Constructing non-positively curved spaces and groups

Day 1: The basics

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Outline

I. CAT($\kappa$) and $\delta$-hyperbolic

II. Curvature conjecture

III. Decidability issues

IV. Length spectrum
I. CAT(0) spaces

**Def:** A geodesic metric space $X$ is called (globally) CAT(0) if

- $\forall$ points $x, y, z \in X$
- $\forall$ geodesics connecting $x, y, z$
- $\forall$ points $p$ in the geodesic connecting $x$ to $y$

\[ d(p, z) \leq d(p', z') \]

in the corresponding configuration in $\mathbb{E}^2$.

**Rem:** CAT(1) and CAT(−1) are defined similarly using $\mathbb{S}^2$ and $\mathbb{H}^2$ respectively - with restrictions on $x, y, z$ in the spherical case, since not all spherical comparison triangles are constructible.
\( \delta \)-hyperbolic spaces

**Def:** A geodesic metric space \( X \) is \( \delta \)-hyperbolic if

- \( \forall \) points \( x, y, z \in X \)
- \( \forall \) geodesics connecting \( x, y, \) and \( z \)
- \( \forall \) points \( p \) in the geodesic connecting \( x \) to \( y \)
- the distance from \( p \) to the union of the other two geodesics is at most \( \delta \).

**Rem:** Hyperbolic \( n \)-space, \( \mathbb{H}^n \) is both \( \delta \)-hyperbolic and \( \text{CAT}(-1) \).
Local curvature

$\delta$-hyperbolic only implies the large scale curvature is negative. We get no information about local structure.

CAT(0) and CAT($-1$) imply good local curvature conditions.

**Lem:** $X$ is CAT(0) [CAT($-1$)] $\iff$ $X$ is locally CAT(0) [CAT($-1$)] and $\pi_1X = 1$ (needs completeness)

**Def:** A locally CAT(0) [CAT($-1$)] space is called *non-positively [negatively] curved*. 
CAT(−1) vs. CAT(0) vs. \( \delta \)-hyperbolic

**Thm:** \( \text{CAT}(\kappa) \Rightarrow \text{CAT}(\kappa') \) when \( \kappa \leq \kappa' \). In particular, \( \text{CAT}(-1) \Rightarrow \text{CAT}(0) \).

**Def:** A flat is an isometric embedding of a Euclidean space \( \mathbb{E}^n, n > 1 \).

**Thm:** \( \text{CAT}(-1) \Rightarrow \text{CAT}(0) \meetsplus \text{no flats} \)

**Thm:** \( \text{CAT}(-1) \Rightarrow \delta \)-hyperbolic

In fact, when \( X \) is \( \text{CAT}(0) \) and has a proper, cocompact group action by isometries, \( X \) is \( \delta \)-hyperbolic \( \iff \) \( X \) has no flats. (Flat Plane Thm)
**CAT(0) groups and hyperbolic groups**

**Def:** A group $G$ is *hyperbolic* if for some $\delta$ it acts properly and cocompactly by isometries on some $\delta$-hyperbolic space.

**Lem:** $G$ is is hyperbolic if for some finite generating set $A$ and for some $\delta$, its Cayley graph w.r.t. $A$ is $\delta$-hyperbolic.

**Def:** A group $G$ is **CAT(0)** if it acts properly and cocompactly by isometries on some CAT(0) space.

**Rem:** Unlike hyperbolicity, showing a group is CAT(0) requires the construction of a CAT(0) space.
CAT(−1) vs. CAT(0) vs. word-hyperbolic

\[
\begin{align*}
\text{CAT}(−1) & \\
\Downarrow & \\
\text{CAT}(0) + \text{no flats} & \implies \text{word-hyperbolic} \\
\Downarrow & \\
\text{CAT}(0) + \text{no } \mathbb{Z} \times \mathbb{Z} & 
\end{align*}
\]

Flat Torus Thm: \( \mathbb{Z} \times \mathbb{Z} \) in \( G \) \( \Rightarrow \) \( \exists \) a flat in \( X \).

Problem: Flat in \( X \) \( \Leftrightarrow \) \( \mathbb{Z} \times \mathbb{Z} \) in \( G \)?

Thm(Wise) \( \exists \) aperiodic flats in CAT(0) spaces which are not limits of periodic flats.

Rem: This is not even known for VH CAT(0) squared complexes.
Constant curvature complexes

Constant curvature models: $\mathbb{S}^n$, $\mathbb{E}^n$, and $\mathbb{H}^n$.

**Def:** A *piecewise spherical / euclidean / hyperbolic complex* $X$ is a polyhedral complex in which each polytope is given a metric with constant curvature $1$ / $0$ / $-1$ and the induced metrics agree on overlaps. In the spherical case, the cells must be convex polyhedral cells in $\mathbb{S}^n$. The generic term is $M_\kappa$-complex, where $\kappa$ is the curvature.

**Thm(Bridson)** Compact $M_\kappa$ complexes are geodesic metric spaces.

**Exercise:** What restrictions on edge lengths are necessary in order for a PS/PE/PH $n$-simplex to be buildable?
II. Curvature conjecture

\[
\begin{align*}
\text{PH CAT}(-1) & \Rightarrow \text{CAT}(-1) \\
(?)(?) & \downarrow \\
\text{PE CAT}(0) & \Rightarrow \text{CAT}(0) \\
\text{no flats} & \downarrow \\
\text{PE CAT}(0) & \Rightarrow \text{CAT}(0) \\
\text{no } \mathbb{Z} \times \mathbb{Z} & \Rightarrow \text{no } \mathbb{Z} \times \mathbb{Z}
\end{align*}
\]

**Conj:** These seven classes of groups are equal.

**Rem 1:** Analogue of Thurston’s hyperbolization conjecture.

**Rem 2:** If Geometrization (Perleman) holds then this is true for 3-manifold groups.
PH CAT(−1) vs. PE CAT(0)

**Thm(Charney-Davis-Moussong)** If $M$ is a compact hyperbolic $n$-manifold, then $M$ also carries a PE CAT(0) structure.

**Rem:** This is open even for compact (variably) negatively-curved $n$-manifolds.

**Thm(N. Brady-Crisp)** There is a group which acts nicely on a 3-dim PH CAT(−1) structure, and on a 2-dim PE CAT(0) structure, but not on any 2-dim PH CAT(−1) structure.

**Moral:** Higher dimensions are sometimes necessary to flatten things out.
Rips Complex

If our goal is to create complexes with good local curvature for an arbitrary word-hyperbolic group, the obvious candidate is the Rips complex (or some variant).

**Def:** Let $P_d(G, A)$ be the flag complex on the graph whose vertices are labeled by $G$ and which has an edge connecting $g$ and $h$ iff $gh^{-1}$ is represented by a word of length at most $d$ over the alphabet $A$.

**Thm:** If $G$ is word-hyperbolic and $d$ is large relative to $\delta$, the complex $P_d(G, A)$ is contractible (and finite dimensional).
Adding a metric to the Rips complex

Let $G$ be a word-hyperbolic group.

Q: Suppose we carefully pick a generating set $A$ and pick a $d$ very large and declare each simplex in $P_d(G, A)$ to be a regular Euclidean simplex with edge length 1. Is the result a CAT(0) space?

Exercise: Is this true when $G$ is free and $A$ is a basis?
Adding a metric to the Rips complex

Let $G$ be a word-hyperbolic group.

**Q:** Suppose we carefully pick a generating set $A$ and pick a $d$ very large and declare each simplex in $P_d(G, A)$ to be a regular Euclidean simplex with edge length 1. Is the result a CAT(0) space?

**Exercise:** Is this true when $G$ is free and $A$ is a basis?

**A:** No one knows!

**Moral:** Our ability to test whether compact constant curvature metric space is CAT(0) or CAT($-1$) is very primitive.
III. Decidability

**Thm(Elder-M)** Given a compact $M_{\kappa}$-complex, there is an “algorithm” which decides whether it is locally $\text{CAT}(\kappa)$.

**Proof sketch:**
- reduce to galleries in PS complexes
- convert to real semi-algebraic sets
- apply Tarski’s “algorithm”
Galleries

A 2-complex, a linear gallery, its interior and its boundary.
Reduction to geodesics in PS complexes

**Rem:** The link of a point in an $M_\kappa$-complex is an PS complex.

**Thm:** An $M_\kappa$-complex is locally $\text{CAT}(\kappa)$
$\iff$ the link of each vertex is globally $\text{CAT}(1)$
$\iff$ the link of each cell is an PS complex which contains no closed geodesic loop of length less than $2\pi$.

**Moral:** Showing that PE complexes are non-positively curved or PH complexes are negatively curved hinges on showing that PS complexes have no short geodesic loops.
Geodesics

**Def:** A *local geodesic* in a $M_k$-complex is a concatenation of paths such that
1) each path is a geodesic in a simplex, and
2) at the transitions, the “angles are large” meaning that the distance between the “in” direction and the “out” direction is at least $\pi$ in the link.

**Rem:** Notice that there is an induction involved in the check for short geodesics. To test whether a particular curve is a short geodesic, you need to check whether it is short and whether it is a geodesic, but the latter involves checking geodesic distances in a lower dimensional PS complex, but this involves checking geodesic distances in a lower dimensional PS complex...
Unshrinkable geodesics

In practice, we will often restrict our search to unshrinkable geodesics.

**Def:** A geodesic is *unshrinkable* if there does not exist a non-increasing homotopy through rectifiable curves to a curve of strictly shorter length.

**Thm(Bowditch)** It is sufficient to search for unshrinkable geodesics.

**Cor:** In a PS complex it is sufficient to search for a geodesic which can neither be shrunk nor homotoped till it meets the boundary of its gallery without increasing length.
Converting to Polynomial Equations, I

Spaces and maps:

\[
\begin{array}{c}
\{x_i\} \rightarrow S^1 \subset \mathbb{R}^2 \\
\downarrow \\
K \leftarrow G \rightarrow S^n \subset \mathbb{R}^{n+1}
\end{array}
\]

For each 0-cell \( v \) in \( G \)
- create a vector \( \vec{u}_v \) in \( \mathbb{R}^{n+1} \)

For each \( x_i \)
- create a vector \( \vec{y}_i \) in \( \mathbb{R}^{n+1} \)
- a vector \( \vec{z}_i \) in \( \mathbb{R}^2 \).

Add equations which stipulate
- they are unit vectors,
- the edge lengths are right,
- \( \vec{y}_i \) is a positive linear comb. of certain \( \vec{u}_v \),
- the \( \vec{z}_i \) march counterclockwise around \( S^1 \) starting at \((1,0)\).
Converting to Polynomial Equations, II

A 1-complex, a gallery and its model space.
Real semi-algebraic sets

**Def:** A *real semi-algebraic set* is a boolean combination (∪, ∩ and complement) of real algebraic varieties.

Inducting through dimensions, it is possible to show that there is a real semi-algebraic set in which the points are in one-to-one correspondence with the closed geodesics in the circular gallery $G$.

**Punchline:** Tarski’s theorem about the decidability of the reals implies that there is an algorithm which decides whether a real semi-algebraic set is empty or not.

**Rem:** It is still not known whether there is an algorithm to decide whether a particular complex supports a CAT(0) metric.
Why is this so hard?

Problems with high codimension ($\geq 2$) can often be quite hard.

Q: What is the unit volume 3-polytope with the smallest 1-skeleton (measured by adding up the edge lengths)?
Why is this so hard?

Problems with high codimension ($\geq 2$) can often be quite hard.

Q: What is the unit volume 3-polytope with the smallest 1-skeleton (measured by adding up the edge lengths)?

A: No one knows, but the best guess is a triangular prism.
IV. Length spectra

**Def:** The lengths of open geodesics from $x$ to $y$ is the *length spectrum from $x$ to $y*."

**Thm (Bridson-Haefliger)** The length spectrum from $x$ to $y$ in a compact $M_\kappa$-complex is discrete.

**Def:** The lengths of closed geodesics in a space is simply called its *length spectrum*.

**Thm (N. Brady-M)** The length spectrum of a compact $M_\kappa$-complex is discrete.

**Proof sketch:**
- Suppose not and reduce to a single gallery.
- Closed geodesics are critical points of $d$.
- $d$ is real analytic on a compact set containing the tail of the sequence.
- $d$ extends to real analytic function on a larger open set.
- $\therefore$ only finitely many critical values.
Totally geodesic surfaces

**Def:** A surface $f : D \to X$ is *totally geodesic* if $\forall d \in D$, $\text{Lk}(d)$ is sent to a local geodesic in $\text{Lk}(f(d))$.

**Cor:** If $D$ is a totally geodesic surface in a NPC PE complex then the points in the interior of $D$ with negative curvature have curvatures bounded away from 0.

**Rem:** In a 2-dimensional NPC PE complex, every null-homotopic curve bounds a totally geodesic surface. This fails in dimension 3 and higher, and is one of the key reasons why theorems in dimension 2 fail to generalize easily to higher dimensions.