Existence of CAT(0) structures for finite type Artin groups

Jon McCammond  
U.C. Santa Barbara
Overview

I. Finite-type Artin groups

II. Brady-Krammer complexes

III. Non-positive curvature

IV. Old and new results
Coxeter and Artin groups

Let $\Gamma$ be a finite graph with edges labeled by integers greater than 1, and let $(a, b)^n$ be the length $n$ prefix of $(ab)^n$.

**Def:** The Artin group $A_\Gamma$ is generated by its vertices with a relation $(a, b)^n = (b, a)^n$ whenever $a$ and $b$ are joined by an edge labeled $n$.

**Def:** The Coxeter group $W_\Gamma$ is the Artin group $A_\Gamma$ modulo the relations $a^2 = 1 \ \forall a \in \text{Vert}(\Gamma)$.

![Graph](image)

**Artin presentation**

$\langle a, b, c \mid aba = bab, ac = ca, bcbc = cbcb \rangle$

**Coxeter presentation**

$\left\langle a, b, c \mid \begin{array}{c} aba = bab, ac = ca, bcbc = cbcb \\ a^2 = b^2 = c^2 = 1 \end{array} \right\rangle$
Finite-type Artin groups

The finite Coxeter groups have been classified. An Artin group defined by the same labeled graph as a finite Coxeter is called a finite-type Artin.
Irreducible Dynkin diagrams

\[
\begin{array}{c}
\begin{array}{ccc}
B_9 & A_9 & D_9 \\
B_8 & A_8 & D_8 & E_8 \\
B_7 & A_7 & D_7 & E_7 \\
B_6 & A_6 & D_6 & E_6 \\
B_5 & A_5 & D_5 \\
F_4 & B_4 & A_4 & D_4 & H_4 \\
B_3 & A_3 & H_3 \\
B_2 & A_2 & I_2(m) \\
A_1
\end{array}
\end{array}
\]
Eilenberg-MacLane spaces for Artin groups

Finite-type Artin groups are fundamental groups of complexified Coxeter hyperplane arrangements quotiented by the action of the Coxeter group.

Each finite type Artin group has a
- finite dimensional $\text{CAT}(0)$ $\text{K}(G,1)$
- finite dimensional compact $\text{K}(G,1)$
but no known
- finite dimensional compact $\text{CAT}(0)$ $\text{K}(G,1)$

Thus they do not yet qualify as $\text{CAT}(0)$ groups, but they are good candidates.
Brady-Krammer Complexes

In 1998 Tom Brady and Daan Krammer independently discovered new complexes on which the braid groups and the other Artin groups of finite type act.

In the case of the braid groups, the link of a vertex in the cross section is the order complex of a well-known combinatorial object known as the noncrossing partition lattice.
Noncrossing Partitions

A noncrossing partition is a partition of the vertices of a regular $n$-gon so that the convex hulls of the partitions are disjoint.

One noncrossing partition $\sigma$ is contained in another $\tau$ if each block of $\sigma$ is contained in a block of $\tau$.
Factors of the Coxeter element

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_3$</td>
<td>1-6-6-1</td>
</tr>
<tr>
<td>$B_3$</td>
<td>1-9-9-1</td>
</tr>
<tr>
<td>$H_3$</td>
<td>1-15-15-1</td>
</tr>
<tr>
<td>$A_4$</td>
<td>1-10-20-10-1</td>
</tr>
<tr>
<td>$B_4$</td>
<td>1-12-24-12-1</td>
</tr>
<tr>
<td>$D_4$</td>
<td>1-16-36-16-1</td>
</tr>
<tr>
<td>$F_4$</td>
<td>1-24-55-24-1</td>
</tr>
<tr>
<td>$H_4$</td>
<td>1-60-158-60-1</td>
</tr>
<tr>
<td>$A_5$</td>
<td>1-15-50-50-15-1</td>
</tr>
<tr>
<td>$B_5$</td>
<td>1-20-70-70-20-1</td>
</tr>
<tr>
<td>$D_5$</td>
<td>1-25-100-100-25-1</td>
</tr>
</tbody>
</table>

General formulae exist for the $A_n$, $B_n$ and $D_n$ types as well as explicit calculations for the exceptional ones, but no general formula explains all of these numbers in a coherent framework.
$F_4$ Poset
**CAT(0)**

Def: A geodesic metric space $C$ is called (globally) CAT(0) if

$\forall$ points $x, y, z \in C$

$\forall$ geodesics connecting $x, y, \text{ and } z$

$\forall$ points $p$ in the geodesic connecting $x$ to $y$

$$d(p, z) \leq d(p', z')$$

in the corresponding configuration in $E^2$. 

![Diagram](image.png)
Piecewise Euclidean Complexes

**Def:** A *piecewise euclidean complex* $X$ is a simplicial complex in which each simplex is given a Euclidean metric and the induced metrics on the intersections always agree.

**Thm:** A PE complex is CAT(0) iff the link of each cell does not contain a closed geodesic loop of length less than $2\pi$. 
**CAT(0) and Artin groups**

**Thm(T. Brady-M)** The finite-type Artin groups with at most 3 generators are CAT(0)-groups and the Artin groups $A_4$ and $B_4$ are CAT(0) groups.

**Proof:** The link of a vertex in the cross section is the order complex of a fairly small poset. It is then relatively easy to check that using a fairly “natural” metric, each of these links satisfy the link condition.

**Conj:** The Brady-Krammer complex is CAT(0) for all Artin groups of finite type.
CAT(0) metrics on $D_4$ and $F_4$

**Thm(Choi):** The Brady-Krammer complexes for $D_4$ and $F_4$ do not support reasonable PE CAT(0) metrics.

Reasonable means that symmetries of the group should lead to symmetries in the metric.

**Proof Idea:** First determine what Euclidean metrics on the 3-dimensional cross-section complex have dihedral angles which make the edge links (which are finite graphs) large.

Then check these metrics in the vertex links (which are 2-dimensional PS complexes).
The software

The program `coxeter.g` is a set of GAP routines used to examine Brady-Kramer complexes. Initially developed to test the curvature of the Brady-Kramer complexes using the “natural” metric, the routines were extensively modified by Woonjung Choi so that they
• find the 3-dimensional structure of the cross-section
• find representative vertex and edge links (up to automorphism)
• find the graphs for the edge links
• find the simple cycles in these graph
• find the linear system of inequalities which need to be satisfied by the dihedral angles of the tetrahedra.
Dihedral angles

**Thm:** Let $\sigma$ and $\tau$ be $n$-simplices and let $f$ be a bijection between their vertices. If the dihedral angle at each codimension 2 face of $\sigma$ is at least as big as the dihedral angle at the corresponding codimension 2 face of $\tau$, then $\sigma$ and $\tau$ are similar (isometry up to a scale factor).

**Proof:** $\exists a_i > 0$ s.t. $\sum a_i \vec{u}_i = \vec{0}$ (Minkowski).

\[
0 = ||\vec{0}||^2 = \sum \sum a_i a_j (\vec{u}_i \cdot \vec{u}_j) \\
\geq \sum \sum a_i a_j (\vec{v}_i \cdot \vec{v}_j) \\
= ||\sum a_i \vec{v}_i||^2 \geq 0
\]

This implies $\vec{u}_i \cdot \vec{u}_j = \vec{v}_i \cdot \vec{v}_j$ for all $i$ and $j$, which shows $\sigma$ and $\tau$ are similar.
CAT(0) and Brady-Krammer complexes
**Types $D_4$ and $F_4$**

$D_4$ has:
- 162 simplices
- 15 columns
- 3 types of tetrahedra in the cross section
- 4 vertex types to check
- 21 inequalities in 9 variables
- 13 simplified inequalities in 9 variables

$F_4$ has:
- 432 simplices
- 18 columns
- 4 types of tetrahedra in the cross section
- 7 vertex types to check
- 81 inequalities in 13 variables
- 27 simplified inequalities in 13 variables
**Type** $H_4$

The case of $H_4$ is hard to resolve because the defining diagram has no symmetries which greatly increases the number of equations and variables involved in the computations.

$H_4$ has:
- 1350 simplices
- 23 columns
- 16 types of tetrahedra in the cross section
- 10 vertex types to check
- 2986 inequalities in 96 variables
- 638 simplified inequalities in 96 variables

The $F_4$ and $D_4$ cases produced systems small enough to analyze by hand. This system is not.