Math 110, Fall 2012, Sections 109-110
Worksheet 5

1. Let A and B be $n \times n$ matrices such that AB is invertible. Prove that A and B are invertible. Give an example to show that arbitrary matrices A and B need not be invertible if AB is invertible.

2. Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ and $\beta = \{(1,2),(-1,-3)\}$ is a basis for $\mathbb{R}^2$ for which $[T]_\beta = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$. Find $T(x,y)$.

3. (a) Find a nonzero $A \in M_{n\times n}(\mathbb{R})$ such that $A^2 = 0$.
   (b) Show that there exists a non-zero linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $T^2 = 0$.
   (c) If V is a finite-dimensional vector space, show that there is a non-zero linear transformation $T : V \to V$ such that $T^2 = 0$.

4. (a) Given a basis $\beta = \{x_1, \ldots, x_n\}$ for V, define the dual basis β^*.
   (b) Let $\beta = \{e_1, e_2\}$ be the standard basis for $\mathbb{R}^2$. What is the dual basis β^*?
   (c) Let $\gamma = \{e_1, e_1 + e_2\}$. What is the dual basis γ^*?

5. Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$. Let $L_A : \mathbb{R}^2 \to \mathbb{R}^2$ be the left-multiplication linear transformation associated to A. Let β be the standard basis for $\mathbb{R}^2$, and let β^* be the dual basis for $(\mathbb{R}^2)^*$.
   (a) What are the domain and codomain of $(L_A)^t$? How is this different than L_{A^t}?
   (b) Compute $(L_A)^t$ on an arbitrary element of its domain.
   (c) Compute $[(L_A)^t]_{\beta^*}$, and comment.