Möbius inversion and combinatorial curvature

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Outline

I. Two Theorems about Curvature

II. Angle Sums in Polytopes

III. A General Theorem
Normalized Angles

Let $F$ be a face of a polytope $P$.

- The normalized *internal angle* $\alpha(F, P)$ is the proportion of unit vectors perpendicular to $F$ which point into $P$ (i.e. the measure of this set of vectors divided by the measure of the sphere of the appropriate dimension).

- The normalized *external angle* $\beta(F, P)$ is the proportion of unit vectors perpendicular to $F$ so that there is a hyperplane with this unit normal which contains $F$ and the rest of $P$ is on the other side.

**Thm:** $\sum_{v \in P} \beta(P, v) = 1$. 
Curvature in PE complexes

Following Cheeger-Müller-Schrader (and Charney-Davis), we can think of the curvature of a piecewise Euclidean cell complex $X$ as concentrated at its vertices.

$$\chi(X) = \sum (-1)^{\text{dim} P}$$

$$= \sum \sum (-1)^{\text{dim} P} \beta(v, P)$$

$$= \sum \sum (-1)^{\text{dim} P} \beta(v, P)$$

$$= \sum \kappa(v)$$

where $\kappa(v) := \sum (-1)^{\text{dim} P} \beta(v, P)$.

Rem: This equation led to the Charney-Davis conjecture.
An Example

If $X$ is the boundary of a dodecahedron, then

$$
\kappa(v) = \beta(v, v) - \sum_{e \ni v} \beta(v, e) + \sum_{f \ni v} \beta(v, f)
$$

$$
= 1 - 5 \left( \frac{1}{2} \right) + 5 \left( \frac{1}{3} \right) = \frac{1}{6}
$$

Since $\chi(X) = \sum_v \kappa(v)$ there must be 12 vertices

$(2 = V/6)$. 
Combinatorial Gauss-Bonnet

An angled 2-complex is one where we arbitrarily assign normalized external angles $\beta(v, f)$ for each vertex-face pair.

Define $\kappa(v)$ as above. Define $\kappa(f)$ as a correction term which measures how far the external vertex angles are from 1.

$$\kappa(f) = 1 - \sum_{v \in f} \beta(v, f)$$

Thm(Gersten,Ballmann-Buyalo,M-Wise)
If $X$ is an angled 2-complex, then

$$\sum_{v} \kappa(v) + \sum_{f} \kappa(f) = \chi(X)$$

Rem: In all these papers the sum was $2\pi \chi(X)$ since the angles were not normalized. As we shall see normalization is crucial for the equations in higher dimensions.
Angle Sums

The sum of the internal angles in a triangle is $\pi$, but the sum of the dihedral angles in a tetrahedron can vary. The relations between the various internal and external angles in a Euclidean polytope are best described via incidence algebras.
Posets and Incidence Algebras

Let $P$ be a finite poset on $[n]$ numbered according to some linearization of $P$, and let $I(P)$ be its incidence algebra.

**Rem:** The elements of $I(P)$ can also be thought of as functions from $P \times P \to \mathbb{R}$.

The identity matrix is the *delta function* where $\delta(x, y) = 1$ iff $x = y$.

The *zeta function* is the function $\zeta(x, y) = 1$ if $x \leq_P y$ and 0 otherwise (i.e. 1’s wherever possible).

The *möbius function* is the matrix inverse of $\zeta$. Note that $\mu \zeta = \zeta \mu = \delta$. 
Incidence Algebras for Polytopes

The faces of a Euclidean polytope under inclusion (including the empty face) is its face lattice.

The set of all normalized internal (external) angles of a polytope $P$ forms a single element $\alpha$ ($\beta$) of the incidence algebra of its face lattice—once we extend these notions so that $\alpha(\emptyset, F)$ and $\beta(\emptyset, F)$ have well-defined values. One possibility is

$$\alpha(\emptyset, F) = \begin{cases} 1 & \text{if } \dim F \leq 0 \\ 0 & \text{if } \dim F > 0 \end{cases}$$

$$\beta(\emptyset, F) = \begin{cases} 1 & \text{if } \dim F < 0 \\ 0 & \text{if } \dim F \geq 0 \end{cases}$$
Equations for Angles

The most interesting of angle identity is the one discovered by Peter McMullen.

**Thm(McMullen)** $\alpha \beta = \zeta$, i.e.

$$
\sum_{F \leq G \leq H} \alpha(F, G) \beta(G, H) = \zeta(F, H)
$$

**Proof Idea:**
- Look at (a polytopal cone) $\times$ (its dual cone)
- Integrate $f(\vec{x}) = \exp(-||\vec{x}||^2)$ over this $\mathbb{R}^{2n}$ in two different ways.
Möbius Functions for Polytopes

Because the value of the möbius function is the reduced Euler characteristic of the geometric realization of interior of the interval, we have:

**Lem:** The möbius function of the face lattice of a polytope is \( \mu(F, G) = (-1)^{\dim G - \dim F} \).

**Proof:** The geometric realization of the portion of the face lattice between \( F \) and \( G \) is a sphere.

**Def:** Let \( \bar{\alpha}(F, G) = \mu(F, G)\alpha(F, G) \), [Hadamard product] (i.e. \( \bar{\alpha} \) is a signed normalized internal angle.

**Thm(Sommerville)** \( \mu\alpha = \bar{\alpha} \) i.e.

\[
\sum_{F \leq G \leq H} \mu(F, G)\alpha(G, H) = \mu(F, H)\alpha(F, H)
\]

**Cor:** \( \bar{\alpha}\beta = \mu\alpha\beta = \mu\zeta = \delta \).
Combinatorial Gauss-Bonnet Revisited

**General CGB Thm** Every factorization $\alpha \beta = \zeta$, gives rise to a Gauss-Bonnet type formula.

In particular, $\tilde{\chi}(X)$ is

$$= \sum_{P \geq \emptyset} (-1)^{\dim P} \sum_{P \geq \emptyset} (-1)^{\dim P} \zeta(\emptyset, P)$$

$$= \sum_{P \geq \emptyset} (-1)^{\dim P} \left( \sum_{Q \in [\emptyset, P]} \alpha(\emptyset, Q)\beta(Q, P) \right)$$

$$= \sum_{Q \geq \emptyset} (-1)^{\dim Q} \alpha(\emptyset, Q) \left( \sum_{P \geq Q} \bar{\beta}(Q, P) \right)$$

$$= \sum_{Q \geq \emptyset} (-1)^{\dim Q} \alpha(\emptyset, Q) \kappa^\uparrow(Q)$$

where $\kappa^\uparrow(Q)$ is defined as the obvious signed sum implicit in the final equality.

**Rem 1:** Factorizations with lots of 0s are best.

**Rem 2:** Both earlier theorems are special cases.