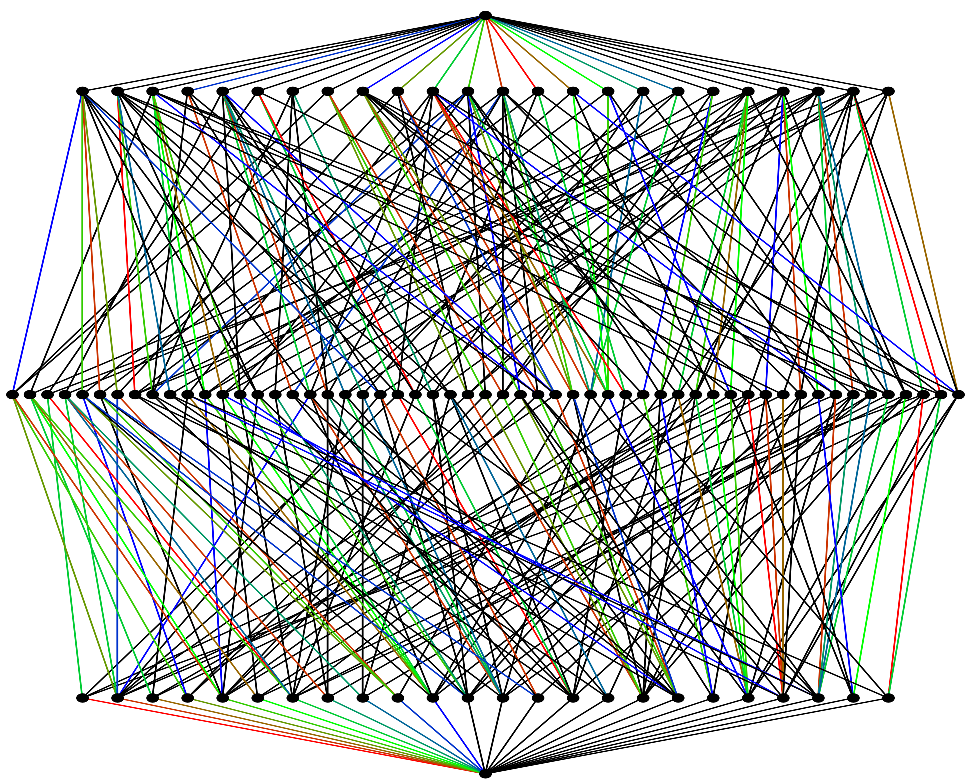
Constructing non-positively curved spaces and groups

Day 3: Artin groups and small-cancellation groups



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Outline

I. CAT(0) and Artin groups

II. CAT(0) and small cancellation groups

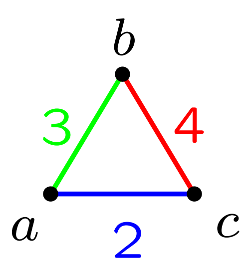
III. CAT(0) and ample twisted face pairings

I. Coxeter and Artin groups

Let Γ be a finite graph with edges labeled by integers greater than 1, and let $\langle a,b\rangle^n$ be the length n prefix of $(ab)^n$.

Def: The *Artin group* A_Γ is generated by its vertices with a relation $\langle a,b\rangle^n = \langle b,a\rangle^n$ whenever a and b are joined by an edge labeled n.

Def: The *Coxeter group* W_Γ is the Artin group A_Γ modulo the relations $a^2 = 1$ $\forall a \in$ **Vert**(Γ).

Graph 

Artin presentation

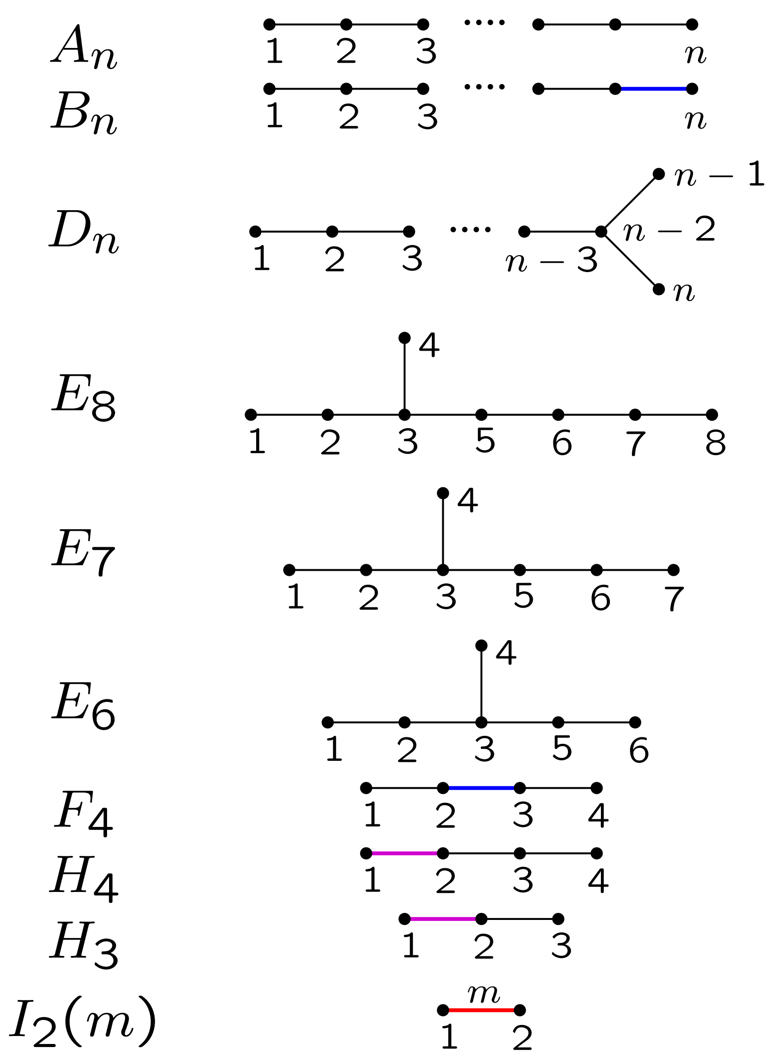
$\langle a,b,c |\ aba = bab, ac = ca, bcbc = cbcb\rangle$

Coxeter presentation

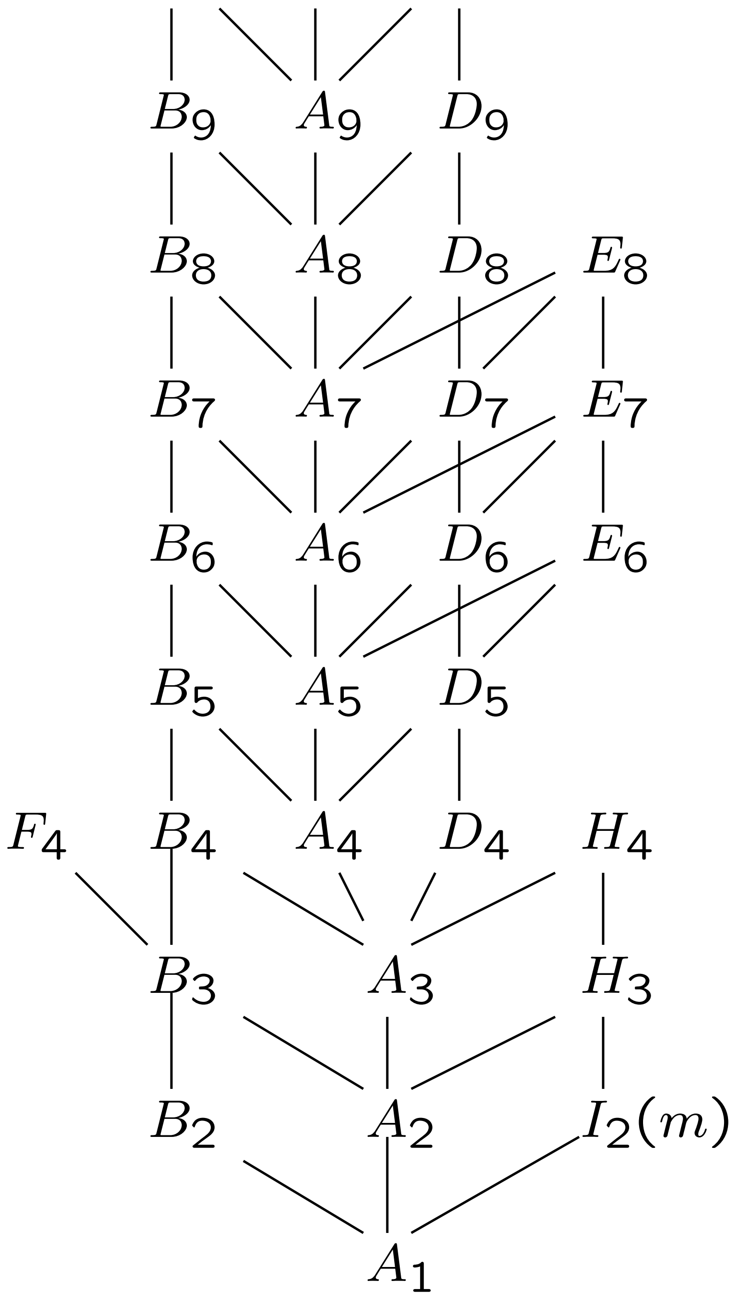
$$\left\langle a,b,c \middle|\ \begin{array}{c} aba = bab, ac = ca, bcbc = cbcb \\ a^2 = b^2 = c^2 = 1 \end{array} \right\rangle$$

Finite-type Artin groups

The finite Coxeter groups have been classified. An Artin group defined by the same labeled graph as a finite Coxeter is called a *finite-type Artin*. (other convention used below)



Irreducible Dynkin diagrams



Eilenberg-MacLane spaces for Artin groups

Finite-type Artin groups are fundamental groups of complexified Coxeter hyperplane arrangements quotiented by the action of the Coxeter group.

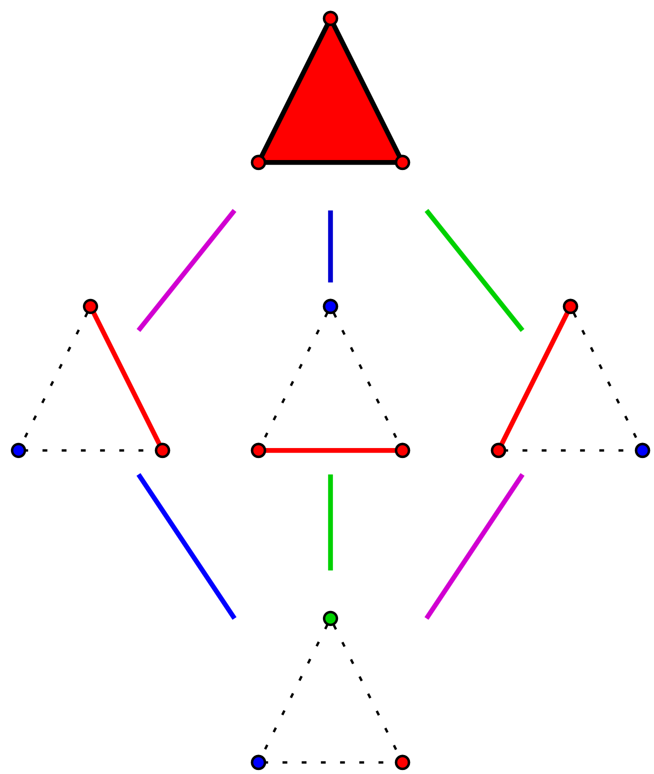
Each finite type Artin group has a
• finite dimensional CAT(0) K(G,1)
(but not complete or compact)
• finite dimensional compact K(G,1)
(with no metric)
but no known
• finite dimensional compact CAT(0) K(G,1)

Thus they do not yet qualify as CAT(0) groups, but they are good candidates.

Brady-Krammer Complexes

In 1998 Tom Brady and Daan Krammer independently discovered new complexes on which the braid groups and the other Artin groups of finite type act.

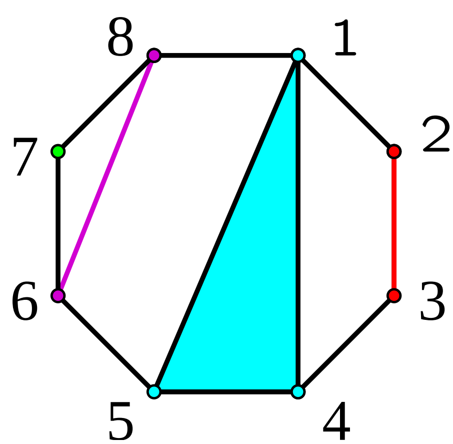
In the case of the braid groups, there is a close connection with a well-known combinatorial object known as the noncrossing partition lattice.



Noncrossing Partitions

A *noncrossing partition* is a partition of the vertices of a regular n-gon so that the convex hulls of the partitions are disjoint.

One noncrossing partition σ is contained in another τ if each block of σ is contained in a block of τ.



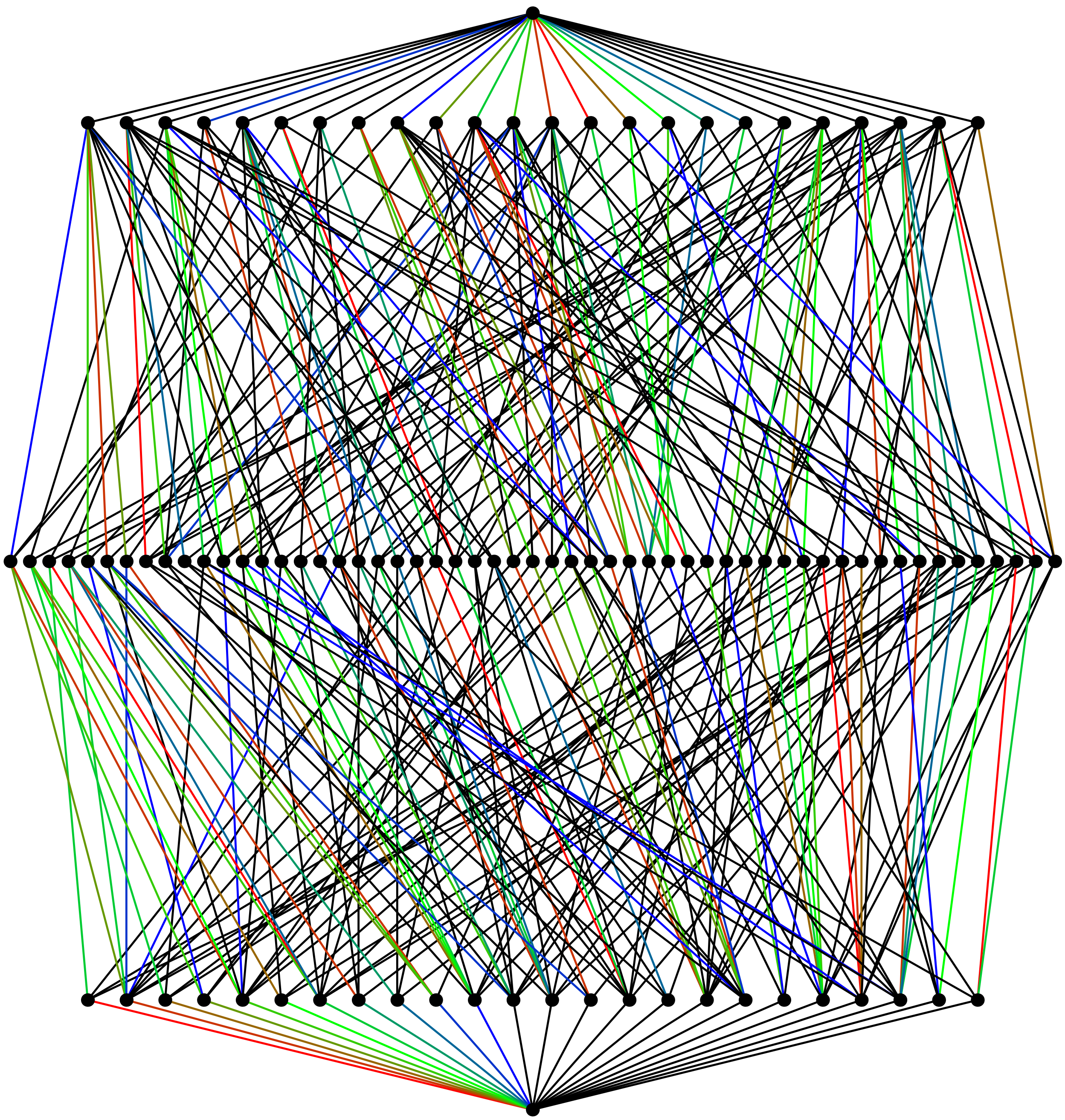
$\{\{1, 4, 5\}, \{2, 3\}, \{6, 8\}, \{7\}\}$

Factors of the Coxeter element

A_3	1-6-6-1
B_3	1-9-9-1
H_3	1-15-15-1
A_4	1-10-20-10-1
B_4	1-12-24-12-1
D_4	1-16-36-16-1
F_4	1-24-55-24-1
H_4	1-60-158-60-1
A_5	1-15-50-50-15-1
B_5	1-20-70-70-20-1
D_5	1-25-100-100-25-1

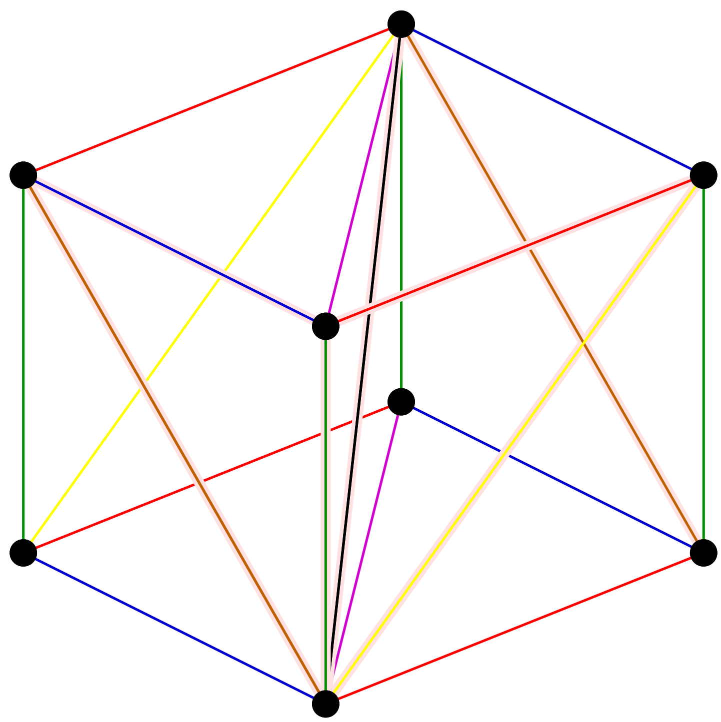
General formulas exist for the A_n, B_n and D_n types as well as explicit calculations for the exceptional ones, but no general formula explains all of these numbers in a coherent framework.

F_4 Poset



Natural metric

The metric: The metric which views the edges in a maximal chain as mutually orthogonal steps in a Euclidean space is natural in the sense that it turns Boolean lattices into Euclidean cubes. Also, the link of the long diagonal in a Boolean lattice is a Coxeter complex for the symmetric group.



CAT(0) and Artin groups

Thm(T.Brady-M) The finite-type Artin groups with at most 3 generators are CAT(0)-groups and the Artin groups A_4 and B_4 are CAT(0) groups.

Proof: The link of a vertex in the cross section is the order complex of a fairly small poset. It is then relatively easy to check that using the "natural" metric, each of these links satisfy the link condition.

Natural Conj: The Brady-Krammer complex is CAT(0) for all Artin groups of finite type.

CAT(0) metrics on D_4 and F_4

Thm(Choi): The Brady-Krammer complexes for D_4 and F_4 do not support reasonable PE CAT(0) metrics.

Reasonable means that symmetries of the group should lead to symmetries in the metric.

Proof Idea: First determine what Euclidean metrics on the 3-dimensional cross-section complex have dihedral angles which make the edge links (which are finite graphs) large.

Then check these metrics in the vertex links (which are 2-dimensional PS complexes).

The software

The program `coxeter.g` is a set of GAP routines used to examine Brady-Kramer complexes. Initially developed to test the curvature of the Brady-Krammer complexes using the "natural" metric, the routines were extensively modified by Woonjung Choi so that they

- find the 3-dimensional simplicial structure of the cross-section
- find representive vertex and edge links (up to automorphism)
- find the graphs for the edge links
- find the simple cycles in these graphs
- find the linear system of inequalities which need to be satisfied by the dihedral angles of the tetrahedra.

(do a demonstration)

Dihedral angle rigidity

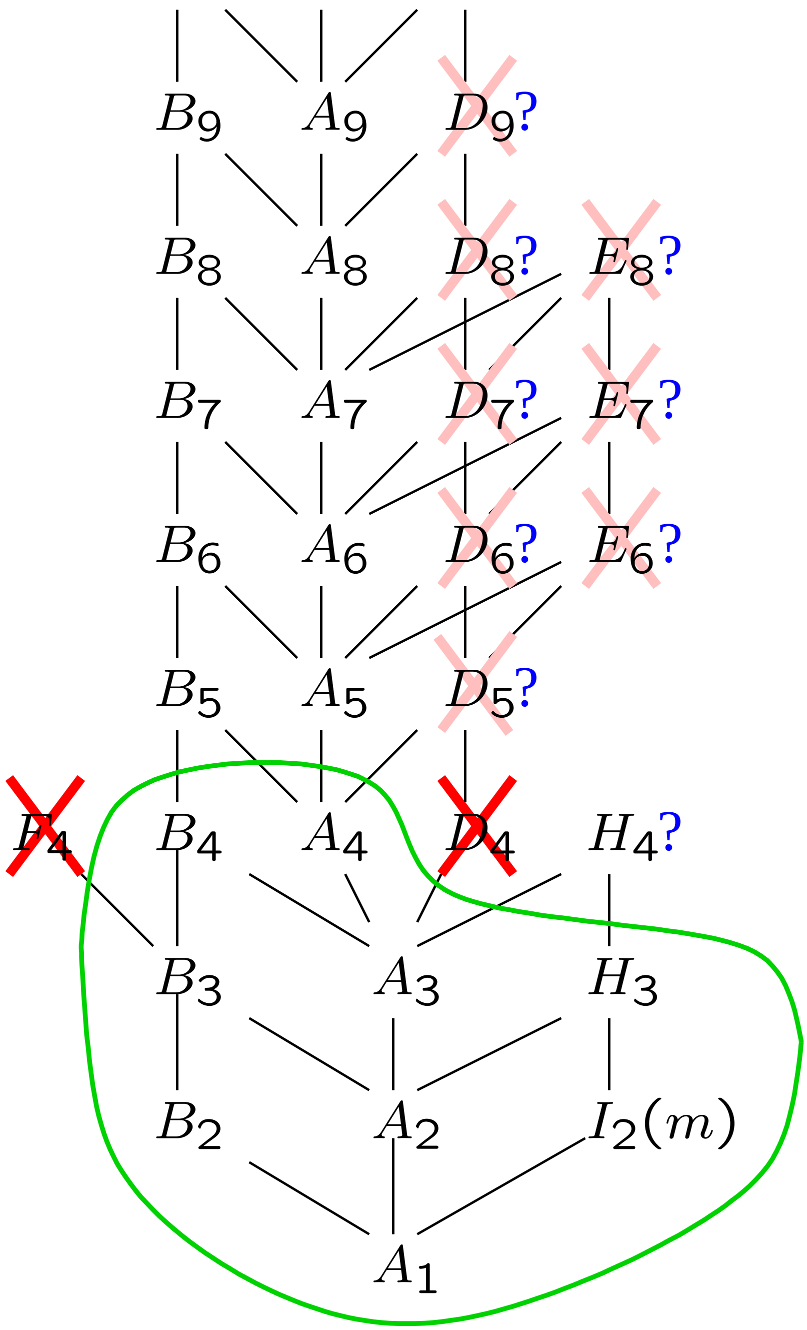
Thm: Let σ and τ be n-simplices and let f be a bijection between their vertices. If the dihedral angle at each codimension 2 face of σ is at least as big as the dihedral angle at the corresponding codimension 2 face of τ, then σ and τ are isometric up to a scale factor.

Proof: $\exists a_i > 0$ s.t. $\sum_i a_i \vec{u_i} = \vec{0}$ (Minkowski).

$$\begin{aligned} 0 = ||\vec{0}||^2 &= \sum_i \sum_j a_i a_j (\vec{u_i} \cdot \vec{u_j}) \\ &\geq \sum_i \sum_j a_i a_j (\vec{v_i} \cdot \vec{v_j}) \\ &= ||\sum_i a_i \vec{v_i}||^2 \geq 0 \end{aligned}$$

This implies $\vec{u_i} \cdot \vec{u_j} = \vec{v_i} \cdot \vec{v_j}$ for all i and j, which shows σ and τ are similar.

CAT(0) and Brady-Krammer complexes



Type H_4

The case of H_4 is hard to resolve because the defining diagram has no symmetries which greatly increases the number of equations and variables involved in the computations.

H_4 has:

- 1350 simplices
- 23 columns
- 16 types of tetrahedra in the cross section
- 10 vertex types to check
- 2986 inequalities in 96 variables
- 638 simplified inequalities in 96 variables

The F_4 and D_4 cases produced systems small enough to analyze by hand. This system is not.

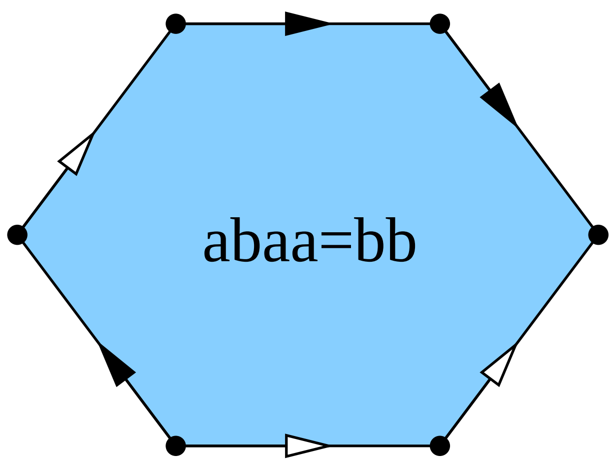
II. Small cancellation groups

Def: A *piece* is a path in the 1-skeleton which can be ϵ-pushed off the 1-skeleton in at least two distinct ways.

Def: A 2-complex is $C(p)$ if each 2-cell boundary cannot be covered with fewer than p pieces.

Def: A 2-complex is $T(q)$ if there does not exist an immersed path in a vertex link with length between 2 and q.

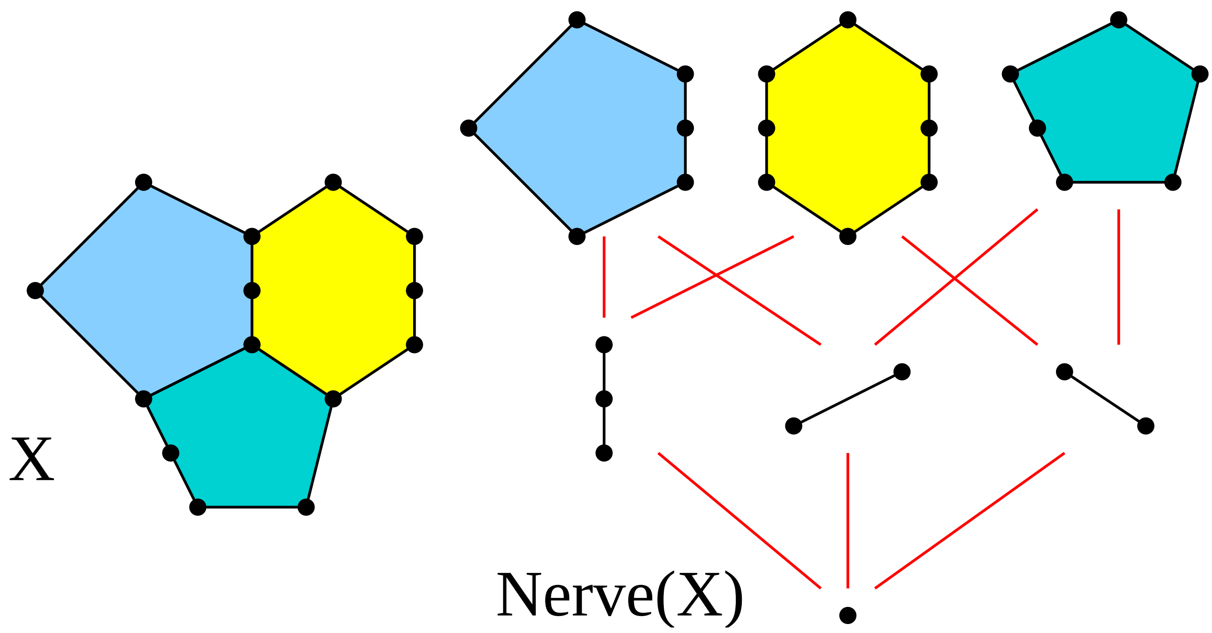
Recall: Higher dimensions help local curvature.



Philosophy

Let X be a finite combinatorial cell complex, let $\mathcal{C}$ be the collection of maximal closed cells in $\widetilde{X}$, and let P be the poset of intersections of elements in $\mathcal{C}$. The poset P is the *nerve*.

The main idea is to replace each maximal cell in X with a high-dimensional cell so that they glue together nicely and the nerve of the result is identical.



“Pieces”

Def: A *piece* is a subcomplex of $\widetilde{X}$ which corresponds to an element of the nerve.

$$P = \cap_{i=1}^{n} C_i \text{ where } C_i \in \mathcal{C}$$

Rem: Notice that this differs from the standard definition of piece in that subcomplexes of pieces are not necessarily pieces.

We will try to find new complexes with the same nerve so that every piece is a face of each maximal closed cell which contains it.

“Small cancellation”

We will be particularly interested in complexes in which

1. each C embeds in $\widetilde{X}$ and

2. each $P \in \text{Pieces}(X)$ is contractible.

Under these types of restrictions, different complexes realizing the same nerve will be homotopy equivalent.

Various small-cancellation-like conditions on X will guarantee both of these properties. For example, overlaps between closed cells are “small” subcomplexes of its boundary and links are “large”.

Sample Theorem

Recall: a cube complex is NPC iff its vertex links are flag.

Thm (Brady-M, Wise) If X is $C'(1/4)-T(4)$ complex then $\pi_1 X$ is the fundamental group of a compact high-dimensional nonpositively curved cube complex.

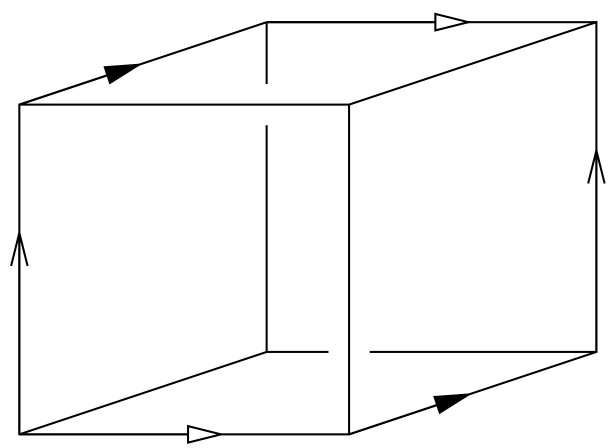
Rem: Actually it is sufficient for the total length of any two consecutive pieces in R to be at most half of $|R|$.

Rem: Dani Wise can extend many of these results to $C'(1/6)$ groups.

Proof

Step 1: Subdivide every edge so that every 2-cell has even length.

Step 2: Identify each 2-cell R with $|\partial R| = 2n$ with a n-dimensional cube.



Step 3: Glue cubes along faces corresponding to the pieces.

It is easy to check that the result is a non-positively curved cube complex with the same nerve as the original 2-complex.

III. Ample twisted face pairings

Noel Brady and I have also shown that one ample twisted face-pairing example is the fundamental group of a high dimensional CAT(0) cube complex.

(3 transparencies with pictures by Cannon, Floyd and Parry)