Math 110, Fall 2012, Sections 109-110
Worksheet 2

1. Prove that if $\{A_1, A_2, \ldots, A_k\}$ is a linearly independent subset of $M_{n\times n}(F)$ then so is $\{A_1^t, \ldots, A_k^t\}$.
2. Determine if the statement is true or false, and justify your answer. Let S and T be subsets of a vector space V.
   (a) If S is linearly independent and $S \subset T$, then T is linearly independent.
   (b) If S is linearly dependent and $S \subset T$, then T is linearly dependent.
   (c) If S spans V and $S \subset T$, then T spans V.
   (d) If S does not span V and $S \subset T$, then T does not span V.
3. Let V be a vector space. Assume $u, v, w \in V$ and $u + v + w = 0$. Let $W_1 = \text{span}\{u, v\}$ and $W_2 = \text{span}\{u, w\}$. Is $W_1 = W_2$? Prove your answer.
4. (a) As a subset of the vector space $\mathbb{R}^3$, is $\{(2, 4, 1), (1, 2, 4)\}$ linearly independent or linearly dependent?
   (b) Compute $4(2, 4, 1) - (1, 2, 4)$ in $(\mathbb{Z}/7)^3$ and comment.
5. Let V be the real vector space of functions from $\mathbb{R}$ to $\mathbb{R}$ with pointwise addition and scalar multiplication. Let $f, g, h \in V$ be the elements $f(x) = e^x$, $g(x) = \cos(x)$, and $h(x) = \sin(x)$. Is $f \in \text{span}\{g, h\}$?

1) Suppose $c_1 A_1^t + \cdots + c_k A_k = 0$. We must show that $c_i = 0$ for all i. By the linearity of the transpose, we have
$$(c_1 A_1 + \cdots + c_n A_n)^t = 0,$$
and taking transposes of both sides yields
$$c_1 A_1 + \cdots + c_n A_n = 0.$$
Since we assumed that $\{A_k\}$ was linearly independent, we must have $c_i = 0$ for all i.

2) a) False. $\{(1, 0)\}$ is a linearly independent subset of $\mathbb{R}^2$ but $\{(1, 0), (0, 0)\}$ is not.

b) True. Since S is linearly dependent, there are $v_1, \ldots, v_k \in S$ and non-zero $c_1, \cdots, c_k \in F$ with

$$c_1v_1 + \cdots c_kv_k = 0.$$

But then every $v_k \in T$, so T is linearly dependent as well.

c) True. For every $v \in V$ there are $v_1, \ldots, c_k \in S$ and $c_1, \ldots, c_k \in F$ with

$$v = c_1v_1 + \cdots c_kv_k.$$

But each $v_k \in T$, so T spans V as well.

d) False. $\{(1, 0)\}$ does not span $\mathbb{R}^2$ but $\{(1, 0), (0, 1)\}$ does.

3) Indeed $W_1 = W_2$. We first show $W_1 \subseteq W_2$. Let $x \in W_1$ and we will show it is in W_2. Since $W_1 = \text{span}\{u, v\}$, there are scalars $c, d \in F$ with $x = cu + dv$. We'd like to show that $x \in W_1$, so we must write it as a linear combination of u and w. Since $w = -u - v$, we also have $x = (c - d)u - dw$ and thus $x \in \text{span}\{u, w\}$.

Conversely, we must also show $W_2 \subseteq W_1$ so let $x \in W_2$. Then there are $c, d \in F$ with $x = cu + dw$. Substituting $w = -u - v$ gives $x = (c - d)u - dv$, so $x \in \text{span}\{u, v\}$. We conclude that $W_2 \subseteq W_1$ and $W_2 = W_1$.

4) a) A set with two elements is linearly independent if and only if one element is a multiple of the other. As that is not the case here, the given set must be linearly independent.

b) However $4(2, 4, 1) - (1, 2, 4) = (0, 0, 0)$ in $(\mathbb{Z}/7)^3$, so $\{(2, 4, 1), (1, 2, 4)\}$ is a linearly dependent subset of $(\mathbb{Z}/7)^3$. This shows us that the even though both $\mathbb{R}^3$ and $(\mathbb{Z}/7)^3$ have vectors called $(2, 4, 1)$ and $(1, 2, 4)$, their properties are determined by which vector space we are working in.

5) It is not true that $f \in \text{span}\{g, h\}$. Suppose to a contradiction that there are real numbers α and β with $f = \alpha g + \beta h$. That is, $f(x) = \alpha g(x) + \beta h(x)$ for all $x \in \mathbb{R}$. Plugging in $x = 0$ yields $1 = \alpha$. On the other hand, plugging in $x = 2\pi$ yields $e^{2\pi} = \alpha$. This is a contradiction, so we must not have $f \in \text{span}\{g, h\}$.