1, 3, 5

Name:
GSI's Name: $\boxed{\text{Solution}}$
Section:

Midterm 2
Math 1B, Fall 2008
Wilkening

0	1	
1	3	
2	3	
3	3	
4	7	
5	8	
6	5	
7	6	
total	36	

0. (1 point) write your name, your GSI's name, and your section number at the top of your exam.

1. (3 points or 0 points). Suppose $|\cos x| \neq 1$. Evaluate $\sum_{n=0}^{\infty} (\cos x)^{2n}$.

a. $\cot x$

b. $\csc^2 x$

c. $\cosh x$

d. $\frac{x}{\sqrt{1-x^2}}$

e. none of the above

$$\sum_{0}^{\infty} (\cos^2 x)^n = \frac{1}{1-\cos^2 x} = \frac{1}{\sin^2 x} = \csc^2 x$$

2. (3 points or 0 points) Describe the behavior of the sequence $a_1 = 0, \quad a_{n+1} = \dfrac{a_n^2+3}{4}$

a. a_n increases monotonically and converges to 1

b. a_n increases monotonically and converges to 3

c. a_n increases monotonically to ∞

d. a_n decreases monotonically to $-\infty$

e. a_n is not monotonic

$$a_n < a_{n+1} < 1$$
$$\Rightarrow a_n^2 < a_{n+1}^2 < 1$$
$$\Rightarrow a_n^2+3 < a_{n+1}^2+3 < 4$$
$$\Rightarrow \frac{a_n^2+3}{4} < \frac{a_{n+1}^2+3}{4} < 1$$

(increasing)
(bounded above by 1)

$$L = \frac{L^2+3}{4}$$
$$4L = L^2+3 \qquad L^2-4L+3=0 \qquad (L-3)(L-1)=0 \qquad L=1 \quad \cancel{3}$$

3. (3 points or 0 points) Supppose $0 \leq a_n < 1$, $a_n < b_n$, and $\sum b_n$ is convergent. Circle all the statements that are necessarily true:

a. $\sum b_n^2$ converges and $\sum b_n^2 < \sum b_n$

b. $\sum \sqrt{b_n}$ converges and $\sum \sqrt{b_n} < \sum b_n$

c. $\sum a_n^2$ converges and $\sum a_n^2 < \sum a_n$

d. $\sum \sqrt{a_n}$ converges and $\sum \sqrt{a_n} < \sum a_n$

e. if $p > 0$ then $\sum (-1)^n a_n^p$ is convergent

Counterexample for (e): $a_n = \begin{cases} 1/n^2 & n \text{ even} \\ 1/n^4 & n \text{ odd} \end{cases}$

$p = 1/2 \qquad (-1)^n a_n^p = \begin{cases} 1/n & n \text{ even} \\ -1/n^2 & n \text{ odd} \end{cases}$

4a. (3 points) Let $f(x) = e^{-x^2}$.
Write down the Maclaurin series for $f(x)$ and evaluate $f^{(99)}(0)$ and $f^{(100)}(0)$.

$$e^x = \sum_0^\infty \frac{x^n}{n!}$$

$$\Rightarrow e^{-x^2} = \sum_0^\infty \frac{(-1)^n x^{2n}}{n!}$$

$$\sum_0^\infty \frac{(-1)^n x^{2n}}{n!} = \sum_0^\infty \frac{f^{(k)}(0)}{k!} x^k$$

match up coefficients 99, 100

(no odd coefficients ~~a~~ left, so $f^{(99)}(0) = 0$)

$$\frac{f^{(100)}(0)}{100!} = \frac{(-1)^{50}}{50!} \Longrightarrow f^{(100)}(0) = \frac{100!}{50!}$$

4b. (4 points) Find all x that satisfy the equation $\sum_{n=1}^{\infty} nx^n = \frac{1}{2}$.

$$\frac{1}{1-x} = \sum_0^\infty x^n$$

$$\frac{1}{(1-x)^2} = \sum_1^\infty n x^{n-1} \Rightarrow \frac{x}{(1-x)^2} = \sum_1^\infty n x^n$$

So set $\frac{x}{(1-x)^2} = \frac{1}{2}$

$$x = \tfrac{1}{2}(1-x)^2 = \tfrac{1}{2} - x + \tfrac{x^2}{2}$$

$$\frac{x^2}{2} - 2x + \frac{1}{2} = 0$$

$$x = \frac{2 \pm \sqrt{3}}{2 \cdot 1/2} = 2 \pm \sqrt{3}$$

need $|x| < 1$, so

$$x = 2 - \sqrt{3}$$

5. (2 points each) For each of the following series, determine whether the series is absolutely convergent (AC), conditionally convergent (CC), or divergent (D). Show some work, but do not spend excessive time justifying all your steps.

$$\sum_{n=1}^{\infty}(-1)^n[\sin(1/n^2)]^{2/3}$$

Check for absolute convergence:

$\sum_1^{\infty} (\sin \frac{1}{n^2})^{2/3}$ $\quad$ $\sin \frac{1}{n^2} \approx \frac{1}{n^2}$ for large n (small x)

lim comparison to $(1/n^2)^{2/3}$: $\lim_{n\to\infty} \frac{(\sin 1/n^2)^{2/3}}{(1/n^2)^{2/3}} = 1$

Since $\sum_1^{\infty} 1/n^{4/3}$ conv abs, $\sum_1^{\infty} (-1)^n \sin(\frac{1}{n^2})^{2/3}$ AC

$$\sum_{n=1}^{\infty} \ln \cos \frac{1}{n}$$

$\ln(\cos(1/n)) \sim \ln(1 - 1/n^2)$

$\sim -1/n^2$ (asymptotically)

more precisely, $\lim_{n\to\infty} \frac{\ln(\cos(1/n))}{-1/n^2} = 1$ (check by l'Hospital's)

$\therefore$ since $\sum_1^{\infty} -1/n^2$ converges absolutely, $\sum_1^{\infty} \ln(\cos 1/n)$ is AC

$$\sum_{n=1}^{\infty}\frac{(2n)!}{3^n(n!)^2}$$

Ratio test: $\frac{a_{n+1}}{a_n} = \frac{(2n+2)(2n+1)}{3(n+1)^2} \longrightarrow 4/3 > 1$

D

$$\sum_{n=0}^{\infty}\binom{5}{n}3^n$$

Note: $\binom{5}{n} = 0$ for $n \geq 5$

So this is a finite sum $\Rightarrow$ AC

6a. (2 pts) Is the following statement True or False? Justify your answer with a proof or counterexample. (Obviously it's true if $a_n \geq 0$ and $b_n \geq 0$, so don't assume this).

If $\sum a_n$ is divergent and $\sum b_n$ is divergent, then $\sum(a_n + b_n)$ is also divergent.

False

Countoex: $a_n = 1$ (for all n)

$b_n = -1$

6b. (3 points) Suppose $\sum c_n x^n$ has radius of convergence 2 while $\sum d_n x^n$ has radius of convergence 5. What is the radius of convergence of the series $\sum(c_n + d_n)x^n$? Explain.

It must be 2. For $|x|<2$, $\Sigma(c_n+d_n)x^n$ is the sum of 2 convergent series, hence convergent. For $2<|x|<5$, $\Sigma(c_n+d_n)x^n$ is the sum of a convergent series and a divergent series, hence divergent. This is enough to force $R = 2$.

7a. (3 points) Prove that $e \geq \left(1+\frac{1}{k}\right)^k$ for $k \geq 1$. (*Hint:* $\ln(1+x) = x - \frac{x^2}{2} + \cdots$)

$$e \geq \left(1+\frac{1}{k}\right)^k$$

$$\iff \ln(e) = 1 \geq k\ln\left(1+\frac{1}{k}\right)$$

$$= k\left[\frac{1}{k} - \frac{1}{2k^2} + \frac{1}{3k^3} - \ldots\right]$$

$$= 1 - \frac{1}{2k} + \frac{1}{3k^2} - \ldots$$

this is an alternating series with decreasing absolute value of terms. Hence

$1 \geq 1 - \frac{1}{2k} + \frac{1}{3k^2} - \ldots$, completing the proof.

7b. (3 points) Use part (*a*) and mathematical induction to prove the following crude version of Stirling's approximation:

$$e^n n! \geq n^n \qquad \text{for all } n \geq 1.$$

Base case $n=1$: $e^1\, 1! \overset{?}{\geq} 1^1$

$(e \geq 1)$ ✓

induction step: suppose $e^k k! \geq k^k$. Then

$$e^{k+1}(k+1)! = e \cdot e^k \cdot (k+1)!$$

(by induction hyp.) $\geq e \cdot \frac{k^k}{k!}(k+1)!$

(by (a)) $\geq (1+1/k)^k k^k \frac{(k+1)!}{k!}$

$$= (k+1)^k (k+1)$$

$$= (k+1)^{k+1}$$ ✓