Math 1B, Fall 2008
Section 107

Quiz 4 Solutions

(1) *Determine whether the given sequence converges or diverges. If it converges, find the limit.*

$$a_n = \frac{(n+2)!}{n!}.$$

Since

$$\begin{aligned}(n+2)! &= 1 * 2 * \cdots * n * (n+1) * (n+2) \\ &= n!(n+1)(n+2),\end{aligned}$$

we can re-write

$$a_n = \frac{(n+1)(n+2)n!}{n!} = (n+1)(n+2).$$

Thus a_n diverges (to ∞).

(2) *Determine whether the given sequence converges or diverges. If it converges, find the limit.*

$$a_n = \frac{\sqrt{n}}{1+\sqrt{n}}.$$

Instead, consider the function $f(x) = \frac{\sqrt{x}}{1+\sqrt{x}}$. By L'Hopital's rule,

$$\lim_{x\to\infty} \frac{\sqrt{x}}{1+\sqrt{x}} = \lim_{x\to\infty} \frac{(2x)^{-1/2}}{(2x)^{-1/2}} = 1.$$

Thus a_n converges and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} f(n) = 1$ as well.

(3) *Prove that*

$$\lim_{x\to\infty} \frac{e^x}{x^n} = \infty$$

for all $n \geq 1$.

Proof by induction. First, the base case $\underline{n=1}$. By L'Hopital's rule,

$$\lim_{x\to\infty} \frac{e^x}{x} = \lim_{x\to\infty} e^x = \infty.$$

Now we prove that $\underline{k \implies k+1}$. That is, assume that

$$\lim_{x\to\infty} \frac{e^x}{x^k} = \infty$$

and we will use this to show that

$$\lim_{x\to\infty} \frac{e^x}{x^{k+1}} = \infty.$$

By L'Hopital's rule,

$$\begin{aligned}
\lim_{x\to\infty} \frac{e^x}{x^{k+1}} &= \lim_{x\to\infty} \frac{e^x}{(k+1)x^k} \\
&= \frac{1}{k+1} \lim_{x\to\infty} \frac{e^x}{x^k} \qquad \text{(pull constant out of limit)} \\
&= \frac{1}{k+1}\infty \qquad \text{(by our assumption about the } k \text{ case)} \\
&= \infty,
\end{aligned}$$

which is what we were trying to prove.