Math 1B, Fall 2008
Section 107

Quiz 5 Solutions

(1) *Determine whether the given sequence converges or diverges. If it converges, find the limit.*

$$a_n = \arctan\left(\frac{2}{n}\right).$$

Since arctan is continuous, we can use the "Direct Substitution Property:"

$$\lim_{n\to\infty} \arctan\left(\frac{2}{n}\right) = \arctan\left(\lim_{n\to\infty} \frac{2}{n}\right) = \arctan 0 = 0.$$

(2) *Determine whether the given series converges or diverges.*

$$\sum_{n=1}^{\infty} \frac{1+2^n}{3^n}.$$

Solution 1: We can write

$$\sum_{n=1}^{\infty} \frac{1+2^n}{3^n} = \sum_{n=1}^{\infty} \frac{1}{3^n} + \sum_{n=1}^{\infty} \frac{2^n}{3^n}.$$

Both $\sum_{n=1}^{\infty} \frac{1}{3^n}$ and $\sum_{n=1}^{\infty} \frac{2^n}{3^n}$ are geometric series with ratio r such that $|r| < 1$ ($r_1 = 1/3$ and $r_2 = 2/3$). Since $\sum_{n=1}^{\infty} \frac{1+2^n}{3^n}$ is the sum of two convergent series, it also converges. Note: you could also find the value of the sum using the formula for geometric series, but that wasn't part of the question.

Solution 2: For large n, $1 + 2^n \approx 2^n$, so let's apply the Limit Comparison Test to

$$a_n = \frac{1+2^n}{3^n}, \qquad b_n = \frac{2^n}{3^n}.$$

Using L'Hopital,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1+2^n}{2^n} = \lim_{n\to\infty} \frac{(\ln 2)2^n}{(\ln 2)2^n} = 1,$$

so the Limit Comparison Test applies. Since $\sum_{n=1}^{\infty} b_n$ is a geometric series with ratio 2/3, it converges. By the Limit Comparison Test, $\sum_{n=1}^{\infty} a_n$ also converges.

(3) *Determine whether the given series converges or diverges.*

$$\sum_{n=1}^{\infty} ne^{-n}.$$

We'll use the Integral Test. Let $f(x) = xe^{-x}$. From looking at the formula, f is continuous everywhere and positive for $x > 0$. Since $f'(x) = (1-x)e^{-x}$, f is decreasing for $x > 1$. Thus the Integral Test applies. Now evaluate

$$\begin{aligned}
\int_5^{\infty} xe^{-x}\,dx &= \lim_{b\to\infty} -xe^{-x} - e^{-x}\Big|_5^b \qquad \text{(use integration by parts)} \\
&= 6e^{-5} + \lim_{b\to\infty} -be^{-b} - e^{-b} \\
&= 6e^{-5} \qquad \text{(use L'Hopital on the first term)}
\end{aligned}$$

Because $\int_5^{\infty} xe^{-x}\,dx$ converges,

$$\sum_{n=1}^{\infty} ne^{-n}.$$

also converges by the Integral Test.

Note: I picked 5 to start the integral to show that it doesn't really matter where you start, as long as you start somewhere after any discontinuities of f.