Constructing non-positively curved spaces and groups

Day 4: Combinatorial notions of curvature

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Outline

I. Angles in Polytopes

II. Combinatorial Gauss-Bonnet

III. Conformally CAT(0)

IV. One-relator groups
I. Angles in Polytopes

Let $F$ be a face of a polytope $P$.

- The normalized \textit{internal angle} $\alpha(F, P)$ is the proportion of unit vectors perpendicular to $F$ which point into $P$ (i.e. the measure of this set of vectors divided by the measure of the sphere of the appropriate dimension).

- The normalized \textit{external angle} $\beta(F, P)$ is the proportion of unit vectors perpendicular to $F$ so that there is a hyperplanes with this unit normal which contains $F$ and the rest of $P$ is on the other side.

\textbf{Thm:} \quad \sum_{v \in P} \beta(P, v) = 1.
Angle Sums

The sum of the internal angles in a triangle is $\pi$, but the sum of the dihedral angles in a tetrahedron can vary.

There are relations between the various internal and external angles in a Euclidean polytope but we will need a digression into combinatorics in order to state the relationship properly.
**Posets and Incidence algebras**

Let $P$ be a finite poset with elements labeled by $[n]$. The set of $n \times n$ matrices with $a_{ij} \neq 0$ only when $i \leq_P j$ is called the *incidence algebra of $P$, $I(P)$.*

For any finite poset $P$ there is a numbering of its elements which is consistent with its order. In this ordering, the incidence algebra is a set of upper triangular matrices.

\[
\zeta_P = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]
Delta, Zeta and Möbius functions

Rem: The elements of $I(P)$ can also be thought of as functions from $P \times P \to \mathbb{R}$.

The identity matrix is the delta function where $\delta(x, y) = 1$ iff $x = y$.

The zeta function is the function $\zeta(x, y) = 1$ if $x \leq_P y$ and 0 otherwise (i.e. 1’s wherever possible).

The möbius function is the matrix inverse of $\zeta$. Note that $\mu \zeta = \zeta \mu = \delta$. 

\[
\mu P = \begin{bmatrix}
1 & -1 & -1 & -1 & 2 \\
0 & 1 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]
Möbius functions and Euler characteristics

Let $P$ be a finite poset and let $\hat{P}$ be the same poset with the addition of a new minimum element $\hat{0}$ and a new maximum element $\hat{1}$. The value of the Möbius function on the interval $(\hat{0}, \hat{1})$ is the reduced Euler characteristic of the geometric realization of the poset $P$.

\[
\hat{P} = \begin{array}{cccc}
2 & 3 & 4 & 5 \\
\end{array} \\
\tilde{\chi}(P) = 2
\]

In this example the realization of $P$ is 3 discrete points.
Digression on $\ell^{(2)}$ Betti numbers

Following the type of philosophy espoused in Wolfgang Lück’s talks, John Meier and I recently calculated the $\ell^{(2)}$ Betti numbers of the pure symmetric automorphism groups with very few calculations.

We used
- a spectral sequence to show that all but the top Betti number was 0,
- the final Betti number must be the Euler characteristic of the fundamental domain,
- which comes from the Möbius function,
- which we computed using techniques from enumerative combinatorics.

\[ HT_4 = \]
Incidence algebras for Polytopes

The faces of a Euclidean polytope under inclusion is its face lattice. Traditionally $\hat{0} = \emptyset$ is added so that the result is a lattice in the combinatorial sense.

The set of all internal (external) angles forms an element of the incidence algebra of the face lattice, $\alpha (\beta)$.

**Rem:** The notion of internal and external angle needs to be extended so that $\alpha(\hat{0}, F')$ and $\beta(\hat{0}, F')$ have values, and there are many natural ways to do this.
Möbius functions for Polytopes

**Lem:** The möbius function of the face lattice of a polytope is \( \mu(F, G) = (-1)^{\dim G - \dim F} \).

**Proof:** The geometric realization of the portion of the face lattice between \( F \) and \( G \) is a sphere.

**Def:** Let \( \bar{\alpha}(F, G) = \mu(F, G)\alpha(F, G) \), [Hadamard product] (i.e. \( \bar{\alpha} \) is a *signed* normalized internal angle.

**Thm (Sommerville)** \( \mu \alpha = \bar{\alpha} \) i.e.

\[
\sum_{F \leq G \leq H} \mu(F, G)\alpha(G, H) = \mu(F, H)\alpha(F, H)
\]
Equations for angles

The most interesting of angle identity is the one discovered by Peter McMullen.

Thm(McMullen) $\alpha \beta = \zeta$, i.e.

$$\sum_{F \leq G \leq H} \alpha(F, G) \beta(G, H) = \zeta(F, H)$$

Proof Idea:
• Look at (a polytopal cone) $\times$ (its dual cone)

• Integrate $f(\mathbf{x}) = \exp(-||\mathbf{x}||^2)$ over this $\mathbb{R}^{2n}$ in two different ways.

Cor: $\mu \alpha \beta = \bar{\alpha} \beta = \delta$. 
Curvature in PE complexes

Following Cheeger-Müller-Schrader (and Charney-Davis), if $X$ is a PE complex

$$
\chi(X) = \sum_P (-1)^{\dim P} \\
= \sum_P \sum_{v \in P} (-1)^{\dim P} \beta(v, P) \\
= \sum_v \sum_{P \ni v} (-1)^{\dim P} \beta(v, P) \\
= \sum_v \kappa(v)
$$

where $\kappa(v) := \sum_{P \ni v} (-1)^{\dim P} \beta(v, P)$.

**Rem 1:** $\kappa(v)$ is similar to (but not) a signed version of $\beta$.

**Rem 2:** The first step is really just replacing $\delta$ with $\bar{\alpha}\beta$ in a very precise sense.
II. Combinatorial Gauss-Bonnet

An angled 2-complex is one where we assign normalized external angles $\beta(v, f)$ for each vertex $v$ in a face $f$.

Define $\kappa(v)$ as above. Define $\kappa(f)$ as a correction term which measures how far the external vertex angles are from 1.

$$\kappa(f) = 1 - \sum_{v \in f} \beta(v, f)$$

**Thm(Gersten,Ballmann-Buyalo,M-Wise)**

If $X$ is an angled 2-complex, then

$$\sum_v \kappa(v) + \sum_f \kappa(f) = \chi(X)$$

**Rem:** In all these papers the sum was $2\pi \chi(X)$ since the angles were not normalized. As we have seen normalization is crucial for the equations in higher dimensions.
Combinatorial Gauss-Bonnet in higher dimensions

The formula $\sum_{v \in P} \beta(v, P) = 1$ is a consequence of McMullen’s theorem under one extension of $\alpha$ and $\beta$ to the intervals $(\hat{0}, F)$.

Similarly, the combinatorial Gauss-Bonnet Theorem on the previous slide comes from reversing the order of summation for another factorization of the zeta function.

**General CGB “Thm”** Given any factorization $\alpha \beta = \zeta$, reversing the order of summation gives a combinatorial Gauss-Bonnet type formula.

**Rem 1**: Only factorizations which produce lots of 0s will be of much use, but there is room to explore.

**Rem 2**: The Regge calculus should also fit into this framework.
III. Conformal CAT(0) structures

A 2-complex $X$ with an angle assigned to each corner is an *angled 2-complex*.

If the vertex links are CAT(1), then $X$ is called *conformally CAT(0)*.

**Thm(Corson):** Conformally CAT(0) 2-complexes are aspherical.

**Example:** The Baumslag-Solitar groups are conformally CAT(0) - even though they are not CAT(0), except in the obvious cases.
**Sectional curvature**

**Def:** Let $X$ be an angled 2-complex. If every connected, 2-connected subgraph of each vertex link is CAT(1), then $X$ has *non-positive sectional curvature*.

**Thm (Wise)** If $X$ is an angled 2-complex with non-positive sectional curvature, then $\pi_1 X$ is coherent.

**Rem:** Using Howie towers, these are the key types of sublinks that need to be considered.
Special polyhedra

**Def:** A 2-complex is called a *special polyhedron* if the link of every point is either a circle, a theta graph, or the complete graph on 4 vertices. These points define the intrinsic 2-, 1- and 0-skeleta of $X$. 
Conformal CAT(0) structures and Special polyhedra

**Lem:** If $X$ is an angled 2-dimensional special polyhedron, then $X$ is conformally CAT(0) if and only if $X$ has non-positive sectional curvature.

**Pf:** The only subgraphs to check are triangles, and whole graph.

**Cor:** If $X$ is a 2-dimensional special polyhedron with a conformal CAT(0) structure, then $\pi_1 X$ is coherent.
IV. One-relator groups

Conj A: Every one-relator group is coherent.

Conj B: Every one-relator group is the fundamental group of a 2-dimensional special polyhedron with a conformal CAT(0) structure.

Rem 1: Conjecture B implies Conjecture A, and it would help explain why one-relator groups tend to “act like” non-positively curved groups.

Rem 2: For Conjecture A it is sufficient to prove Conjecture B for 2-generator one-relator groups since every one-relator group is a subgroup of a 2-generator one-relator group. Moreover, the inequalities are tight (and become equations) in this case.
Special polyhedra for one-relator groups

**Def:** If $x$ is a point in $X^{(2)}$ such that $X - x$ deformation retracts onto a graph, then $x$ is a **puncture point**.

**Rem:** If $X$ has a puncture point then $\pi_1 X$ is a one-relator group.

**Thm (N. Brady-M)** If $X$ is the presentation 2-complex for a one-relator group, then $X$ is simply-homotopy equivalent to a 2-dimensional special polyhedron $Y$ with a puncture point. In addition, $Y$ can be chosen so that it has no monogons, bigons, or untwisted triangles.
Additional remarks

The puncture point and $\chi(X) = 0$ allow you to remove most portions of the 2-skeleton which are not discs.

The game is to use the flexibility in the special polyhedron construction to manipulate the linear system so that it has a solution. Since this system has $3n$ variables and $2n$ equations, our odds are good in general – we only need to avoid contradictions.

The first several examples we tried by hand produced conformal CAT(0) structures, even when we proceeded “randomly”.

A computer program to check all the one-relator groups out to a modest size is high on my to-do-list.