Math 1B, Fall 2008
Section 108

Quiz 4 Solutions

(1) *If* $f(x) = xe^x$*, prove that* $f^{(n)}(x) = (x+n)e^x$*. (Note on notation:* $f^{(n)}$ *is the* n*th derivative of* f*.)*

Proof by induction. First, we show the base case $\underline{n = 1}$. Using the product rule,

$$f^{(1)}(x) = f'(x) = xe^x + e^x = (x+1)e^x.$$

Now we show the inductive step $\underline{k \implies k+1}$. That is, we assume that

$$f^{(k)}(x) = (x+k)e^x$$

and we try to show that

$$f^{(k+1)}(x) = (x+k+1)e^x.$$

To do this, we will think of $f^{(k+1)}$ (the $(k+1)$st derivative of f) as the derivative of the kth derivative of f. Written out, we get

$$\begin{aligned} f^{(k+1)}(x) &= \frac{d}{dx} f^{(k)}(x) \\ &= \frac{d}{dx}(x+k)e^x \qquad \text{(by our assumption about the } k \text{ case)} \\ &= (x+k)e^x + e^x \qquad \text{(product rule)} \\ &= (x+k+1)e^x, \end{aligned}$$

which is what we were trying to show.

(2) *Determine whether the given sequence converges or diverges. If it converges, find the limit.*

$$a_n = \frac{(n+2)!}{n!}.$$

Since

$$\begin{aligned} (n+2)! &= 1 * 2 * \cdots * n * (n+1) * (n+2) \\ &= n!(n+1)(n+2), \end{aligned}$$

we can re-write

$$a_n = \frac{(n+1)(n+2)n!}{n!} = (n+1)(n+2).$$

Thus a_n diverges (to ∞).

(3) *Let a_n be the sequence given by*

$$\begin{aligned} a_1 &= 1 \\ a_{n+1} &= \frac{1}{1+a_n}. \end{aligned}$$

Assuming that $\{a_n\}$ converges, find $\lim_{n\to\infty} a_n$.

Let $L = \lim_{n\to\infty} a_n$. Since the first term of the sequence does not affect the limit, we also have $L = \lim_{n\to\infty} a_{n+1}$ (exercise: prove this from the definition of the limit). Starting from $a_{n+1} = 1/(1+a_n)$ and taking limits of both sides gives

$$\begin{aligned} L &= \lim_{n\to\infty} a_{n+1} && \text{(justified above)} \\ &= \lim_{n\to\infty} \frac{1}{1+a_n} && \text{(definition of } a_{n+1}\text{)} \\ &= \frac{1}{1+\lim_{n\to\infty} a_n} && \text{(limit rules)} \\ &= \frac{1}{1+L}. \end{aligned}$$

Thus $L = 1/(1+L)$. Solving for L gives $L^2 + L - 1 = 0$, so L is a root of the polynomial $x^2 + x - 1$. The roots of $x^2 + x - 1$ are

$$\frac{-1 \pm \sqrt{5}}{2}.$$

Now we need to find which one of these two solutions is L. Looking at the formula for a_n, we can see that $a_n \geq 0$ for all n (exercise: prove this by induction). Thus $\lim_{n\to\infty} a_n \geq 0$. But only one root of x^2+x-1 is positive, so we conclude

$$L = \frac{-1+\sqrt{5}}{2}.$$