. The domain Γ_j is well-defined, but the expression γ_j is only defined up to cyclic conjugate. For each Γ_j place one of the copies of γ_j inside a pair of parentheses. The result is a term α' of rank $i+1$. The normal form for α' will be called the *rank $i+1$ structure* for W.

A similar procedure can be attempted for str$_i$-reduced words whose rank is less than $i+1$. In that case, no such supports exist and no parentheses will be added to the rank i structure for W. Moreover, since the rank i structure for W is already in normal form, the final result will be the rank i structure for W. Thus, if the str-rank of W is less than $i+1$, the rank $i+1$ structure for W and the rank i structure for W will be identical.

Finally, this definition can be extended to arbitrary words W by defining the rank $i+1$ structure for W to be the rank $i+1$ structure for its str$_i$-reduction.

Lemma 12.7. *If W is any word in any Burnside semigroup $\mathcal{B}_n(m)$ with $m \geq 6$, then the rank $i+1$ structure of W is well-defined.*

Proof. The only ambiguous point in the definition is where to place the parentheses. In the language of the earlier sections, all of the possibilities differ by rank $i+1$ shifts. Since terms which differ by shifts are equivalent, and since by Theorem 7.2 every equivalence class of terms has a unique normal form, the final result will be uniquely defined. □

Lemma 12.8. *If S is an arbitrary word in $\mathcal{B}_n(m)$ of cir-rank i, then there exist terms γ, ϵ and ϵ' such that γ has rank i, ϵ and ϵ' have rank at most i and the rank i structure for S^j, $j \geq 4$, is $\epsilon\gamma^{j-3}\epsilon'$. Moreover, if S^j $(j \geq 4)$ is contained in a word W, then the rank i structure for W will also contain γ^{j-3} as a subexpression.*

Proof. Since S has cir-rank i, S^j will contain a minimal support for U^m where U is a rank $i-1$ base. Moreover, by Lemma 2.9, the length of U^m will be at most $|S|+|U|$. In particular, there will be at least $j-2$ copies of this minimal support in S^j. Consider only the subword T of S^j from the first copy to the last copy. This subword T has a list of domains which form a repeating pattern. In particular, the exact same pattern of domains is repeated $j-3$ times. By Lemma 12.9, in the previous rank, this repetition also occurs in the rank i structure. That is, it has the same sequence of crucial portions repeated $j-3$ times. If the last rank i limit term of this last crucial portion is ignored, the result is $j-3$ copies of the same rank i term γ. This proves the first result. The second statement is immediate once it is realized that by Lemma 9.9 the subword T survives in the minimal support for S^j and thus by Lemma 10.4 it survives in the str_i-reduction of W. □

Lemma 12.9. *If W is a str_i-reduced word of str-rank at least $i+1$, then the portions of the rank $i+1$ structure for W are in one-to-one correspondence with the rank $i+1$ portions of W. Moreover, the former can be constructed, one at a time, from the latter.*

Proof. The first assertion is clear from the definition of the rank $i+1$ structure. It only remains to show that rank $i+1$ portions of W are enough to reconstruct the rank $i+1$ structure of W.

By applying this lemma in the previous rank we know that the rank i portions of the rank i structure of W are in one-to-one correspondence with the rank i portions of W and that they are enough to reconstruct the rank i structure.

Let S be one particular domain which begins with a minimal support for U^m and ends with an incompatible minimal support for V^m. This particular domain of W is enough to reconstruct a section of the rank i structure. Moreover, by Lemma 12.8, the rank i structure of this domain contains $m-3$ copies of some rank i term γ at the beginning and $m-3$ of some rank i term δ at the end. At the intermediate stage, rank $i+1$ parentheses are introduced around a copy of γ at the beginning and a copy of δ at the end. This is true regardless of whether the crucial portions of the rank i structure of S are viewed on their own or as part of the rank i structure for the entire word. Thus the sole crucial portion of the intermediate stage in the construction of the rank $i+1$ structure for S corresponds to a possible crucial portion of the intermediate stage in the construction of the rank $i+1$ structure for W. Given the local nature of the normal form algorithm (Lemma 5.1), the rank $i+1$ crucial portion which results will be the same in both cases. A similar argument holds for the initial and final portions. □

Lemma 12.10. *Let α be a rank $i+1$ term in normal form and let M be a number such that for all $m \geq M$, α is the rank $i+1$ structure for $\phi_m(\alpha)$. If α' is another rank $i+1$ term in normal form all of whose rank $i+1$ portions occur in α, then for all $m \geq M$, α' is the rank $i+1$ structure for $\phi_m(\alpha')$. In other words, the lower bound M only depends on the rank $i+1$ portions which occur in α and not on α itself.*

Proof. This is an immediate consequence of Lemma 12.9. □

Lemma 12.11. *Let W be a word in $\mathcal{B}_n(m)$ with $m \geq 6$. If W' is the str_{i+1}-reduction of W, then the rank $i+1$ structure of W and of W' will be identical.*

Proof. Without loss of generality assume that both W and W' are str_i-reduced and that W' is obtained from W by the application of a single reduction of rank $i+1$. In other words, assume that there is a rank i base word U and that W' is obtained from W by replacing a subword which is the minimal support for U^{m+1} with a minimal support for U^m. This operation cannot change the list of domains of minimal supports of m-th powers of rank i base word since the one which contains the minimal support of U^m certainly remains and none of the others are affected. More specifically, by Lemma 2.9 and Lemma 2.10 they overlap so little with the domain of $\text{MinSup}(U^m)$ as to have their domains unchanged. Next, notice that by Lemma 10.4 and Lemma 12.8, there is a rank i term γ such that W will contain γ^{m-3} and by Lemma 12.9 the rank i structure for W' can be obtained from the rank i structure for W by removing one of the copies of γ. Finally, note that if the parentheses are placed appropriately, the intermediate stage of W and of W' will differ by a single contraction of type 4. That they are equivalent in F_n at this stage guarantees that the final results will be identical. The full strength of the statement is obtained by repeating this procedure. □

Lemma 12.12. *Let α be a rank $i+1$ final portion in normal form. Specifically, let $\alpha = (\gamma)\epsilon$, where ϵ has rank at most i and where (γ) is a rank $i+1$ limit term in preferred cyclic normal form. Moreover, let l be 1 more than the i-length of $\gamma\epsilon$, and let β be a rank i expansion of α. If j is any number greater than l, then every j-th power of a rank i term which is contained in β will be associated with the rank $i+1$ limit term (γ). A similar statement is true for rank $i+1$ initial portions in normal form.*

Proof. Let Γ denote the maximal γ-periodic initial segment of $\beta = \gamma^r\epsilon$. If there were a subexpression η^j in $\gamma^r\epsilon$ which was not contained in Γ, then by Lemma 2.9 or Lemma 2.10, the overlap between η^j and γ^r would be at most $|\gamma| + |\eta|$. In particular, the final segment $\gamma\epsilon$ would contain η^{j-1}, but this is not possible given the choice of j. The proof of the second statement is analogous. □

Lemma 12.13. *Let $\alpha = (\gamma)\epsilon(\delta)$ be a rank $i+1$ crucial portion in normal form, let l denote the portion length of α, and let β be a rank i expansion of α. If $j > l$, then every j-th power of a rank i term which is contained in β will be associated with one of the two limit terms of α.*

Proof. Let Γ denote the domain of (γ) and let Δ denote the domain of (δ) in $\beta = \gamma^r\epsilon\delta^s$. If there were a subexpression η^j in $\gamma^r\epsilon\delta^s$ which was not contained in Γ or Δ, then by Lemma 2.9 or Lemma 2.10, the overlap between η^j and γ^r would be at most $|\gamma| + |\eta|$ and the overlap between η^j and δ^s would be at most $|\delta| + |\eta|$. In particular, the segment $\gamma\epsilon\delta$ would contain η^{j-2}, but this is not possible given the choice of j. □

Lemma 12.14. *Let α be a rank $i+1$ term in normal form, let l be a portion bound of α, and let β be a rank i expansion of α. If $j > l$, then every j-th power in β which does not strictly contain the domain of one of the limit terms will be associated with one of the limit terms of α.*

Proof. If there were a subexpression η^j in β which did not strictly contain the domain of one of the limit terms, then η^j would be a subexpression of a rank i expansion of the initial portion, of the final portion, or of one of the crucial portions of α. By Lemma 12.12 and Lemma 12.13 the proof is complete. □

Lemma 12.15. *If α is a rank $i+1$ term in normal form, then for sufficiently large m, α is the rank $i+1$ structure for $\phi_m(\alpha)$. Moreover, α will also be the rank $i+1$ structure for the str_{i+1}-reduction of $\phi_m(\alpha)$. In other words, for large values of m, α can be reconstructed from its image in $\mathcal{B}_n(m)$.*

Proof. Let β_m be the rank i expansion of α formed by replacing each rank $i+1$ limit term with m copies of its contents. By Lemma 10.7, β_m is in normal form for all $m \geq 1$. Next, by Lemma 12.3 (in the previous rank) there is a number M such that for $m \geq M$, β_2 is the rank i structure for $\phi_m(\beta_2)$. Since β_2 and β_m have the same rank i portions for all $m \geq 2$, it follows from Lemma 12.10 that for all $m \geq M$, β_m is the rank i structure for $\phi_m(\beta_m)$. Moreover, since $\phi_m(\beta_m)$ and $\phi_m(\alpha)$ are identical words, β_m is, in fact, the rank i structure for $\phi_m(\alpha)$.

Let (γ_j) denote the the rank $i+1$ limit term in α which is the j-th one from the left and let $U_j = \phi_m(\gamma_j)$. By Lemma 12.2 and Lemma 10.4, the str-reduction of $\phi_m(\alpha)$ will contain a minimal support for V_j^m where V_j is the cir-reduction of U_j for each j. Thus each of the images of the rank $i+1$ limit terms will produce a pair of parentheses in the rank $i+1$ structure. Let Γ_j be the domain of (γ_j) in β_m. By Lemma 12.8, Γ_j will contain $m-3$ copies of γ_j.

If the normal form for $\phi_m(\alpha)$ contained a minimal support for any other m-th powers of rank i base words, then by Lemma 12.8, β_m would contain a subexpression of the form η^{m-3} where η was a rank i term, and η^{m_3} would not strictly contain any of the Γ_j. For sufficiently large m, this is impossible by Lemma 12.14. Thus exactly one pair of parentheses needs to be added to β_m for each of the domains Γ_j.

Finally, let α' be the term which results when, starting from α, $m-1$ expansions of type 4 have been applied to each rank 1 limit term (γ_j). Notice that α' is equivalent in F_n to α, and that α' is a possible intermediate step in the creation of the rank $i+1$ structure for $\phi_m(\alpha)$ from β_m. Lemma 12.7 shows that all possible intermediate steps lead to the same rank $i+1$ structure in normal form. Since α' is equivalent to α and α is the unique normal form in this equivalence class, the rank $i+1$ structure for $\phi_m(\alpha)$ must be α. This completes the proof of the first assertion. The second one now follows from Lemma 12.11. □

Lemma 12.16. *If α and β are rank $i+1$ terms in normal form, and for all m, $\phi_m(\alpha)$ and $\phi_m(\beta)$ are equivalent in $\mathcal{B}_n(m)$, then α and β are in fact identical.*

Proof. By Lemma 12.3, for all sufficiently large m, the words $\phi_m(\alpha)$ and $\phi_m(\beta)$ have str-rank $i+1$ in $\mathcal{B}_n(m)$. Similarly, by Lemma 12.15, for all sufficiently large m, α is the rank $i+1$ structure of the str_{i+1}-reduction of $\phi_m(\alpha)$ and β is the rank

$i+1$ structure of the str_{i+1}-reduction of $\phi_m(\beta)$. Let m be chosen large enough to make all four of these facts true. For this m, $\phi_m(\alpha)$ and $\phi_m(\beta)$ have str-rank $i+1$ and thus their str_{i+1}-reductions are in fact their normal forms. Moreover, since $\phi_m(\alpha)$ and $\phi_m(\beta)$ are equivalent in $\mathcal{B}_n(m)$, they have the same normal form. Finally since the rank $i+1$ structure for this common normal form is both α and β, α and β must be identical. □

This completes the induction.

13. Main Theorems about $\Omega_n^\kappa \mathbf{A}$

Now that the induction is complete, we can quickly complete the proofs of the main theorems.

Theorem 13.1. *If α and β are the normal forms for distinct elements of F_n, then there is an integer m such that $\phi_m(\alpha)$ and $\phi_m(\beta)$ are distinct elements of $\mathcal{B}_n(m)$.*

Proof. Let i and j be the ranks of α and β, respectively, and assume without loss of generality that $i \geq j$. If $i > j$ then the result follows from Lemma 12.5. If $i = j$ it follows from Lemma 12.16. □

Theorem 13.2. *The map ψ from the κ-semigroup F_n to the κ-semigroup $\Omega_n^\kappa \mathbf{A}$ is an isomorphism.*

Proof. The map ψ is clearly onto, so we only need to show that it is one-to-one. Suppose, on the contrary, that the map ψ is not injective. Then there would be two distinct elements of F_n represented by their normal forms α and β which are identified under the map ψ. By the definition of $\Omega_n^\kappa \mathbf{A}$, this would imply that α and β are identified under all κ-homomorphisms into finite aperiodic semigroups. On the other hand, by Theorem 13.1, there is an m such that $\phi_m(\alpha)$ and $\phi_m(\beta)$ are distinct elements of $\mathcal{B}_n(m)$. Since by Theorem 8.11, $\mathcal{B}_n(m)$ is finite $\mathcal{J}$-above, there is a finite quotient of $\mathcal{B}_n(m)$ in which the images of α and β remain distinct. This contradiction shows that ψ is injective and that F_n and $\Omega_n^\kappa \mathbf{A}$ are in fact identical. □

Corollary 13.3. *The n-generated κ-semigroup $\Omega_n^\kappa \mathbf{A}$ can be defined by a finite list of pseudoidentities and has a decidable term problem. Consequently the pseudovariety $\mathbf{A}$ is κ-recursive.*

Proof. This follows immediately from Theorem 13.2, the definition of F_n, and Theorem 7.2. □

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